Limit theorems for a class of critical superprocesses with stable branching

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Background

Theorem (R. Slack (1968))

Let $\{(Z_n)_{n\geq 0}; P\}$ *be a critical Galton-Watson process with offspring generating function*

$$f(s) = s + (1-s)^{\alpha} l(1-s), \quad s \geq 0$$

where $\alpha \in (1,2]$ *and* l *is a slowly varing function at* 0.
Then

$$P(Z_n > 0) = n^{-1/(\alpha-1)} L(n)$$

where L *is slowly varying at* ∞; *and*

$$\left\{P(Z_n > 0)Z_n; P(\cdot|Z_n > 0)\right\} \xrightarrow[n\to\infty]{\text{law}} \mathbf{z}^{(\alpha-1)},$$

where $\mathbf{z}^{(\alpha-1)}$ *is a positive random variable with Laplace transform*

$$E[e^{-u\mathbf{z}^{(\alpha-1)}}] = 1 - (1 + u^{-(\alpha-1)})^{-1/(\alpha-1)}, \quad u \geq 0.$$

Background

	$\alpha = 2$: Analytical method	$\alpha = 2$: Probabilistic method	$\alpha \in (1, 2)$
GW	A. Kolmogorov (1938) A. Yaglom (1947) H. Kesten, P. Ney and F. Spitzer (1966)	R. Lyons, R. Pemantle and Y. Peres (1995) J. Geiger (1999) J. Geiger (2000) Y.-X. Ren, R. Song and Z. Sun (2018a)	V. Zolotarev (1957) R. Slack (1968)
Multitype GW	A. Joffe and F. Spitzer (1967)	V. Vatutin and E. Dyakonova (2001)	M. Goldstein and F. Hoppe (1978)
Continuous time GW	K. Athreya and P. Ney (1972)	-	V. Vatutin (1977)
Continuous time Multitype GW	K. Athreya and P. Ney (1974)	-	V. Vatutin (1977)
Branching Markov processes	S. Asmussen and H. Hering (1983)	E. Powell (2015)	S. Asmussen and H. Hering (1983)
CSBP	Z. Li (2000) A. Lambert (2007)	Y.-X. Ren, R. Song and Z. Sun (2018b+)	A. Kyprianou and J. Pardo (2008) Y.-X. Ren, T. Yang and G.-H. Zhao (2014)
Superprocesses	Evans and Perkins (1990) Y.-X. Ren, R. Song and R. Zhang (2015)	Y.-X. Ren, R. Song and Z. Sun (2018b+)	Y.-X. Ren, R. Song and Z. Sun (2018c+)

Settings

- E: locally compact separable metric space.
- $\mathcal{M}_f$: the collection of all the finite Borel measures on E.
- **Spatial motion** (ξ_t): an E-valued Hunt process with transition semigroup (P_t).
- **Branching mechanism** $\psi : E \times [0,\infty) \to [0,\infty)$ s.t.

$$\psi(x,z) := -\beta(x)z + \alpha(x)z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(x,dy).$$

  where $\beta \in b\mathscr{B}_E$; $\alpha \in bp\mathscr{B}_E$; π is a kernel from E to $(0,\infty)$ s.t. $\sup_{x\in E} \int_{(0,\infty)} (y \wedge y^2)\pi(x,dy) < \infty$.

Superprocesses

- $\mu(f) := \int f d\mu$.

Definition (Superprocesses)

Say a $\mathcal{M}_f$-valued Markov process $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f(E)}\}$ is a (ξ,ψ)-superprocess if

$$\mathbf{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)},$$

where $(t,x)\mapsto V_t f(x)$ on $[0,\infty)\times E$ is the unique locally bounded positive solution to the equation

$$V_t f(x) + \int_0^t P_{t-s}\psi\big(\cdot, V_s f(\cdot)\big)(x)ds = P_t f(x).$$

Superprocesses

- Superprocess arose as high-density limits of branching particle systems. (Watanabe 1968).

Example

Consider a Branching Brownian motion with

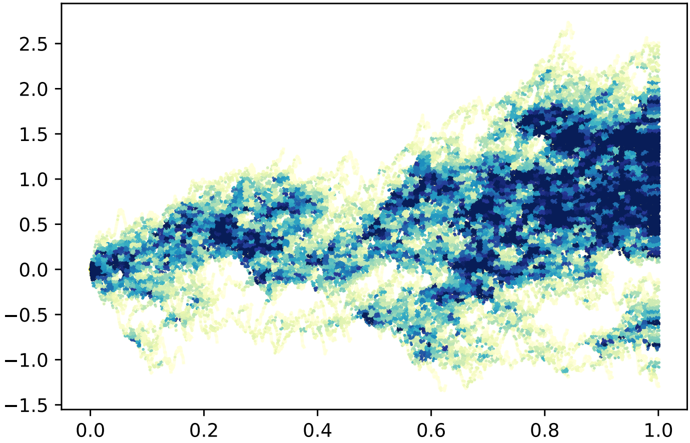
- k initial particles;
- killing rate $2k$;
- critical binary branching.

