Topics in probability theory: Brownian Motions

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September 23, 2024

Gaussian processes

Theorem (Existence of Gaussian process)

Let T be any index set and let $M : T \to \mathbb{R}$ by any function. Let $K : T \times T \to \mathbb{R}$ be a non-negative definite function. Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $(X_t)_{t\in T}$ such that $(X_t)_{t\in T}$ is a Gaussian process with mean M and covariance K.

- Non-negative definite function?
- Gaussian process?
- Mean and covariance of a Gaussian process?

Non-negative definite function

- Let $A = (a_{i,j})_{i,j=1}^n$ be an $n \times n$ symmetric matrix with real entries.
- We say A is non-negative definite if for any $x \in \mathbb{R}^n$, we have

$$\langle x, Ax \rangle = \sum_{i=1}^d x_i (Ax)_i = \sum_{i=1}^d \sum_{j=1}^d x_i a_{i,j} x_j \geq 0.$$

- We say a function $K : T \times T \to \mathbb{R}$ is non-negative definite if, for every $t_1, \ldots, t_n$ in T, the matrix $(K(t_i, t_j))_{i,j=1}^n$ is non-negative definite.

Gaussian process

- Let T be any index set.
- We say $(X_t)_{t\in T}$ is a Gaussian process, if for any $n \in \mathbb{N}$, $t_1, \ldots, t_n$ in T, and $\lambda_1, \ldots, \lambda_n$ in $\mathbb{R}$, the random variable

$$\lambda_1 X_{t_1} + \cdots + \lambda_n X_{t_n}$$

has a normal distribution.

Mean and covariance of a Gaussian process

- Let T be any index set.
- Let $(X_t)_{t\in T}$ be a Gaussian process.
- We say a function $M : T \to \mathbb{R}$ is the mean of the Gaussian process X, if $\mathbb{E}[X_t] = M(t)$ for every $t \in T$.
- We say a function $K : T \times T \to \mathbb{R}$ is the covariance of the Gaussian process X, if $\mathrm{Cov}(X_t, X_s) = K(t, s)$ for every t, s in T.

Gaussian processes

Theorem (Existence of Gaussian process)

Let T be any index set and let $M : T \to \mathbb{R}$ by any function. Let $K : T \times T \to \mathbb{R}$ be a non-negative definite function. Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $(X_t)_{t \in T}$ such that $(X_t)_{t \in T}$ is a Gaussian process with mean M and covariance K.

- This allows the modeling of a wide range of stochastic processes with spatial and time correlations, making Gaussian processes extremely flexible in applications like machine learning, statistics, and stochastic modeling.

Independent increments

Theorem 14.4

Let $(X_t)_{t\geq 0}$ be a continuous process in $\mathbb{R}^d$ with $X_0 = 0$. Then these conditions are equivalent:

1. X has independent increments,
2. X is Gaussian, and there exist some continuous functions $(b_t)_{t\geq 0}$ in $\mathbb{R}^d$ and $(a_t)_{t\geq 0}$ in $\mathbb{R}^d \times \mathbb{R}^d$ such that $X_t - X_s$ has mean $b_t - b_s$ and covariance $a_t - a_s$ for every $s \leq t$.

- Independent increments?

Independent increments

- Let $(X_t)_{t\geq 0}$ be a process in $\mathbb{R}^d$.
- We say $(X_t)_{t\geq 0}$ has independent increments, if for every $n \in \mathbb{N}$ and $0 \leq t_0 \leq \cdots \leq t_n$, the random variables

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

are independent.

Independent increments

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- Continuous processes with independent increments must be Gaussian!
- This is =crucial in the modeling of various stochastic phenomena.

Brownian motion

Theorem 14.5 (Definition of BM 1)

There exists a continuous process $(B_t)_{t\geq 0}$ in $\mathbb{R}$ with stationary, independent increments such that $B_0 = 0$, $\mathbb{E}[B_1] = 0$ and $\mathbb{E}[B_1^2] = 1$. Such process is unique in law.

- Stationary, independent increments?

Stationary, independent increments

- Let $(X_t)_{t\geq 0}$ be a process in $\mathbb{R}^d$.
- We say $(X_t)_{t\geq 0}$ has stationary increments, if for every $h \geq 0$ and $t \geq 0$, $\mathcal{L}(X_{t+h} - X_t) = \mathcal{L}(X_h - X_0)$.
- We say $(X_t)_{t\geq 0}$ has stationary independent increments, if it has stationary increments and independent increments.

Brownian motion

Theorem 14.5 (Definition of BM 1)

There exists a continuous process $(B_t)_{t\geq 0}$ in $\mathbb{R}$ with stationary, independent increments such that $B_0 = 0$, $\mathbb{E}[B_1] = 0$ and $\mathbb{E}[B_1^2] = 1$. Such process is unique in law.

