Topics in probability theory: Measure theory

Zhenyao Sun

Beijing Institute of Technology

September 9, 2024

Restriction of outer measure

Theorem 2.1

Let μ be an outer measure on S, and write $\mathcal{S}$ for the class of μ-measurable sets. Then $\mathcal{S}$ is a σ-field. The restriction of μ to $\mathcal{S}$ is a measure.

- outer measure?
- μ-measurable sets?
- σ-field?
- measure?

Outer measure

- Let S be a space.
- Let 2^S be the collection of subsets of S.
- Let $\mu : 2^S \to [0, \infty]$ be a set function.
- We say μ is an outer measure if
  - μ is non-decreasing: $A \subset B$ implies $\mu(A) \leq \mu(B)$;
  - μ is countably sub-additive: $\mu(\cup_{k=1}^{\infty} A_i) \leq \sum_{k=1}^{\infty} \mu(A_i)$;
  - $\mu(\emptyset) = 0$.

μ-measurable sets

- Let μ be an outer measure for a space S.
- We say a subset $A \subset S$ is μ-measurable if for every $B \subset S$,

$$\mu(B) = \mu(B \cap A) + \mu(B \cap A^c).$$

σ-field

- Let S be a given set.
- Let $2^S := \{A : A \subset S\}$ the collection of subsets of S.
- We say $\mathcal{S}$ is a σ-field for S, if
  - $\mathcal{S} \subset 2^S$;
  - $\emptyset \in \mathcal{S}$;
  - $\mathcal{F}$ is closed under countable unions: $A_1, A_2, \dots \in \mathcal{S}$ implies $\cup_{k=1}^{\infty} A_k \in \mathcal{S}$;
  - $\mathcal{S}$ is closed under countable intersection: $A_1, A_2, \dots \in \mathcal{S}$ implies $\cap_{k=1}^{\infty} A_k \in \mathcal{S}$.
  - $\mathcal{S}$ is closed under complementation: $A \in \mathcal{S}$ implies $A^c \in \mathcal{S}$.
- We say $(S, \mathcal{S})$ is a measurable space if $\mathcal{S}$ is a σ-field of S.

Measure

- Let $\mathcal{S}$ be a σ-field of a space S.
- Let μ be a function from $\mathcal{S}$ to $[0, \infty]$.
- We say μ is a measure on $(S, \mathcal{S})$ if
    - $\mu(\emptyset) = 0$;
    - μ is countably additive: Let $A_1, A_2, \cdots \in \mathcal{S}$ be disjoint, then $\mu(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} A_k$.
- We say $(S, \mathcal{S}, \mu)$ is a measure space if $(S, \mathcal{S})$ is a measurable space and μ is a measure on $(S, \mathcal{S})$.

Restriction of outer measure

Theorem 2.1

Let μ be an outer measure on S, and write $\mathcal{S}$ for the class of μ-measurable sets. Then $\mathcal{S}$ is a σ-field. The restriction of μ to $\mathcal{S}$ is a measure.

- The theorem above is one of the main reason why we need the concept of σ-field.
- It is a powerful tool to construct measures like Lebesgue's measure and Hausdorff measure.

Monotone-class theorem

Theorem 1.1

For any π-system $\mathcal{C}$ and λ-system $\mathcal{D}$ in a space S, we have

$$\mathcal{C} \subset \mathcal{D} \implies \sigma(\mathcal{C}) \subset \mathcal{D}$$

- π-system?
- λ-system?
- What is $\sigma(\mathcal{C})$, the σ-field generated by $\mathcal{C}$?

π-system

- Let S be a given set.
- Let $2^S := \{A : A \subset S\}$ be the collection of subsets of S.
- We say $\mathcal{C}$ is a π-system w.r.t. S, if
  - $\mathcal{C} \subset 2^S$ and
  - $\mathcal{C}$ is closed under intersection: $A, B \in \mathcal{C}$ implies $A \cap B \in \mathcal{C}$.

λ-system

- Let S be a given set.
- Let $2^S := \{A : A \subset S\}$ be the collection of subsets of S.
- We say $\mathcal{D}$ is a λ-system w.r.t. S, if
  - $\mathcal{D} \subset 2^S$;
  - $S \in \mathcal{D}$;
  - $\mathcal{D}$ is closed under proper difference: $A, B \in \mathcal{D}$ and $A \subset B$ implies $B \setminus A \in \mathcal{D}$;
  - $\mathcal{D}$ is closed under increasing limits: $A_1, A_2, \cdots \in \mathcal{D}$ with $A_k \subset A_{k+1}$ for every $k \in \mathbb{N}$ implies $A_\infty := \cup_{k=1}^{\infty} A_k \in \mathcal{D}$.