$X_t^{(k)}(A)$: number of particles the Borel set A at time t.

$$\left(\frac{1}{k}X_t^{(k)}(\cdot)\right)_{t\geq 0} \xrightarrow[k\to\infty]{law} (\xi,\Psi)\text{-superprocess}$$

with ξ being a Brownian motion and $\Psi(z) = z^2$.

Branching Brownian Motion with large k



Assumptions

- The mean behavior of Superprocess:

$$\mathbf{P}_{\delta_x}[X_t(f)] = P_t^{\beta} f(x) := \Pi_x[e^{\int_0^t \beta(\xi_r)dr} f(\xi_t)].$$

Assumption 1. (Compact operators)

There exist a σ-finite measure m with full support on E and a family of strictly positive, bounded continuous functions $\{p_t(\cdot,\cdot) : t > 0\}$ on $E \times E$ such that

- $P_t f(x) = \int_E p_t(x,y) f(y) m(dy)$
- $\int_E p_t(y,x) m(dy) \le 1$
- $\int_E \int_E p_t(x,y)^2 m(dx) m(dy) < \infty$

and the functions $x \mapsto \int_E p_t(x,y)^2 m(dy)$ and $x \mapsto \int_E p_t(y,x)^2 m(dy)$ are both continuous.

Assumptions

- $(P_t^{\beta})_{t\geq 0}$ and its disjoint $(P_t^{\beta *})_{t\geq 0}$ are strongly continuous semigroups of compact operators in $L^2(E,m)$.
- Transition density p_t^{β}: $P_t^{\beta} f(x) = \int_E p_t^{\beta}(x,y) f(y) m(dy)$
- L and L^*: the generators of $(P_t^{\beta})_{t\geq 0}$ and $(P_t^{\beta *})_{t\geq 0}$, respectively.
- $\sigma(L)$ and $\sigma(L^*)$: the spectra of L and L^*, respectively.
- $\lambda := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(L^*))$, a common eigenvalue of multiplicity 1.
- ϕ and ϕ^*: the eigenfunction of L and L^* associated with the eigenvalue λ.
- Normalize ϕ and ϕ^* by $\langle \phi, \phi \rangle_m = \langle \phi, \phi^* \rangle_m = 1$.

Assumptions

Assumption 2. (Critical and Intrinsic Ultracontractive)

- $\lambda = 0$.
- $\forall t > 0, \exists c_t > 0, \forall x, y \in E, \quad p_t^{\beta}(x, y) \leq c_t \phi(x)\phi^*(y)$.

Assumption 3. (Stable branching)

The branching mechanism ψ is of the form:

$$\psi(x, z) = -\beta(x)z + \kappa(x)z^{\gamma(x)},$$

where $\beta \in \mathscr{B}_b(E), \gamma \in \mathscr{B}_b^+(E)$, $\kappa \in \mathscr{B}_b^+(E)$ with $1 < \gamma(\cdot) < 2$, $\gamma_0 := \operatorname{ess\,inf}_{m(dx)} \gamma(x) > 1$ and $\kappa_0 := \operatorname{ess\,inf}_{m(dx)} \kappa(x) > 0$.

Results

Theorem 1. (Y.-X. Ren, R. Song and Z. Sun, 2018c+)

(1) $\mathbf{P}_{\delta_x}(\|X_t\| = 0) > 0$, for each $t > 0$ and $x \in E$.

(2) For each $\mu \in \mathcal{M}_E^1$, $\mathbf{P}_\mu(\|X_t\| \neq 0) = t^{-1/(\gamma_0-1)} L(t)$ where $L(t)$ is a slowly varing function at ∞.

- $C_X := \langle \mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa \cdot \phi^{\gamma_0}, \phi^* \rangle_m$
- $\eta_t := \big(C_X(\gamma_0 - 1)t\big)^{-1/(\gamma_0-1)}$

Theorem 2. (Y.-X. Ren, R. Song and Z. Sun, 2018c+)

Suppose that $m(x : \gamma(x) = \gamma_0) > 0$.

(3) $\lim_{t\to\infty} \eta_t^{-1}\mathbf{P}_\mu(\|X_t\| \neq 0) = \mu(\phi)$.

(4) For each $f \in \mathscr{B}^+(E)$ with $\langle f, \phi^*\rangle_m > 0$ and $\|\phi^{-1} f\|_\infty < \infty$, $\{\eta_t X_t(f); \mathbf{P}_\mu(\cdot | \|X_t\| \neq 0)\} \xrightarrow[t\to\infty]{\text{law}} \langle f, \phi^*\rangle_m \mathbf{z}^{(\gamma_0-1)}$.

