On the coming down from infinity of coalescing Brownian Motions

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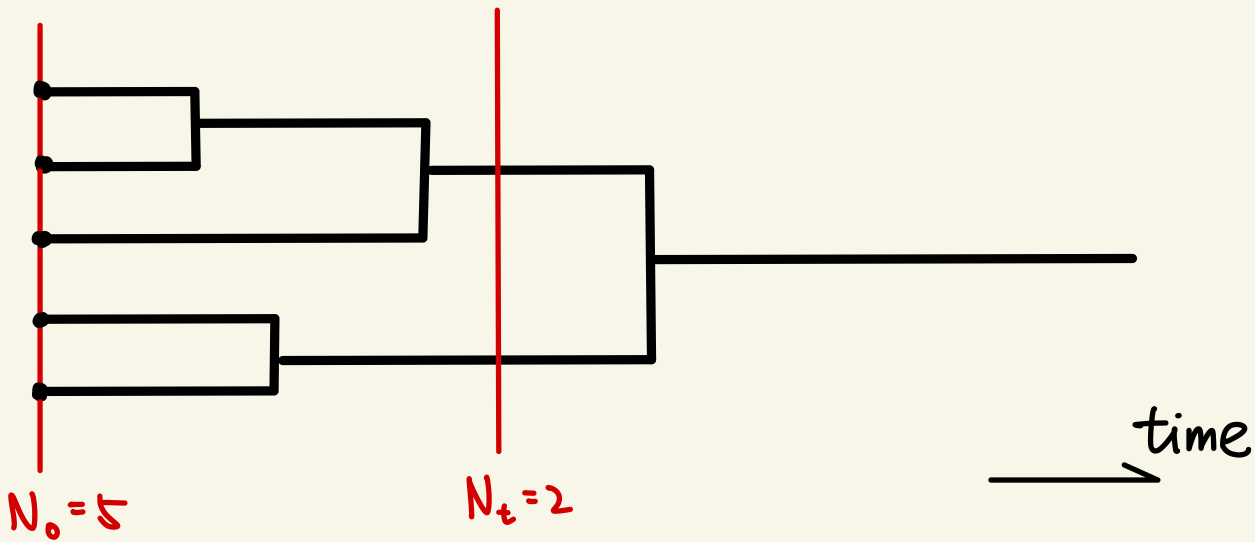
Kingman (1982) → **Kingman's coalescent**

Consider a system of finitely many particles where each pair of particles coalesces into one particle with rate 1, independent of other pairs. Then the total number of particles $(N_t)_{t\geq 0}$ is a continuous time Markov chain on $\mathbb{N}$ which jumps from n to $n-1$ with rate $n(n-1)/2$.



a graphical construction.

Aldous (1999)

Kingman's coalescent can come down from $+\infty$, i.e. $\exists$ a $\mathbb{N}$-valued Markov chain $(N_t^\infty)_{t>0}$ which evolves as Kingman's coalescent and satisfies $t N_t^\infty \to 2$ when $t \downarrow 0$ in probability.

equal in law.

Berestycki, Berestycki & Limic (2010)

There exists a process $(N_t^{(n)})_{t\geq 0, n\in\mathbb{N}}$ s.t. $\forall n \in \mathbb{N}$, $(N_t^{(n)})_{t\geq 0}$ is a Kingman's coalecent with $N_0^{(n)} = n$; and $\forall t \geq 0$, $N_t^{(n)} \leq N_t^{(n+1)}$. The increasing limit $N_t^\infty := \lim_{n\uparrow+\infty} N_t^{(n)}$ satisfies $t N_t^\infty \to 2$ when $t \downarrow 0$ a.s. and in L^p for $p \geq 1$.

Constructed using partitions of $\mathbb{N}$.

Spatial kingman's coalescent is a system of particles living on a Graph $\mathcal{G}$ s.t.

- At each site, particles coalesce according to kingman's coalescent.
- Each particle moves as an independent random walk on $\mathcal{G}$.

