Topics in probability theory: Itô calculus

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September 25, 2024

Covariation

Theorem 18.5

For any continuous local martingales $M = (M_t)_{t \geq 0}, N = (N_t)_{t \geq 0}$, there exists a continuous process $[M, N]$ with locally finite variation and $[M, N]_0 = 0$, such that $MN - [M, N]$ is a local martingale.

- Locally finite variation?

Locally finite variation?

- We say a function f on $[0,t]$ has finite variation $V_t(f)$, if

$$V_t(f) := \sup\left\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : 0 = t_0 < \cdots < t_n = t, n \in \mathbb{N}\right\}$$

  is finite.
- We say a function f on $[0,\infty)$ has locally finite variation, if $V_t(f) < \infty$ for every $t \geq 0$.
- We say a stochastic process $(A_t)_{t\geq 0}$ has locally finite variation, if almost surely its sample path has locally finite variation.

Covariation

Theorem 18.5

For any continuous local martingales $M = (M_t)_{t \geq 0}, N = (N_t)_{t \geq 0}$, there exists a continuous process $[M, N]$ with locally finite variation and $[M, N]_0 = 0$, such that $MN - [M, N]$ is a local martingale.

- Quadratic variation $[M] := [M, M]$.
- The existence of the covariation process is a cornerstone of stochastic calculus, allowing for the detailed study of the interactions between continuous local martingales and their products.

Approximation of covariation

Proposition 18.17

For any continuous martingales X, Y on $[0,t]$ and partitions $0 = t_0^n < \cdots < t_{k_n}^n = t, n \in \mathbb{N}$, with $\max_k(t_k^n - t_{k-1}^n) \to 0$, we have

$$\sum_{k=1}^{k_n} \left(X_{t_k^n} - X_{t_{k-1}^n} \right) \left(Y_{t_k^n} - Y_{t_{k-1}^n} \right) \to [X,Y]_t$$

in probability when $n \to \infty$.

- Convergence in probability? (Review)

Convergence in probability

- Let $(X_n)_{n=1}^{\infty}$ be a sequence of random elements in a complete separable metric space (S, d). Let X be a random element in S.
- We say $(X_n)_{n=1}^{\infty}$ converges to X in probability if for any $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}(d(X_n, X) \geq \epsilon) = 0.$$

- Convergence in probability is weaker than a.s. convergence and $L^p, p \geq 1$, convergence.

Approximation of covariation

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$$\sum_{k=1}^{k_n} \left(X_{t_k^n} - X_{t_{k-1}^n} \right) \left(Y_{t_k^n} - Y_{t_{k-1}^n} \right) \to [X, Y]_t$$

in probability when $n \to \infty$.

- This result explains the choice of the terminology.

Finite-variation martingales

Proposition 18.2

Let M be a continuous local martingale. Then

$$M \text{ has locally finite variation} \iff M \text{ is a.s. constant.}$$

- These two statements are also equivalent to $[M] = 0$.

Itô's Integral

Theorem 18.11

For any continuous local martingale M and process $V \in \mathscr{L}(M)$, there exists an a.s. unique continuous local martingale $V \cdot M$ with $(V \cdot M)_0 = 0$, such that for any continuous local martingale N,

$$[V \cdot M, N] = V \cdot [M, N], \quad a.s.$$

where the right hand side is Stieltjes' integral of V against $[M, N]$.

- $\mathscr{L}(M)$, M-integrable processes?

M-integrable process

- Let $(M_t)_{t\geq 0}$ be a continuous local martingale, defined in a filtered probability space, say $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
- Let $(V_t)_{t\geq 0}$ be a real-valued adapted process.
- We require that $(V_t)_{t\geq 0}$ is progressive, that is to say, for any $t \geq 0$, the map $(\omega, s) \mapsto V_s(\omega)$ from the product space $(\Omega \times [0,t], \mathcal{F}_t \otimes \mathcal{B}_{[0,t]})$ to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is measurable.
- We say a progressive $(V_t)_{t\geq 0}$ is M-integrable if for every $t > 0$, almost surely, $(V^2 \cdot [M])_t < \infty$, where $V^2 \cdot [M]$ is Stieltjes' integral.

Itô's Integral

Theorem 18.11

For any continuous local martingale M and process $V \in \mathscr{L}(M)$, there exists an a.s. unique continuous local martingale $V \cdot M$ with $(V \cdot M)_0 = 0$, such that for any continuous local martingale N,

$$[V \cdot M, N] = V \cdot [M, N], \quad a.s.$$

where the right hand side is Stieltjes' integral of V against $[M, N]$.