Size-biased transform

- $(\Omega, \mathscr{F}, Q)$: a measure space.
- $G \in \mathscr{F}^+$: $Q(G) \in (0, \infty)$.

Definition

A probability measure Q^G is called the G-size-biased transform (or simply, G-transform) of Q if
$$dQ^G = \frac{G}{Q(G)} dQ.$$

- Let Q be a probability measure. $\{(X_t)_{t\in\Gamma}; Q\}$ be a stochastic process.
- We say a process $(Y_t)_{t\in\Gamma}$ is the G-transform of process (X_t) if

$$(Y_t)_{t\in\Gamma} \overset{d}{=} \{(X_t)_{t\in\Gamma}; Q^G\}.$$

Size-biased add-on

- X: an non-negative r.v. with finite mean.
- $\dot{X}$: an X-transform of X.
- We say $F(\theta) := E[e^{-\theta \dot{X}}]/E[e^{-\theta X}]$ is the size-biased add-on function.

Lemma (Size-biased add-on, easy to verify)

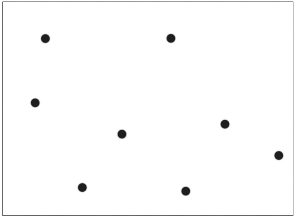
$$-\log \mathbf{P}[e^{-\theta X}] = \mathbf{P}[X] \int_0^{\theta} F(r)dr, \quad \theta \geq 0.$$

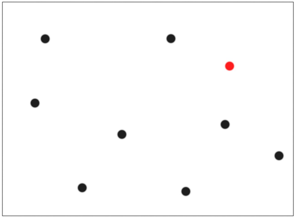
Poisson random measure

- $\mathcal{N}$: a Poisson random measure on a measurable space $(S, \mathscr{S})$ with intensity measure N.
- $F \in \mathscr{S}^+$: $0 < N(F) < \infty$.

Theorem (Ren, Song and Sun (2018b+))

$$\{\mathcal{N}; P^{\mathcal{N}(F)}\} \overset{d}{=} \{\mathcal{N}; P\} \otimes \{\delta_s; N^F(ds)\}.$$





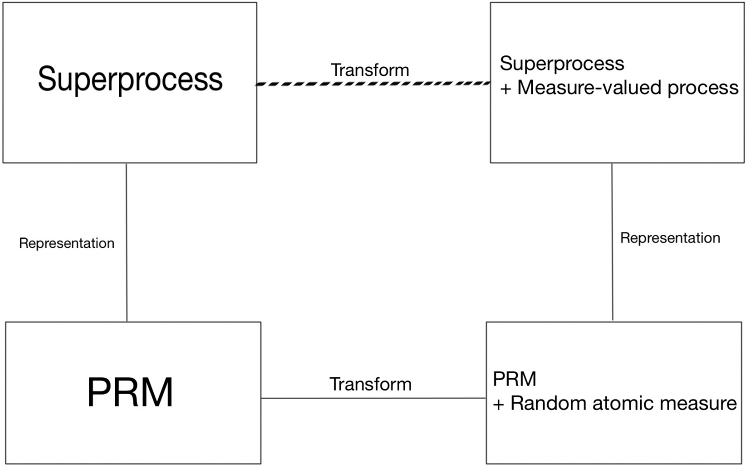
Superprocesses as PRMs

- $\mathcal{W}$: Skorokhod space of $\mathcal{M}_f$-valued càdlàg paths.
- $(\mathbb{N}_x)_{x\in E}$: Kuznetsov measure (N-measure, excursion measure) of superprocess (X_t).
- $\mu \in \mathcal{M}_f$.
- $\mathcal{N}_\mu$: a Poisson random measure on $\mathcal{W}$ with intensity measure $\int_E \mathbb{N}_x[\cdot]\mu(dx)$.

Theorem (Superprocesses as PRMs, see Li (2011) for example.)

$$\{(X_t)_{t>0}; \mathbf{P}_\mu\} \stackrel{d}{=} \left(\int_{\mathcal{W}} w_t(\cdot)\mathcal{N}_\mu(dw)\right)_{t>0}.$$

Idea


Superprocess
Transform
Superprocess
+ Measure-valued process
Representation
Representation
PRM
Transform
PRM
+ Random atomic measure

Size-biased transforms of Superprocesses

- F: a non-negative measurable function on $\mathcal{W}$ s.t. $\mathbb{N}_\mu[F] \in (0, \infty)$.

