Topics in probability theory: Martingale

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Conditional expectation

Theorem 8.1

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For any sub σ-field $\mathcal{F} \subset \mathcal{A}$, there exists an a.s. unique map $\mathbb{E}^{\mathcal{F}} = \mathbb{E}[\cdot|\mathcal{F}]$ from $L^1(\mathcal{A})$ to $L^1(\mathcal{F})$, such that

$$\mathbb{E}[\mathbb{E}^{\mathcal{F}}[\xi]\mathbf{1}_A] = \mathbb{E}[\xi \mathbf{1}_A], \quad \xi \in L^1(\mathcal{A}), A \in \mathcal{F}.$$

- Sub σ-field?
- $L^1(\mathcal{A})$ and $L^1(\mathcal{F})$?

Sub-σ field

- Let $(\Omega, \mathcal{A})$ be a measurable space.
- We say $\mathcal{F}$ is a sub-σ field, if $\mathcal{F}$ is a σ-field and $\mathcal{F} \subset \mathcal{A}$.

$L^1(\mathcal{A})$ and $L^1(\mathcal{F})$

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.
- Let $\mathcal{F}$ be a sub-σ-field of $\mathcal{A}$.
- It can be verified that $\mathbb{P}|_{\mathcal{F}}$, the restriction of $\mathbb{P}$ on $\mathcal{F}$ is a probability measure on the measurable space $(\Omega, \mathcal{F})$.
- With an abuse of notation, we still write $\mathbb{P}|_{\mathcal{F}}$ by $\mathbb{P}$.
- We say $\xi \in L^1(\mathcal{A})$ if it is an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$.
- We say $\xi \in L^1(\mathcal{F})$ if it is an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

Conditional expectation

Theorem 8.1

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For any sub σ-field $\mathcal{F} \subset \mathcal{A}$, there exists an a.s. unique linear operator $\mathbb{E}^{\mathcal{F}} = \mathbb{E}[\cdot|\mathcal{F}]$ from vector spaces $L^1(\mathcal{A})$ to $L^1(\mathcal{F})$, such that

$$\mathbb{E}[\mathbb{E}^{\mathcal{F}}[\xi]\mathbf{1}_A] = \mathbb{E}[\xi \mathbf{1}_A], \quad \xi \in L^1(\mathcal{A}), A \in \mathcal{F}.$$

- Conditional expectation is the expected value of a random variable given the information of the sub-σ-field.

Martingale convergence

Theorem 9.19

Let $(X_n)_{n=1}^{\infty}$ be an L^1-bounded sub-martingale w.r.t. a filtration $(\mathcal{F}_n)_{n=1}^{\infty}$. Then $(X_n)_{n=1}^{\infty}$ converges in $\mathbb{R}$ almost surely.

- Filtration.
- Sub-martingale/martingale.
- L^1-bounded.

Filtration

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.
- Supoose that $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a sequence of σ-fields.
- Suppose that $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{A}$ for every $n \in \mathbb{N}$.
- Then we say $(\mathcal{F}_n)_{n=1}^{\infty}$ is a filtration.

Sub-martingale

- Suppose that $(X_n)_{n=1}^{\infty}$ is a stochastic process, i.e. a sequence of random variables, on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_n)_{n=1}^{\infty}$.
- Suppose that the process $(X_n)_{n=1}^{\infty}$ is adapted to the filtration $(\mathcal{F}_n)_{n=1}^{\infty}$. That is to say, $\sigma(X_n) \subset \mathcal{F}_n$ for every $n \in \mathbb{N}$.
- Suppose that for every $n \in \mathbb{N}$, almost surely,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n,$$

- Then we say $(X_n)_{n=1}^{\infty}$ is a submartingale w.r.t. filtration $(\mathcal{F}_n)_{n=1}^{\infty}$.

Super-martingale and martingale

- Suppose that $(X_n)_{n=1}^{\infty}$ is a stochastic process, adapted to a filtration $(\mathcal{F}_n)_{n=1}^{\infty}$.
- We say $(X_n)_{n=1}^{\infty}$ is a super-martingale if $(-X_n)_{n=1}^{\infty}$ is a sub-martingale.
- We say $(X_n)_{n=1}^{\infty}$ is a martingale, if it is sub-martingale and a super-martingale.

L^1-bounded

- Suppose that $(X_t)_{t\in T}$ is a family of random variables.
- We say $(X_t)_{t\in T}$ is L^1-bounded, if

$$\sup_{t\in T} \|X_t\|_{L^1} = \sup_{t\in T} \mathbb{E}[|X_t|] < \infty.$$

Martingale convergence

Theorem 9.19

Let $(X_n)_{n=1}^{\infty}$ be an L^1-bounded sub-martingale w.r.t. a filtration $(\mathcal{F}_n)_{n=1}^{\infty}$. Then $(X_n)_{n=1}^{\infty}$ converges in $\mathbb{R}$ almost surely.

- Of cause, the result still holds if 'sub-martingale' is replaced by 'super-martingale' or 'martingale'.