- We call such process a standard Brownian motion.
- $(B_t)_{t\geq 0}$ is a continuous Gaussian process with mean 0 and covariance $\mathbb{E}[B_t B_s] = t \wedge s$.
- Brownian motion is the fundamental continuous stochastic process.

Invariance Principle

Theorem 23.6 (Definition of BM 2)

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with $\mathbb{E}[\xi_i] = 0$ and $\mathbb{E}\xi_i^2 = 1$. Let

$$S_t := \begin{cases} \sum_{k=1}^{n} \xi_k, & t = n \in \mathbb{Z}_+, \\ \text{continuous linear}, & t \in [n, n+1], n \in \mathbb{Z}_+. \end{cases}$$

Then, after the parabolic rescaling,

$$\left(\frac{1}{\sqrt{n}} S_{nt} \right)_{t \geq 0} \xrightarrow{d} \text{standard Brownian motion } (B_t)_{t \geq 0}, \quad n \to \infty$$

as random elements in Wiener space $C([0, \infty), \mathbb{R})$.

- This demonstrates the universality of Brownian motion as a continuous limit of random walks.

Brownian motion as Markov process

Theorem (Definition of BM 3)

Let $(\mu_t)_{t\geq 0}$ be the heat kernels of $\mathbb{R}$. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion $(B_t)_{t\geq 0}$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of $(B_t)_{t\geq 0}$. Then $(B_t)_{t\geq 0}$ is a homogeneous Markov process with transition kernels $(\mu_t)_{t\geq 0}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$, i.e.

$$\mathbb{P}(B_t \in A|\mathcal{F}_s) = \mu_{t-s}(B_s, A), \quad a.s. \ A \in \mathcal{B}_{\mathbb{R}}, 0 \leq s \leq t.$$

- Heat kernels?
- Natural filtration?

Heat kernels

- We say a family of probability kernels $(\mu_t)_{t\geq 0}$ on $\mathbb{R}$ are heat kernels, if $\mu_0(x,\cdot)=\delta_x(\cdot)$ and for $t>0$,

$$\mu_t(x,dy)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(y-x)^2}{2t}}\mathrm{d}y.$$

- Heal kernels are the fundamental solutions to the heat equation.
- Consider the heat equation

$$\begin{cases}\partial_t u_t(x)=\frac{\Delta}{2}u_t(x)\\ u_0(x)=f(x).\end{cases}$$

It is known that the solution is the mixture of fundamental solutions:

$$u_t(x)=\int f(y)\mu_t(x,\mathrm{d}y).$$

Natural filtration

- Let $(X_t)_{t\geq 0}$ be a given process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
- Define $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$.
- Then we say $(\mathcal{F}_t^X)_{t\geq 0}$ is the natural filtration of $(X_t)_{t\geq 0}$.

Brownian motion as Markov process

Theorem (Definition of BM 3)

Let $(\mu_t)_{t\geq 0}$ be the heat kernels of $\mathbb{R}$. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion $(B_t)_{t\geq 0}$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of $(B_t)_{t\geq 0}$. Then $(B_t)_{t\geq 0}$ is a homogeneous Markov process with transition kernels $(\mu_t)_{t\geq 0}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$, i.e.

$$\mathbb{P}(B_t \in A|\mathcal{F}_s) = \mu_{t-s}(B_s, A), \quad a.s.\ A \in \mathcal{B}_{\mathbb{R}}, 0 \leq s \leq t.$$

- This result connects Brownian motions to the heat equations $\partial_t u = \frac{\Delta}{2} u$.
- This gives another definition of Brownian motion.

Strong Markov property

Theorem 14.11

For every $x \in \mathbb{R}$, denote by P_x be the Wiener measure on $C_{\mathbb{R}}$ with initial value x. Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. For an $\mathcal{F}$-Brownian motion $(B_t)_{t\geq 0}$ in $\mathbb{R}$ and an $\mathcal{F}$-optional time $\tau < \infty$, $(B_t)_{t\geq 0}$ satisfies the strong Markov property at τ w.r.t. kernel P.

- Wiener measure?
- $\mathcal{F}$-Brownian motion?

Wiener measure

- Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.
- Since $(B_t)_{t\geq 0}$ has continuous paths, we can regard B as a random elements taking values in the Wiener space $C_{\mathbb{R}}$.
- For every $x \in \mathbb{R}$, the Wiener measure P_x is defined as the law of the $C_{\mathbb{R}}$-valued random element $(B_t + x)_{t\geq 0}$.

$\mathcal{F}$-Brownian motion

- Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.
- Let $(\mu_t)_{t\geq 0}$ be the heat kernels of $\mathbb{R}$.
- We say an $\mathcal{F}$-adapted continuous process $(B_t)_{t\geq 0}$ is an $\mathcal{F}$-Brownian motion, if it is a homogeneous Markov process with transition kernels $(\mu_t)_{t\geq 0}$ w.r.t. filtration $\mathcal{F}$.