- This result gives the mathematical definition of Itô's integral.
- Sometimes, we write

$$(V \cdot M)_t = \int_0^t V_s \mathrm{d}M_s.$$

Chain rule

Lemma 18.14

For any continuous semi-martingale X and progressive U, V with $V \in \mathscr{L}(X)$, we have

1. $U \in L(V \cdot X) \iff UV \in L(X)$, and
2. $U \cdot (V \cdot X) = (UV) \cdot X$ a.s.

- Continuous semi-martingale?
- $\mathscr{L}(X)$, integrable processes for semi-martingale X?
- $V \cdot X$, the integral against a semi-martingale?

Continuous semi-martingale

- Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.
- We say an adapted continuous process $(X_t)_{t\geq 0}$ is a continuous semi-martingale, if it admits a decomposition $X = M + A$ into a continuous local martingale M and a continuous adapted process A of locally finite variation starting at 0.
- The decomposition is unique: if $M + A = M' + A'$, then $M - M' = A' - A$ is a martingale with locally finite variation starting at 0, so it must be the case that $M = M'$.

Stochastic integration against a semi-martingale

- Suppose that X is a continuous semi-martingale with decomposition $X = M + A$.
- We say $V \in \mathscr{L}(M)$ is integrable against X, if Stieltjes' integrals

$$(V \cdot A)_t = \int_0^t V_s \mathrm{d}A_s$$

  is well-defined for every $t \geq 0$ almost surely.
- In this case, we write $V \in \mathscr{L}(X)$ and define

$$V \cdot X := V \cdot M + V \cdot A.$$

Chain rule

Lemma 18.14

For any continuous semi-martingale X and progressive U, V with $V \in \mathscr{L}(X)$, we have

1. $U \in L(V \cdot X) \iff UV \in L(X)$, and
2. $U \cdot (V \cdot X) = (UV) \cdot X$ a.s.

- A fundamental result in stochastic analysis.

Itô's formula

Theorem 18.18

For any continuous semi-martingale X in $\mathbb{R}^d$ and function $f \in C^2(\mathbb{R}^d)$, we have almost surely that

$$f(X) = f(X_0) + \sum_{i=1}^{d} \partial_i f(X) \cdot X^i + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j f(X) \cdot [X^i, X^j].$$

- This second-order correction arises because semimartingales exhibit random fluctuations, and their 2rd order variation contributes to the overall change in $f(X)$.

Elementary stochastic integral

Theorem

For any elementary process

$$V_t = \sum_{k=1}^{n} \xi_k \mathbf{1}_{(t_k, t_{k+1}]}(t), \quad t \geq 0$$

and continuous semi-martingale $(X_t)_{t\geq 0}$, we have

$$(V \cdot X)_t = \sum_{k=1}^{n} \xi_k (X_{t\wedge t_{k+1}} - X_{t\wedge t_k}), t \geq 0, a.s.$$

- Elementary process?

Elementary stochastic integral

- Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.
- We say a process $(V_t)_{t\geq 0}$ is elementary, if

$$V_t = \sum_{k=1}^{n} \xi_k \mathbf{1}_{(t_k, t_{k+1}]}(t), \quad t \geq 0$$

where $n \in \mathbb{N}$, $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ are non-random, and $(\xi_k)_{k=1}^n$ is a family of bounded random variables, furthermore, ξ_k is $\mathcal{F}_{t_k}$-measurable for each k.

Elementary stochastic integral

Theorem

For any elementary process

$$V_t = \sum_{k=1}^{n} \xi_k \mathbf{1}_{(t_k, t_{k+1}]}(t), \quad t \geq 0$$

and continuous semi-martingale $(X_t)_{t \geq 0}$, we have

$$(V \cdot X)_t = \sum_{k=1}^{n} \xi_k (X_{t \wedge t_{k+1}} - X_{t \wedge t_k}), t \geq 0, a.s.$$

- This is known as the elementary stochastic integral.