Regularization of martingale

Theorem 9.28

Let $(X_t)_{t\geq 0}$ be a sub-martingale w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypothesis. Then $(X_t)_{t\geq 0}$ has a càdlàg modification iff $(\mathbb{E}[X_t])_{t\geq 0}$ is right continuous, hence in particular when $(X_t)_{t\geq 0}$ is a martingale.

- Continuous-time filtration?
- Usual hypothesis?
- Continuous-time sub-martingale/super-martingale/martingales?
- Càdlàg modification?

Continuous-time filtration

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.
- Let $(\mathcal{F}_t)_{t \geq 0}$ be a family of σ-fields of Ω.
- We say $(\mathcal{F}_t)_{t \geq 0}$ is a (continuous-time) filtration, if $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{A}$ for every $s \leq t$.

Usual hypothesis

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space.
- We say $U \in \mathcal{A}$ is a null set, if $\mathbb{P}(U) = 0$. The collection of all null sets are denoted by $\mathcal{N}$.
- We say a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual hypothesis, if the following condition holds:
    - The probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is complete. That is to say if $V \subset U$ and $U \in \mathcal{N}$, then $V \in \mathcal{N}$.
    - The filtration is right-continuous. That is to say $\mathcal{F}_s = \cap_{t>s} \mathcal{F}_t$ for every $s \geq 0$.
    - $\mathcal{N} \subset \mathcal{F}_0$.

Continuous sub-martingales

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$.
- Let $(X_t)_{t\geq 0}$ be a real-valued stochastic process, i.e. a family of random variables indexed by $[0, \infty)$.
- Let $(X_t)_{t\geq 0}$ be adapted. That is to say, $\sigma(X_t) \subset \mathcal{F}_t$ for every $t \geq 0$.
- We say $(X_t)_{t\geq 0}$ is a sub-martingal, if $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ almost surely for every $s \leq t$.
- We say $(X_t)_{t\geq 0}$ is a super-martingal, if $(-X_t)_{t\geq 0}$ is a sub-martingale.
- We say $(X_t)_{t\geq 0}$ is a martingal, if $(X_t)_{t\geq 0}$ is a sub-martingale and a super-martingale.

Càdlàg version

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.
- Let $(X_t)_{t\geq 0}$ and $(\tilde{X}_t)_{t\geq 0}$ be two stochastic processes on Ω.
- We say $(\tilde{X}_t)_{t\geq 0}$ is a version of $(X_t)_{t\geq 0}$ if

$$\mathbb{P}(X_t = \tilde{X}_t) = 1, \forall t \geq 0.$$

- We say $(\tilde{X}_t)_{t\geq 0}$ is càdlàg, if

$$\mathbb{P}(X_{t-}, X_{t+} \in \mathbb{R}, X_t = X_{t+}, \forall t \geq 0) = 1$$

  where $X_{t-} := \lim_{s\uparrow t} X_s$ and $X_{t+} := \lim_{s\downarrow t} X_s$.
- We say $(X_t)_{t\geq 0}$ has a càdlàg modification if it has a version which is càdlàg.

Regularization of martingale

Theorem 9.28

Let $(X_t)_{t\geq 0}$ be a sub-martingale w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypothesis. Then (X_t) has a càdlàg version iff $(\mathbb{E}[X_t])_{t\geq 0}$ is right continuous, hence in particular when (X_t) is a martingale.

- In the rest of this course, any filtered probability space, i.e. a probability space equipped with a (continuous time) filtration, will be assumed to satisfy the usual hypothesis, unless stated otherwise.
- Any (continuous time) sub-martingale/ super-martingale/ martingale will be assumed to be càdlàg, unless stated otherwise.

Optional sampling

Theorem 9.30

Let $(X_t)_{t\geq 0}$ be a martingale w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Consider optional times σ and τ, where τ is bounded. Then X_τ is integrable, and almost surely

$$X_{\sigma\wedge\tau} = \mathbb{E}[X_\tau|\mathcal{F}_\sigma].$$

- Optional time?
- Sigma filed $\mathcal{F}_\sigma$?

Optional time

- Suppose that τ is a random variable taking values in $[0, \infty)$ in a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.
- We say τ is an optional time, if $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Sigma field associated with an optional time

- Suppose that τ is an optional time w.r.t. a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
- Define a sub-σ-field

$$\mathcal{F}_\tau := \{A \in \mathcal{A} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}.$$

- We say $\mathcal{F}_\tau$ is the sigma-field associated with the optional time τ.

Optional sampling

Theorem 9.30

Let $(X_t)_{t\geq 0}$ be a martingale w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Consider optional times σ and τ, where τ is bounded. Then X_τ is integrable, and almost surely

$$X_{\sigma\wedge\tau} = \mathbb{E}[X_\tau|\mathcal{F}_\sigma].$$

- In the context of gambling and financial mathematics, the OST ensures that if a gambler follows a fair game (modeled by a martingale) and stops playing at a bounded random time, their expected wealth at the stopping time equals their initial wealth.
- This result is important for understanding the behavior of fair games and in the pricing of financial derivatives.

