Effect of Small Noise on the Speed of Reaction-Diffusion Equations

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The Voter Model (Single Site)

Fix $N \in \mathbb{N}$. Let $\boldsymbol{\eta}(t) = (\eta_1(t), \dots, \eta_N(t)) \in \{0,1\}^N$ be a continuous-time Markov chain. Each voter $i \in \{1, \dots, N\}$ has **two independent** Poisson clocks:

- **Neutral clock** (rate R): when it rings, voter i samples a uniformly random voter $j \neq i$ and sets $\eta_i := \eta_j$.
- **Biased clock** (rate $s > 0$): when it rings, voter i samples a uniformly random voter $j \neq i$. If $\eta_j = 1$, voter i adopts opinion 1 (sets $\eta_i := 1$); otherwise nothing changes.

The **empirical fraction of** 1**-voters** is

$$p_t^N := \frac{1}{N} \sum_{i=1}^{N} \eta_i(t).$$

Scaling Limit → Wright–Fisher Diffusion

Theorem (Scaling Limit)

Fix $s > 0$. Let $R = R_N$ satisfy $R_N/N \to \gamma/2 \in [0,\infty)$ as $N \to \infty$. Then $p^N_\cdot \implies p_\cdot$ in $D([0,\infty),[0,1])$ with the Skorokhod topology, where p_t solves

$$dp_t = s\,p_t(1-p_t)\,dt + \sqrt{\gamma\,p_t(1-p_t)}\;dB_t.$$

Remark

- $\gamma > 0$: ***Wright–Fisher diffusion with selection.***
- *The diffusion coefficient vanishes at $p = 0$ and $p = 1$; these are absorbing boundaries.*
- *Durrett–Fan (2015) extended this to a spatial setting, yielding a Wright–Fisher SPDE $\partial_t u = \Delta u + s\,u(1-u) + \sqrt{\gamma\,u(1-u)}\;\dot{W}$.*

The Fisher–KPP Equation

Setting $s = 1$ and $\gamma = 0$ (pure deterministic limit):

$$\partial_t u(t,x) = \Delta u(t,x) + u(t,x)\big(1 - u(t,x)\big), \qquad t \geq 0,\ x \in \mathbb{R}.$$

Three mechanisms:

- **Diffusion** (Δu): spatial spreading
- **Exponential growth** ($+u$): unchecked reproduction at low density
- **Saturation** ($-u^2$): logistic competition at high density

This is the **Fisher–KPP equation**

(Fisher, 1937; Kolmogorov–Petrovsky–Piscounov, 1937).

Traveling Waves and Minimal Speed Selection

Definition (Traveling Wave)

A function $u(t,x) = W_c(x - ct)$ is a traveling wave of speed c if $W_c \in C^2(\mathbb{R})$, $0 \le W_c \le 1$, and

$$W_c(-\infty) = 1, \qquad W_c(+\infty) = 0.$$

Theorem (Fisher 1937; KPP 1937)

1. *Traveling wave solutions to the FKPP equation exist for every speed $c \ge 2$.*
2. *For* ***Heaviside-type*** *initial data $u(0,\cdot) = \mathbf{1}_{(-\infty,0]}$, the solution converges to a traveling wave solution with minimal speed $c_{\min} = 2$.*

McKean's Duality (1975)

Let {BBM} be branching Brownian motion:

- Particles move independently as Brownian motions with generator Δ (variance rate 2).
- Each particle splits into two at rate 1.

Let $M_t := \max\{\text{position of all particles alive at time } t\}$, and write $\mathbb{P}_x$ for the law starting from a single particle at x.

Theorem (McKean, 1975)

The function

$$u(t,x) := \mathbb{P}_x(M_t \leq 0)$$

solves the KPP equation $\partial_t u = \Delta u + u(1-u)$ *with Heaviside-type initial condition* $u(0,\cdot) = \mathbf{1}_{(-\infty,0]}(\cdot)$.

Bramson's Asymptotics

Using McKean's duality and fine analysis of the BBM front:

Theorem (Bramson, 1978, 1983)

Let M_t be the position of the rightmost particle in BBM. Then ξ_t converges in distribution as $t \to \infty$ where

$$M_t = 2t - \frac{3}{2}\log t + \xi_t \quad \text{as } t \to \infty.$$

Remark

The $-\frac{3}{2}\log t$ correction is now known as the Bramson shift. The law of ξ depends on the initial condition.

The Question: Speed Correction at Finite Population

The KPP minimal speed is $c = 2$. What happens when the population is finite? Brunet and Derrida (1997) studied this through **three distinct models with population constraint**:

1. A **cutoff PDE** (Artificial analytical model)
2. An N**-particle selection system** (particle model)
3. An **SPDE** (More realistic PDE model with random fluctuations)

Each model produces a front that propagates at a speed $v_N < 2$.

Model 1: Cutoff FKPP Equation

Definition (Cutoff FKPP)

For $N \geq 2$, let $u_N(t,x)$ solve

$$\partial_t u_N = \partial_{xx} u_N + u_N(1-u_N) \cdot \mathbf{1}_{\{u_N \geq 1/N\}}.$$

- The reaction term is **disabled** when $u_N < 1/N$.
- Physically: a population of size N cannot sustain densities below $1/N$, where N is called the **carrying capacity** (effective population scale).