Approximation by the elementary stochastic integrals

Lemma 18.23

For any continuous semi-martingale $X = M + A$ and process $V \in \mathscr{L}(X)$, there exists a sequence of elementary processes $V^1, V^2, \ldots,$ such that a.s., simultaneously for any $t > 0$,

$$\int_0^t (V_s^n - V_s)^2 \mathrm{d}[M]_s + \sup_{r \in [0,t]} \left| \int_0^r (V_s^n - V_s) \mathrm{d}A_s \right| \to 0, \quad n \to \infty.$$

And in this case, for every $t > 0$,

$$\sup_{r \in [0,t]} \left| \int_0^r V_s^n \mathrm{d}X_s - \int_0^r V_s \mathrm{d}X_s \right| \xrightarrow{p} 0, \quad n \to \infty.$$

- This result gives us another definition of Itô's integral.

Stochastic integral and random time change

Theorem 18.24

Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Let τ be a finite random time change with induced filtration $\mathcal{G}$. Let $X = M + A$ be a τ-continuous $\mathcal{F}$-semi-martingale. Then

- $X \circ \tau$ is a continuous $\mathcal{G}$-semi-martingale withdecomposition $M \circ \tau + A \circ \tau$, such that $[X \circ \tau] = [X] \circ \tau$ a.s.
- $V \in \mathscr{L}(X)$ implies $V \circ \tau \in \mathscr{L}(X \circ \tau)$ and

$$(V \circ \tau) \cdot (X \circ \tau) = (V \cdot X) \circ \tau \quad a.s.$$

- finite random time change?
- induced filtration by the random time change?
- continuous w.r.t. a time change τ?
- time changed process $X \circ \tau$?

Finite random time change

- Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.
- Let $(\tau_s)_{s\geq 0}$ be a family of optional times.
- We say $\tau = (\tau_s)_{s\geq 0}$ is a finite random time change, if τ_s is non-decreasing in s, right-continuous in s, and $\tau_s < \infty$ for every $s \geq 0$ almost surely.

Filtration induced by random time change

- Let $(\tau_s)_{s\geq 0}$ be a finite random time change in a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
- Define $\mathcal{G}_s := \mathcal{F}_{\tau_s}, s \geq 0$. It can be verified that $\mathcal{G}_s$ is also a filtration.
- We call $(\mathcal{G}_s)_{s\geq 0}$ the filtration induced by the time change τ.

Continuous w.r.t. a time change τ

- Let $(\tau_s)_{s\geq 0}$ be a finite random time change in a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
- A process X is said to be τ-continuous, if a.s. it is constant on every interval $[\tau_{s-}, \tau_s], s \geq 0$.
- Here, $\tau_{s-} := \lim_{r\uparrow s} \tau_r$, and $\tau_{0-} := 0$.

Time-changed process

- Let $(\tau_s)_{s\geq 0}$ be a finite random time change in a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
- Let $X = (X_t)_{t\geq 0}$ be an $\mathcal{F}$-adapted continuous process.
- Define a new process $Y_s = (X \circ \tau)_s = X_{\tau_s}$.
- We say Y is the time-changed process of X under the random time change τ.

Stochastic integral and random time change

Theorem 18.24

Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Let τ be a finite random time change with induced filtration $\mathcal{G}$. Let $X = M + A$ be a τ-continuous $\mathcal{F}$-semi-martingale. Then

- $X \circ \tau$ is a continuous $\mathcal{G}$-semi-martingale withdecomposition $M \circ \tau + A \circ \tau$, such that $[X \circ \tau] = [X] \circ \tau$ a.s.
- $V \in \mathscr{L}(X)$ implies $V \circ \tau \in \mathscr{L}(X \circ \tau)$ and

$$(V \circ \tau) \cdot (X \circ \tau) = (V \cdot X) \circ \tau \quad a.s.$$

- The structure of semi-martingale, quadratic variation, and stochastic integral is preserved under the random time change.

Martingale as time-changed Brownian motion

Theorem 19.4

Let M be a continuous local martingale w.r.t. filtration $(\mathcal{F}_t)_{t\geq 0}$ and $M_0 = 0$. Define random time change

$$\tau_s := \inf\{t \geq 0 : [M]_t > s\}, \quad s \geq 0,$$

and the induced filtration

$$\mathcal{G}_s := \mathcal{F}_{\tau_s}, \quad s \geq 0.$$

Then there exists a $\mathcal{G}$-Brownian motion such that almost surely

$$B_s = (M \circ \tau)_s = M_{\tau_s}, \quad s \in [0, [M]_\infty),$$

and

$$M_t = (B \circ [M])_t = B_{[M]_t}, \quad t \geq 0.$$

Thanks!