Generated σ-field

- Let S be a given set.
- Let $2^S := \{A : A \subset S\}$ be the collection of subsets of S.
- Let $\mathcal{C} \subset 2^S$ be non-empty.
- Define

$$\sigma(\mathcal{C}) := \bigcap_{\mathcal{A} \text{ is a } \sigma\text{-field of } S;\ \mathcal{C} \subset \mathcal{A}} \mathcal{A}.$$

- It is known that $\sigma(\mathcal{C})$ is a σ-field.
- $\sigma(\mathcal{C})$ is called the σ-field generated by $\mathcal{C}$.

Monotone-class theorem

Theorem 1.1

For any π-system $\mathcal{C}$ and λ-system $\mathcal{D}$ in a space S, we have

$$\mathcal{C} \subset \mathcal{D} \implies \sigma(\mathcal{C}) \subset \mathcal{D}$$

- It can be used to prove many useful result like:
  - If two measures agrees on a π-system $\mathcal{C}$ then they agrees on $\sigma(\mathcal{C})$.
  - If the sets from two π-systems $\mathcal{C}_1$ and $\mathcal{C}_2$ are independent under a given probability measure, then the sets from $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent.

Polish and Borel spaces

Theorem 1.8

Any uncountable Polish space S is Borel isomorphic to $\mathbb{R}$.

- Polish space?
- Borel sets?
- Borel isomorphic?

Topological space

- Let S be a space.
- Let $\mathcal{S} \subset 2^S$.
- We say $(S, \mathcal{S})$ is a topological space if
  - $\emptyset \in \mathcal{S}$, $S \in \mathcal{S}$;
  - $\mathcal{S}$ is closed under arbitrary union: $\{A_\lambda : \lambda \in \Lambda\} \subset \mathcal{S}$ implies $\cup_{\lambda \in \Lambda} A_\lambda \in \mathcal{S}$;
  - $\mathcal{S}$ is closed under finite intersection: $A_1, \ldots, A_n \in \mathcal{S}$ implies $\cap_{k=1}^n A_k \in \mathcal{S}$.
- If $(S, \mathcal{S})$ is a topological space, then the elements in $\mathcal{S}$ are called open sets.

Separable

- Let $S = (S, \mathcal{S})$ be a topological space.
- We say a countable subset $\{x_n : n \in \mathbb{N}\}$ of S is dense if for any nonempty open set $A \in \mathcal{S}$, there exists an $m \in \mathbb{N}$ such that $x_m \in A$.
- We say the topological space S is separable, if there exists a countable dense subset.

Complete metric space

- Let S be a space.
- Let $d : S^2 \to [0, \infty)$ be a metric on S, i.e.
  - $d(x, x) = 0$ for all $x \in S$;
  - $d(x, y) > 0$ for all $x \neq y$ in S;
  - $d(x, y) = d(y, x)$ for all x, y in S;
  - $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z in S.
- We say a sequence $(x_i)_{i=1}^{\infty}$ is Cauchy if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, $\forall m, n > N$, $d(x_m, x_n) \leq \epsilon$.
- We say a metric space (S, d) is complete, if for every Cauchy sequence $(x_i)_{i=1}^{\infty}$, there exists an $x \in S$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

Completely metrizable

- Let (S, d) be a metric space.
- We say $A \subset S$ is open w.r.t. d if $\forall x \in A$, $\exists \epsilon > 0$ such that

$$\{y \in S : d(x, y) < \epsilon\} \subset A.$$

- Denote by $\mathcal{S}$ the collection of all open set w.r.t. d. Then, $(S, \mathcal{S})$ is a topological space.
- We say $\mathcal{S}$ is the topology induced by the metric d.
- A topology is called completely metrizable, if it can be induced by some compelete metric.

Polish space

- Let $(S, \mathcal{S})$ be a topological space.
- We say $(S, \mathcal{S})$ is Polish if
  - it is separable; and
  - it is completely metrizable.

Borel sets

- Let $(S, \mathcal{S})$ be a topological space.
- Let $\mathcal{B}_S = \sigma(\mathcal{S})$ be the σ-field generated by $\mathcal{S}$.
- We call $\mathcal{B}_S$ the Borel σ-field w.r.t. $(S, \mathcal{S})$.
- Elements in $\mathcal{B}_S$ are called Borel sets.

Measurable map

- Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces.
- We say a map $f : X \to Y$ is measurable map from $(X, \mathcal{X})$ to $(Y, \mathcal{Y})$ if for every $B \in \mathcal{Y}$,

$$f^{-1}(B) := \{x \in X : f(x) \in B\} \in \mathcal{X}.$$

- In particular, if $(Y, \mathcal{Y}) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, then f is called a measurable function.