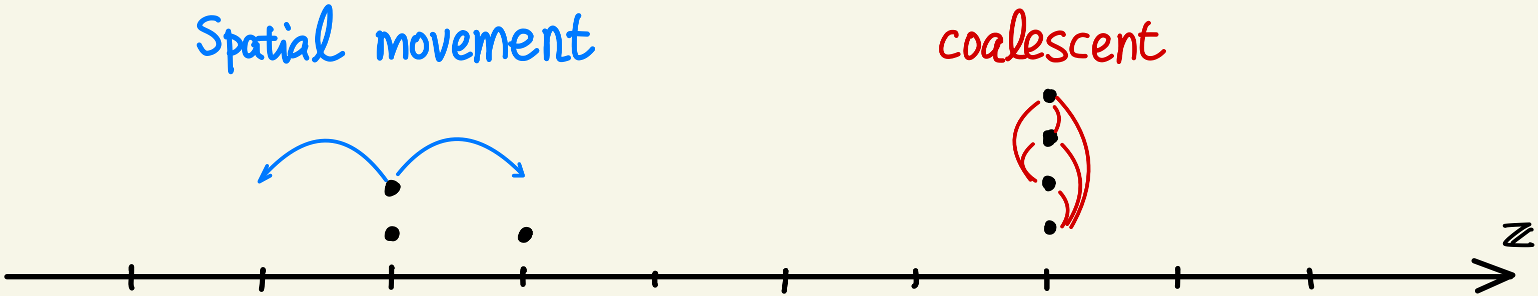
$N_t(v)$:= number of particles at site v and time t.

Then $(N_t)_{t\geq 0}$ is a Markov Process.

One can construct a process $(N_t^{(n)})_{t\geq 0, n\in\mathbb{N}}$ s.t.

- $\forall n\in\mathbb{N}$, $(N_t^{(n)})_{t\geq 0}$ is a spatial kingman's coalescent with n initial particles located at a distinguished root site;
- $\forall t\geq 0$, $N_t^{(n)}(v) \leq N_t^{(n+1)}(v)$, $v\in\mathcal{G}$, $n\in\mathbb{N}$.

Define $N_t^{(\infty)}(v) := \lim\limits_{n\uparrow +\infty} N_t^{(n)}(v)$ and the total population $N_t^{(\infty)}(\mathcal{G}) := \sum\limits_{v\in\mathcal{G}} N_t^{(\infty)}(v)$.



Limic & Sturm (2006)

If $\mathcal{G}$ is a finite graph, then $\mathbb{P}(N_t^{(\infty)}(\mathcal{G}) < +\infty) = 1$, $\forall t > 0$.

Angel, Berestycki & Limic (2012)

If $\mathcal{G}$ is an infinite connected graph with bounded degree,

then $\mathbb{P}(N_t^{(\infty)}(\mathcal{G}) = +\infty) = 1$, $\forall t \geq 0$.

Is there an analogy of Kingman's coalescent in the continumm spatial setting?

The Wright-Fisher diffusion

Well-known duality

Let $(N_t)_{t\geq 0}$ be Kingman's coalescent with initial value $N_0 = n$.

Let $(X_t)_{t\geq 0}$ be the solution of the stochastic differential equation

$$dX_t = \frac{1}{2}\sqrt{X_t(1-X_t)}\, dB_t\,; \quad X_0 = p \in [0,1].$$

Then $\mathbb{E}\left[(1-X_0)^{N_t}\right] = \mathbb{E}\left[(1-X_t)^{N_0}\right]$.

Brownian motion.

Is there an analogy of Kingman's coalescent in the continuum spatial setting? Yes!

Shiga's coalescing Brownian motions.

Shiga (1988)