Remark

This is mathematically convenient but physically artificial. A more faithful discrete model is needed.

Model 2: N-BBM (Selection)

Definition (N-BBM)

*A system of **exactly** N **particles** on $\mathbb{R}$:*

- *Each particle moves as Brownian motion with generator Δ.*
- *Each particle splits into two at rate 1.*
- *Immediately after each branching, the <u>leftmost</u> particle is killed, restoring the total to N.*

Remark

- *N is a **hard constraint** – the actual number of particles is exactly N at all times.*
- *This models genetic drift: only the "fittest" (rightmost) individuals survive.*
- *The selection mechanism pushes the front <u>slower</u> than the unconstrained BBM.*

Model 3: Wright–Fisher SPDE

In the Durrett–Fan framework, Brunet–Derrida suggest taking $R = 1$ (fixed) and $N \to \infty$:

Definition (Stochastic FKPP)

For each $N \geq 1$, let $w_N(t, x)$ solve

$$\partial_t w_N = \partial_{xx} w_N + w_N(1 - w_N) + \sqrt{\frac{w_N(1 - w_N)}{N}}\, \dot{W},$$

where $\dot{W}$ is space–time white noise.

- Noise intensity $\sim 1/\sqrt{N}$: the finite-population fluctuation.
- $N \to \infty$ formally recovers the deterministic FKPP.
- **Well-posedness**: Shiga (1988) via moment duality with coalescing Brownian motion.

The Brunet–Derrida Conjecture

Conjecture (Brunet–Derrida, 1997; BDMM, 2006)

For any of the three models above, the asymptotic front velocity $v_N := \lim_{t\to\infty} \frac{1}{t} X_N(t)$ *(where* $X_N(t)$ *is the front position, or p-quantile position, whichever makes sense) admits the expansion:*

$$v_N = 2 - \frac{\pi^2}{(\log N)^2} + \frac{6\pi^2 \log\log N}{(\log N)^3} + o\left(\frac{\log\log N}{(\log N)^3}\right) \quad \text{as } N \to \infty.$$

The first correction term was proposed by Brunet–Derrida (1997).

The second-order term was proposed by Brunet, Derrida, Mueller & Munier (2006).

Status of the Conjecture (from what I know)

Term	**Status**
2 (FKPP minimal speed)	Trivial
$-\dfrac{\pi^2}{(\log N)^2}$	**Proved** Bérard–Gouéré 2010 ($\asymp$ order only) MMQ 2011 (precise coefficient)
$+\dfrac{6\pi^2 \log\log N}{(\log N)^3}$	**Open** Consistent with Maillard 2016 (fluctuations)
$o\left(\frac{\log\log N}{(\log N)^3}\right)$	**Open**

2010: Bérard–Gouéré – First Rigorous Result

Theorem (Bérard–Gouéré, 2010)

For N-BRW (discrete-time branching random walk on $\mathbb{R}$ with selection):

$$v_N \to 2 \quad \textit{and} \quad 2 - v_N \asymp \frac{1}{(\log N)^2}.$$

- First proof of the $(\log N)^{-2}$ **scaling**.
- Does **not** establish the precise coefficient π^2.
- Technique: coupling with a branching random walk killed upon exiting a moving interval.

2011: Mueller–Mytnik–Quastel – Precise Coefficient

Theorem (MMQ, Inventiones Math. 2011)

For the Wright–Fisher SPDE model:

$$v_N = 2 - \frac{\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{\log\log N}{(\log N)^3}\right).$$

- First rigorous proof of the **precise coefficient** π^2.
- Key insight: the front is shaped by the behavior near "the tip" at distance $O(\log N)$ from the leading edge.
- The error term $\mathcal{O}(\frac{\log\log N}{(\log N)^3})$ is of the *same order* as the conjectured second correction. Resolving whether the $\mathcal{O}$ hides a $6\pi^2 \log\log N$ term remains open.

2012–2016: Maillard – Matching the second term

Theorem (Maillard, PTRF 2016)

For N-BBM, let ν_t^N be the counting measure at time t:

$$\nu_t^N = \sum_{i=1}^{N} \delta_{X_i(t)}.$$

Define the α-quantile: $\mathrm{med}_\alpha(\nu_t^N) := \inf\{x : \nu_t^N([x,\infty)) < \alpha N\}$. *Then*

$$\frac{\mathrm{med}_\alpha(\nu^N_{\log^3 N})}{\log^3 N} = 2 - \frac{\pi^2}{(\log N)^2} + \frac{6\pi^2 \log\log N + \zeta_N}{(\log N)^3},$$

where $\zeta_N \Rightarrow \zeta$ converges weakly as $N \to \infty$.