Strong Markov property

Theorem 14.11

For every $x \in \mathbb{R}$, denote by P_x be the Wiener measure on $C_{\mathbb{R}}$ with initial value x. Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. For an $\mathcal{F}$-Brownian motion $(B_t)_{t\geq 0}$ in $\mathbb{R}$ and an $\mathcal{F}$-optional time $\tau < \infty$. Then $(B_t)_{t\geq 0}$ satisfies the strong Markov property at τ w.r.t. kernel P.

- The result says that, given B_τ, the future path $\theta_\tau B$ has the conditional distribution P_{B_τ}, and is independent of the past $\mathcal{F}_\tau$.

Lévy's martingale characterization of Brownian motion

Theorem 19.3 (Definition of BM 4)

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. The following two statements are equivalent:

1. Let $(B_t)_{t\geq 0}$ be a standard $\mathcal{F}$-Brownian motion. Then $(B_t)_{t\geq 0}$ and $(B_t^2 - t)_{t\geq 0}$ are $\mathcal{F}$-martingales.
2. Let $(M_t)_{t\geq 0}$ be a continuous $\mathcal{F}$-martingale satisfies the property that $(M_t^2 - t)_{t\geq 0}$ is also a $\mathcal{F}$-martingale. Then $(M_t)_{t\geq 0}$ is a standard $\mathcal{F}$-Brownian motion.

Regularity and Irregularity of Brownian path

Theorem 14.10

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then

$$\mathbb{P}\left((B_t)_{t\in[0,1]} \in C^p([0,1],\mathbb{R})\right) = \begin{cases} 1, & p \in (0,1/2), \\ 0, & p > 1/2. \end{cases}$$

In particular, $(B_t)_{t\geq 0}$ is not differentiable.

- Hölder space $C^p([0,1],\mathbb{R})$.

Hölder space

- Let $p \in [0,1)$. The Hölder space $C^p([0,1],\mathbb{R})$ is the space of continuous functions $f:[0,1]\to\mathbb{R}$ such that there exists a constant $C>0$ satisfying the Hölder condition:

$$|f(t)-f(s)| \leq C|t-s|^p, \quad s,t \in [0,1].$$

- If $p = n \in \mathbb{N}$, $C^p([0,1],\mathbb{R})$ represents the space of n-times continuously differentiable functions.
- If $p = n+\alpha$ where $n \in \mathbb{N}$ and $\alpha \in (0,1)$, then $C^p([0,1],\mathbb{R})$ represents the space of functions in C^n whose n-th derivative is in C^α.

Regularity and Irregularity of Brownian path

Theorem 14.10

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then

$$\mathbb{P}\left((B_t)_{t\in[0,1]} \in C^p([0,1],\mathbb{R})\right) = \begin{cases} 1, & p \in (0,1/2), \\ 0, & p > 1/2. \end{cases}$$

In particular, $(B_t)_{t\geq 0}$ is not differentiable.

- This result roughly says that $dB_t = B_{t+dt} - B_t \approx \sqrt{dt}$.

Quadratic variation

Theorem 14.9

Let B be a standard Brownian motion, and fix any $t > 0$. Then for any partitions $0 = t_{n,0} < t_{n,1} < \cdots < t_{n,k_n} = t$ with $h_n := \max\{t_{n,i} - t_{n,i-1} : i = 1, \dots, k_n\} \to 0$, we have

$$\sum_{i=1}^{k_n} (B_{t_{n,i}} - B_{t_{n,i-1}})^2 \to t, \quad \text{in} \quad L^2.$$

The convergence holds a.s. when the partitions are nested.

- Convergence in $L^p, p \geq 1$?

Convergence in $L^p, p \geq 1$

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.
- For a random variable ξ on Ω, we say $\xi \in L^p$ with $p \geq 1$ if

$$\|\xi\|_p := \mathbb{E}[|\xi|^p]^{1/p} < \infty.$$

- For a sequence of random variables $\xi_1, \xi_2, \dots$ and ξ in L^p, we say $\xi_n \to \xi$ in L^p if

$$\|\xi_n - \xi\|_p \to 0, \quad n \to \infty.$$

Quadratic variation

Theorem 14.9

Let B be a Brownian motion, and fix any $t > 0$. Then for any partitions $0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = t$ with $h_n := \max\{t_{n,i} - t_{n,i-1} : i = 1, \ldots, k_n\} \to 0$, we have

$$\sum_{i=1}^{k_n} (B_{t_{n,i}} - B_{t_{n,i-1}})^2 \to t, \quad \text{in} \quad L^2.$$

The convergence holds a.s. when the partitions are nested.

- This result basically says that $(dB_t)^2 = (B_{t+dt} - B_t)^2 = dt$.

Thanks!