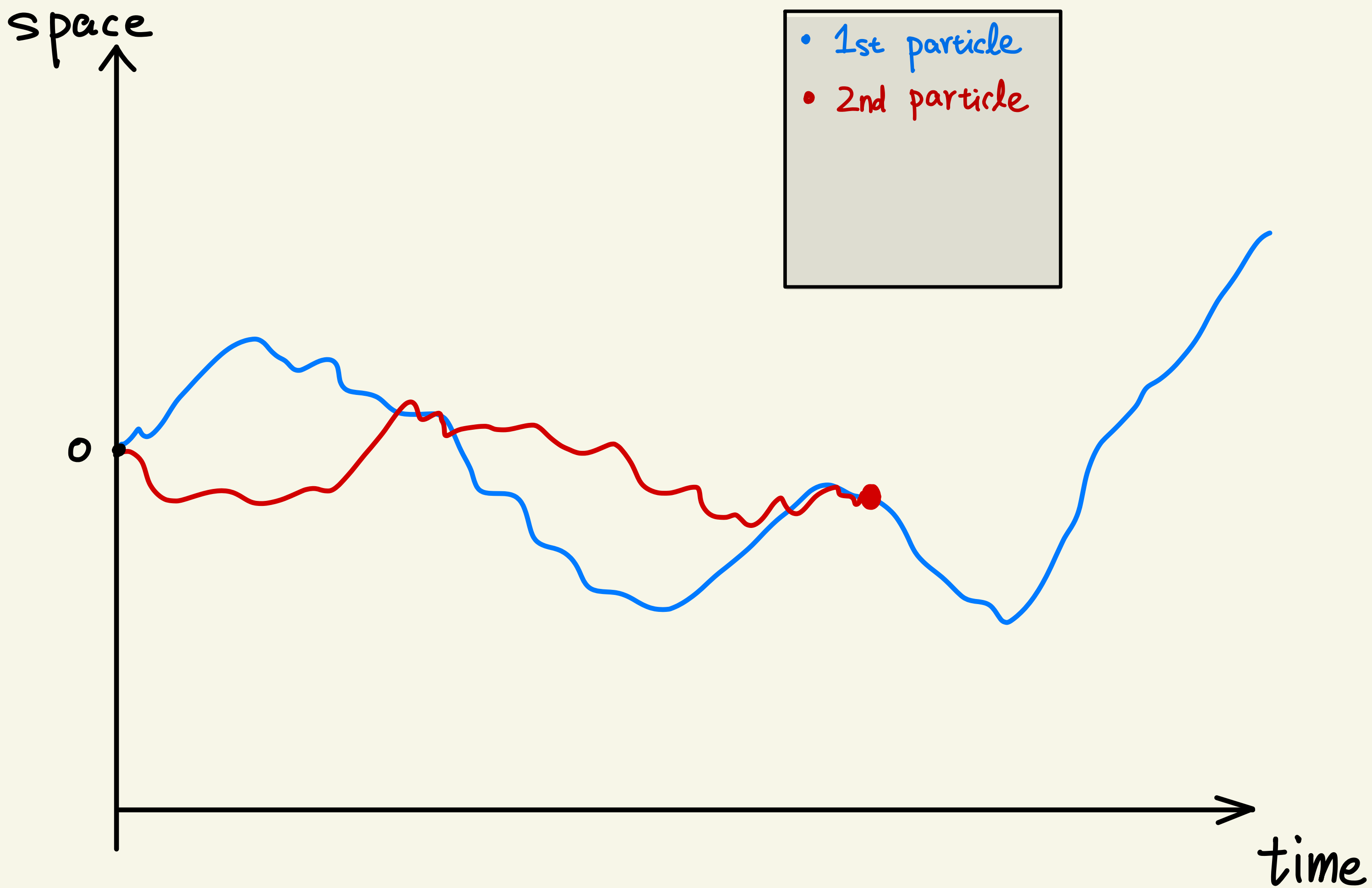
Consider a system of finitely many Brownian particles on $\mathbb{R}$ where each pair of particles coalesces into one particle with rate 1/2 according to their intersection local time.

Denote by $(B_t^{(i)})_{i \in I_t}$ the positions of the particles at time t.

Let $(u_t(x))_{t \geq 0, x \in \mathbb{R}}$ be the solution to the stochastic partial differential equation

$$\partial_t u_t(x) = \frac{1}{2} \partial_x^2 u_t(x) + \sqrt{u_t(x)(1-u_t(x))}\, \underbrace{\dot{W}_{t,x}}_{\text{Space-time white noise}}, \quad u_0 = f \in C(\mathbb{R},[0,1]).$$

Then $\mathbb{E}\Big[\prod_{i \in I_t} \big(1 - u_0(B_t^{(i)})\big)\Big] = \mathbb{E}\Big[\prod_{i \in I_0} \big(1 - u_t(B_0^{(i)})\big)\Big].$


space
0
time
1st particle
2nd particle

Can we initiate shiga's coalescing Brownian motions with infinetly many initial particles? Yes!