What Maillard's Result Does NOT Prove

Important Distinction

Maillard's result is about the front position at a fixed (albeit large) time $T_N = \log^3 N$. It does **not** establish the existence of an asymptotic **velocity**:

$$v_N := \lim_{t \to \infty} \frac{X_N(t)}{t}.$$

- The conjectured expansion is for v_N, the long-time speed.
- Maillard's expansion is for $X_N(T_N)/T_N$ at a specific time scale.
- These are **related but not equivalent**: the random fluctuation ζ_N could vanish or shift in the $t \to \infty$ limit.
- Hence: the second-order term for the **velocity** remains open.

Motivation: Non-Lipschitz Drift

Consider the single-site voter model with **state-dependent** biased voting rate:

- **Neutral**: voter i samples a uniformly random $j \neq i$ and sets $\eta_i := \eta_j$ at rate R.
- **Biased (state-dependent)**: voter i samples a uniformly random $j \neq i$, and if $\eta_j = 1$ adopts opinion 1 (sets $\eta_i := 1$). The attempt rate is $s(p_t^N) = (p_t^N)^{-1/2}$.

If $s(p) = 1/\sqrt{p}$, then under the same scaling $R/N \to \gamma/2$, p_t^N converges in distribution to p_t solving

$$dp_t = \sqrt{p_t}(1 - p_t)\,dt + \sqrt{\gamma\, p_t(1 - p_t)}\,dB_t.$$

The drift $b(p) = \sqrt{p}(1 - p)$ is **non-Lipschitz** at $p = 0$: $b'(0+) = \infty$. This stronger drift near $p = 0$ reflects *intensified selection when the favored allele is rare*.

Wright–Fisher SPDE with Non-Lipschitz Drift

Generalizing the voter model to a spatial setting yields:

Definition

For $\alpha \in (0,1)$ *and* $N \geq 1$, *consider*

$$\partial_t u = \partial_{xx} u + b_\alpha(u) + \sqrt{\frac{u(1-u)}{N}}\,\dot{W},$$

where $b_\alpha(u) = u^\alpha(1-u)$.

- b_α is **non-Lipschitz** at $u = 0$ when $\alpha < 1$: $b'_\alpha(0+) = \infty$.
- This raises fundamental **well-posedness questions**.
- Mueller–Mytnik–Ryzhik (2021): weak uniqueness for $\alpha \geq 1/2$.

Well-Posedness and Wave Speed

Theorem (Barnes–Mytnik–Sun, 2024 – Wave Speed)

For $\alpha \in [1/2, 1)$, the asymptotic speed satisfies:

$$V_{\alpha,N} \asymp N^{\frac{1-\alpha}{1+\alpha}} \quad \text{as } N \to \infty.$$

Theorem (Barnes–Mytnik–Sun, 2026)

For every $\alpha \in [0, 1)$, the SPDE

$$\partial_t u = \partial_{xx} u + b_\alpha(u) + \sqrt{\frac{u(1-u)}{N}}\, \dot{W}$$

*with $b_\alpha(u) = u^\alpha(1-u)$ for $\alpha \in (0,1)$ and $b_0(u) = \mathbf{1}_{(0,1]}(u) - u$ admits a **unique-in-law** martingale solution.*

Conjecture: Power-Law Speed for All $\alpha \in [0, 1)$

Conjecture (Barnes–Mytnik–Sun, 2024)

The scaling $V_{\alpha,N} \asymp N^{(1-\alpha)/(1+\alpha)}$ persists for all $\alpha \in [0, 1)$.

Current status:

- $\alpha \in [1/2, 1)$: proved (BMS 2024).
- $\alpha \in [0, 1/2)$: open – non-Lipschitz singularity makes the analysis more delicate.

Open Problem: Heavy-Tailed Offspring N-BBM

Consider N-BBM with heavy-tailed offspring:

$$\mathbb{P}(\text{offspring} = k) \sim C_\alpha \, k^{-1-\alpha}, \quad k \to \infty, \quad \alpha \in (0,1).$$

What is the scaling of the front velocity $V_{\alpha,N}^{\text{BBM}}$?

Private Communication (Berestycki-Maillard-Pain-Sun, 2024), Simulation Evidence (Sun Jian, Master's Thesis 2025)

$$V_{\alpha,N}^{\text{BBM}} \asymp N^{\frac{1-\alpha}{2}} \ll N^{\frac{1-\alpha}{1+\alpha}} \asymp V_{\alpha,N}^{\text{SPDE}}.$$

Why the Particle System Must Be Slower

Consider one particle at random produces K_α offspring with

$$\mathbb{P}(K_\alpha = k) \sim C_\alpha k^{-1-\alpha}, \quad \alpha \in (0,1).$$

Immediately after branching, we have $N + K_\alpha - 1$ particles. However, the selection kills the **leftmost** particles to maitain the total population to be N, so the effective number of offspring is no more than

$$\tilde{K}_\alpha = \min(K_\alpha, N).$$

Therefore, the model can be dominated by a BBM with offspring distribution $\tilde{K}_\alpha$ whose velocity is

$$2\sqrt{\mathbb{E}[\tilde{K}_\alpha]} \asymp \sqrt{N^{1-\alpha}} = N^{\frac{1-\alpha}{2}}.$$

Thank You!