Borel isomorphic

- Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be two topological spaces.
- Let $\mathcal{B}_S$ and $\mathcal{B}_T$ be the Borel σ-fields generated by $\mathcal{S}$ and $\mathcal{T}$ respectively.
- We say the topological space $(S, \mathcal{S})$ is Borel isomorphic to $(T, \mathcal{T})$, if there exists a bijection $f : S \to T$ such that
  - f is a measurable map from $(S, \mathcal{B}_S)$ to $(T, \mathcal{B}_T)$; and
  - f^{-1} is a measurable map from $(T, \mathcal{B}_T)$ to $(S, \mathcal{B}_S)$.

Theorem 1.8

Any uncountable Polish space S is Borel isomorphic to $\mathbb{R}$.

- This is a powerful tool when handling measure theory on Polish space.
- Many important state spaces for random elements are Polish, for example, the Wiener space $C([0,1],\mathbb{R})$ equipped with the supremum norm.

Monotone convergence

Theorem 1.21

For any non-negative measurable functions $f, f_1, f_2, \dots$ on a measure space $(S, \mathcal{S}, \mu)$, we have

$$0 \le f_n \uparrow f \implies \mu(f_n) \uparrow \mu(f).$$

- $0 \le f_n \uparrow f$ means that for every $s \in S$, $0 \le f_n(s) \uparrow f(s)$.
- Integral $\mu(f_n)$ and $\mu(f)$?

Integral for simple functions

- We will define the integral

$$\mu(f) = \int f \mathrm{d}\mu = \int f(\omega)\mu(\mathrm{d}\omega).$$

- We say a measurable function ϕ on a measure space $(S, \mathcal{S}, \mu)$ is simple if $\exists n \in \mathbb{Z}_+$, $c_1, \dots, c_n \in [0, \infty)$ and $A_1, \dots, A_n \in \mathcal{S}$ such that

$$\phi = c_1 \mathbf{1}_{A_1} + \cdots + c_n \mathbf{1}_{A_n},$$

and in this case,

$$\mu(\phi) := c_1 \mu(A_1) + \dots c_n \mu(A_n).$$

Integral for non-negative measurable functions

- Let f be a non-negative measurable function on a measure space $(S, \mathcal{S}, \mu)$.
- It can be shown that there exists a sequence of simple measurable functions $f_1, f_2, \ldots$ such that $0 \leq f_n \uparrow f$.
- It can be verified that $\mu(f_n)$ is non-decreasing.
- Define $\mu(f) := \lim_{n \to \infty} \mu(f_n) \in [0, \infty]$.
- We say f is integrable if $\mu(f) < \infty$.

Integral for integrable functions

- Let f be a measurable function on a measure space $(S, \mathcal{S}, \mu)$.
- Define $f^+ = \max\{f, 0\}$ and $f^- = (-f)^+$. Then $f = f^+ - f^-$.
- We say f is integrable if f^+ and f^- are integrable.
- If f is integrable, define $\mu(f) := \mu(f^+) - \mu(f^-)$.

Monotone convergence

Theorem 1.21

For any non-negative measurable functions $f, f_1, f_2, \ldots$ on a measure space $(S, \mathcal{S}, \mu)$, we have

$$0 \le f_n \uparrow f \implies \mu(f_n) \uparrow \mu(f).$$

- A powerful tool to exchange order between expectation and limits.
- Some times is used to construct random objects.

Product measure

Theorem 1.29

For any σ-finite measure spaces $(S, \mathcal{S}, \mu)$ and $(T, \mathcal{T}, \nu)$, there exists a unique measure $\mu \otimes \nu$ on $(S \times T, \mathcal{S} \otimes \mathcal{T})$, such that

$$(\mu \otimes \nu)(B \times C) = \mu(B) \times \nu(C), \quad B \in \mathcal{S}, C \in \mathcal{T}.$$

- We say a measure space $(S, \mathcal{S}, \mu)$ is σ-finite if there exists $S_1, S_2, \dots$ in $\mathcal{S}$ such that $S = \bigcup_{k=1}^{\infty} S_k$ and $\mu(S_k) < \infty$ for every $k \in \mathbb{N}$.
- For given sets A and B, the product space $A \times B := \{(a, b) : a \in A, b \in B\}$.
- $\mathcal{S} \otimes \mathcal{T} := \sigma(\mathcal{S} \times \mathcal{T})$.

Fubini's theorem

Theorem 1.29

For any σ-finite measure spaces $(S, \mathcal{S}, \mu)$ and $(T, \mathcal{T}, \nu)$, and any measurable function $f : S \times T \to \mathbb{R}$ integrable against $\mu \otimes \nu$, we have

$$(\mu \otimes \nu)(f) = \int_S \left(\int_T f(s,t)\nu(\mathrm{d}t) \right) \mu(\mathrm{d}s)$$
$$= \int_T \left(\int_S f(s,t)\mu(\mathrm{d}s) \right) \nu(\mathrm{d}t).$$

- Fubini's theorem is a powerful tool to exchange the order of integrals.

Thanks!