First entry

Theorem 9.7

Let the set $A \subset \mathbb{R}_+ \times \Omega$ be progressive w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Then the first entry time $\tau(\omega) := \inf\{t \geq 0 : (t,\omega) \in A\}$ of A is optional w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

- Progressive?

Progressive

- Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.
- Let $A \subset \Omega \times [0, \infty)$.
- We say A is progressive if $A_t := A \cap (\Omega \times [0, t])$ is a measurable subset of the product space $(\Omega \times [0, t], \mathcal{F}_t \otimes \mathcal{B}_{[0,t]})$ for every $t \geq 0$.
- The collection of all progressive subset of $\Omega \times [0, \infty)$ is denoted by $\mathcal{P}$.
- It can be verified that $\mathcal{P}$ is a σ-field.

First entry

Theorem 9.7

Let the set $A \subset \mathbb{R}_+ \times \Omega$ be progressive w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Then the first entry time $\tau(\omega) := \inf\{t \geq 0 : (t,\omega) \in A\}$ of A is optional w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

- This theorem gives a criteria for optional times.

Doob-Meyer decomposition

Theorem 10.5

For an adapted càdlàg process $(X_t)_{t\geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$, these conditions are equivalent:

- $(X_t)_{t\geq 0}$ is a local sub-martingale,
- $X_t = M_t + A_t$ for all $t \geq 0$ almost surely for a local martingale $(M_t)_{t\geq 0}$ and a locally integrable, non-decreasing, predictable process $(A_t)_{t\geq 0}$ with $A_0 = 0$.

The process $(M_t)_{t\geq 0}$ and $(A_t)_{t\geq 0}$ are then almost surely unique.

- Local sub-martingale?
- Local martingale?
- Locally integrable process?
- Predictable process?

Local sub-martingale

- Let $(X_t)_{t\geq 0}$ be an adapted càdlàg process w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$.
- Suppose that there exists a sequence of optional times $(\tau_n)_{n\in\mathbb{N}}$ satisfying that
  - almost surely, $\tau_1 \leq \tau_2 \leq \dots$;
  - almost surely, $\tau_n \uparrow \infty$;
  - for every $n \in \mathbb{N}$, $(X_{t\wedge\tau_n})_{t\geq 0}$ is a sub-martingale.
- Then, we say $(X_t)_{t\geq 0}$ is a local sub-martingale.
- Local martingale and local super-martingale are defined in a similar way.

Locally integrable non-decreasing process

- Let $(A_t)_{t\geq 0}$ be a non-decreasing stochastic process on a filtered probability space.
- We say $(A_t)_{t\geq 0}$ is an integrable non-decreasing process, if $\mathbb{E}[A_\infty] < \infty$.
- We say $(A_t)_{t\geq 0}$ is a locally integrable non-decreasing process, if there exists a sequence of optional times $(\tau_n)_{n\in\mathbb{N}}$ satisfying that
  - almost surely, $\tau_1 \leq \tau_2 \leq \dots$;
  - almost surely, $\tau_n \uparrow \infty$; and
  - for every $n \in \mathbb{N}$, $(A_{t\wedge\tau_n})_{t\geq 0}$ is a integrable non-decreasing process.

Predictable process

- Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.
- We say a map $X : (\omega, t) \mapsto X(t, \omega)$, from $\Omega \times [0, \infty)$ to $\mathbb{R}$, is associated with an adapted continuous process, if
  - For every $t \geq 0$, the map $X_t : \omega \mapsto X(t, \omega)$ is a random variable on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$; and
  - for almost surely all ω, the path $t \mapsto X(t, \omega)$ is continuous.
- Denote by $\mathcal{O}$ be the σ-field of $\Omega \times [0, \infty)$ generated by all the maps X which are associated with an adapted continuous processes.
- We say a stochastic process $(Y_t)_{t\geq 0}$ is predictable, if the map $Y : (\omega, t) \mapsto Y_t(\omega)$ is a measurable map from $(\Omega \times [0, \infty), \mathcal{O})$ to $\mathbb{R}$.

Doob-Meyer decomposition

Theorem 10.5

For an adapted càdlàg process $(X_t)_{t\geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypothesis, these conditions are equivalent:

- $(X_t)_{t\geq 0}$ is a local sub-martingale,
- $X_t = M_t + A_t$ for all $t \geq 0$ almost surely for a local martingale $(M_t)_{t\geq 0}$ and a locally integrable, non-decreasing, predictable process $(A_t)_{t\geq 0}$ with $A_0 = 0$.

The process $(M_t)_{t\geq 0}$ and $(A_t)_{t\geq 0}$ are then almost surely unique.

- Doob-Meyer decomposition is essential in defining quadratic variation and covariation of processes, which are fundamental to understanding the dynamics of continuous-time stochastic processes.

Thanks!