Theorem (Ren, Song and Sun (2018b+))

$$\{(X_t)_{t\geq 0}; \mathbf{P}_\mu^{\mathcal{N}(F)}\} \overset{d}{=} \{(X_t)_{t\geq 0}; \mathbf{P}_\mu\} \otimes \{(w_t)_{t\geq 0}; \mathbb{N}_\mu^F(dw)\}.$$

- While considering the transform of superprocesses, we only have to characterize the corresponding transform of the $\mathbb{N}$-measures.
- We can characterize $\{(w_t)_{t\geq 0}; \mathbb{N}_\mu^F(dw)\}$ while
  - $F(w) = w_t(\phi)$ using the Spine Decomposition Theorem.
  - $F(w) = w_t(f)$ using a generalized Spine Decomposition Theorem.
  - $F(w) = w_t(\phi)^2$ using a 2-Spine Decomposition Theorem.
  - ......

Spine decomposition: A classical example

Example

Suppose the branching mechanism $\psi(x,z) = z^2$ and the underlying process is conservative. Then $\phi(x) = 1$.

- **The spine** (ξ_t): a process with law Π_x;
- **The immigration** $\mathbf{n}_T^{\xi}(ds, dw)$: Conditioned on (ξ_t), a Poisson random measure on $(0, T] \times \mathcal{W}$ with intensity measure $2ds \times \mathbb{N}_{\xi_s}(dw)$;
- Then

$$\{(w_t)_{0<t\leq T}; \mathbb{N}_x^{w_t(1)}(dw)\} \overset{d}{=} \left(\int_{(0,t]\times\mathcal{W}} w_{t-s}\mathbf{n}_T^{\xi}(ds, dw) \right)_{0<t\leq T}.$$

- Spine Decomposition Theorem are developed by Engländer and Kyprianou (2004), Liu, Ren and Song (2009), and Eckhoff, Kyprianou and Winkel (2015).

Size-biased add-on of Superprocesses

- Note that $V_T(\theta f)(x) = -\log \mathbf{P}_{\delta_x}[e^{-\theta X_t(f)}]$.

Theorem (Y.-X. Ren, R. Song and Z. Sun (2018c+))

For any $f \in \mathscr{B}_b^+(E), \theta \geq 0, x \in E$ and $T > 0$, we have

$$V_T(\theta f)(x) = \phi(x) \int_0^\theta \Pi_x^{(\phi)} \left[\frac{f(\xi_T)}{\phi(\xi_T)} e^{-\int_0^T \left(\kappa\gamma V_{T-s}(rf)^{\gamma-1} \right)(\xi_s) ds} \right] dr.$$

- The red part is the size-biased add-on function of $X_t(f)$.

Conparison of equations

After rescaling we have

$$\frac{V_T(\theta\eta_T f)(x)}{\eta_T\phi(x)} = \int_0^\theta \Pi_x^{(\phi)}\left[\frac{f(\xi_T)}{\phi(\xi_T)} e^{-T\int_0^1 \left(\kappa\gamma V_{uT}(r\eta_T f)^{\gamma-1}\right)(\xi_{(1-u)T})du}\right] dr.$$

Without loss of generality we set $\langle f, \phi^* \rangle = 1$, then using the following Lemma:

Lemma (Target distribution, Ren, Song and Sun, 2018c+)

The non-linear delay equation

$$G(\theta) = \int_0^\theta e^{-\frac{\gamma_0}{\gamma_0-1}\int_0^1 G(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u}} dr, \quad \theta \geq 0,$$

has a unique solution: $G(\theta) = (1+\theta^{-(\gamma_0-1)})^{-1/(\gamma_0-1)}, \quad \theta \geq 0.$

we can show that

$$\frac{V_T(\theta\eta_T f)(x)}{\eta_T\phi(x)} \xrightarrow{T\to\infty} G(\theta).$$

Results

Theorem 1. (Y.-X. Ren, R. Song and Z. Sun, 2018c+)

(1) $\mathbf{P}_{\delta_x}(\|X_t\| = 0) > 0$, for each $t > 0$ and $x \in E$.

(2) For each $\mu \in \mathcal{M}_E^1$, $\mathbf{P}_\mu(\|X_t\| \neq 0) = t^{-1/(\gamma_0 - 1)} L(t)$ where $L(t)$ is a slowly varing function at ∞.

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Theorem 2. (Y.-X. Ren, R. Song and Z. Sun, 2018c+)

Suppose that $m(x : \gamma(x) = \gamma_0) > 0$.

(3) $\lim_{t \to \infty} \eta_t^{-1} \mathbf{P}_\mu(\|X_t\| \neq 0) = \mu(\phi)$.

(4) For each $f \in \mathscr{B}^+(E)$ with $\langle f, \phi^* \rangle_m > 0$ and $\|\phi^{-1} f\|_\infty < \infty$,
$\{\eta_t X_t(f); \mathbf{P}_\mu(\cdot | \|X_t\| \neq 0)\} \xrightarrow[t \to \infty]{\text{law}} \langle f, \phi^* \rangle_m \mathbf{z}^{(\gamma_0 - 1)}$.

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