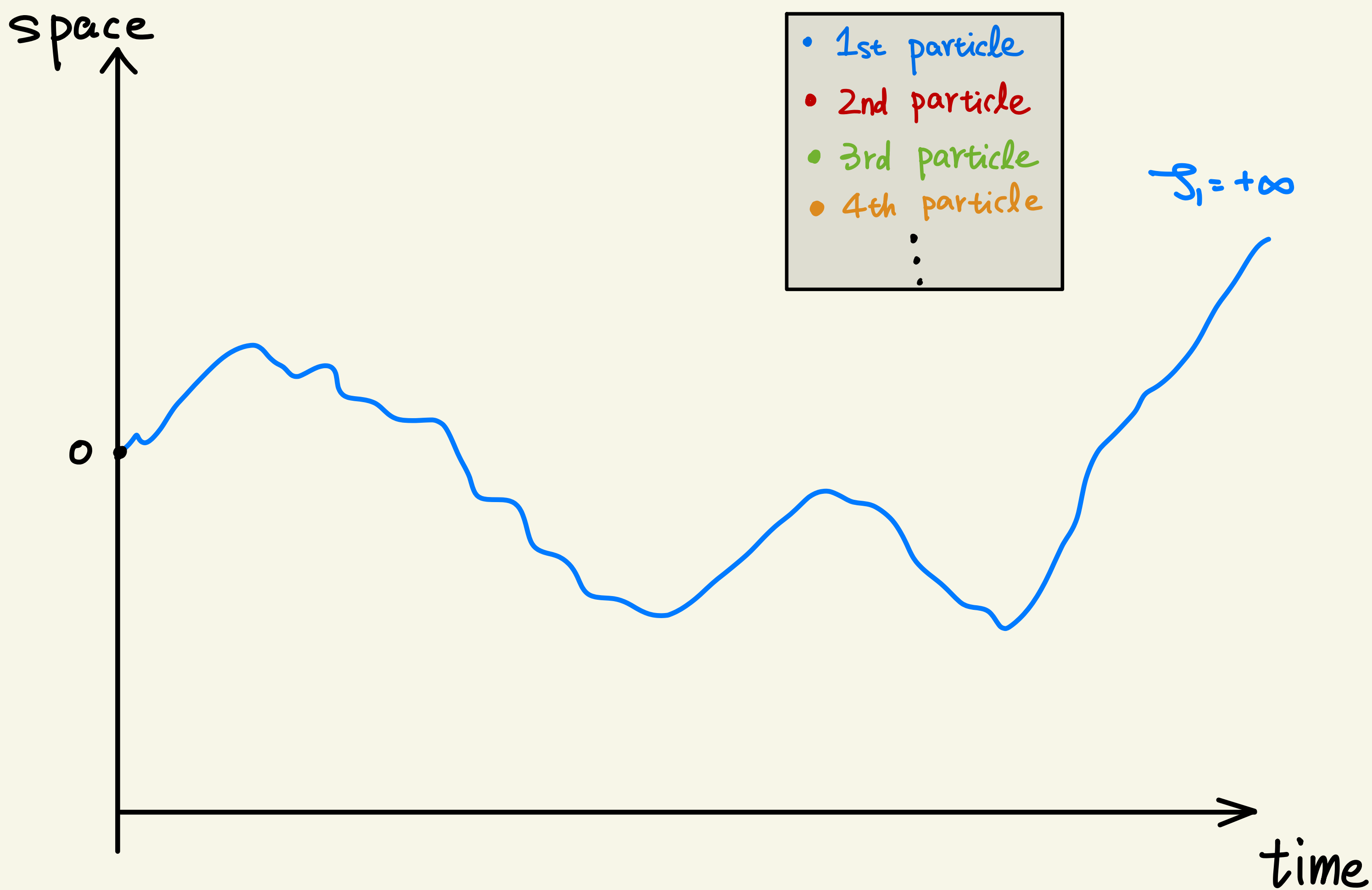
Tribe's construction (1995)

of shiga's coalescing Brownian motions $(Z_t)_{t\geq 0}$ with initial configuration $(x_i)_{i\in\mathbb{N}} \subset \mathbb{R}$:

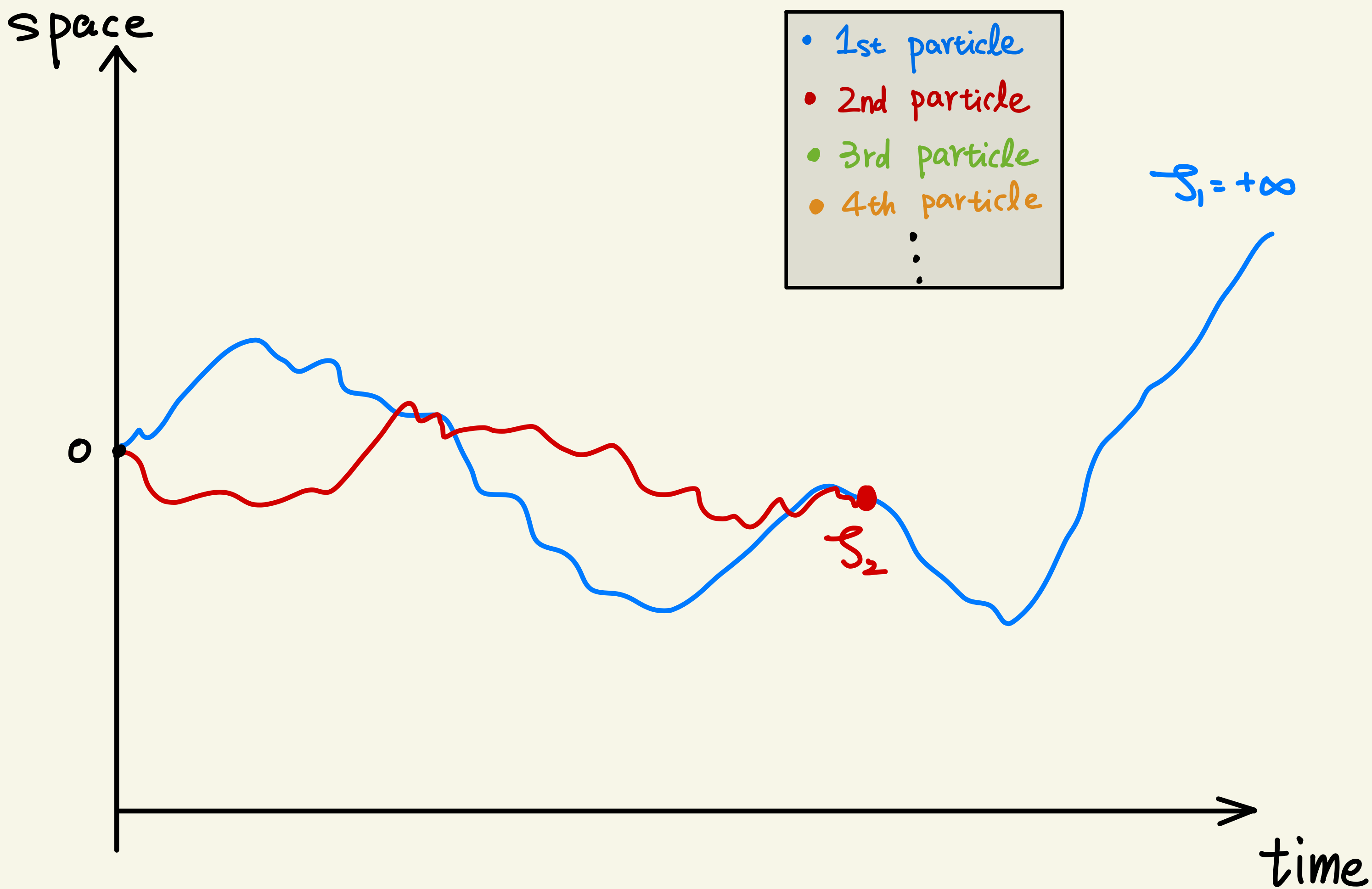
- Let $(B_t^{(i)})_{i\in\mathbb{N}}$ be a sequence of independent Brownian motions s.t. $\forall i\in\mathbb{N}$, $B_0^{(i)} = x_i$.
- Let $(e^{(i)})_{i\in\mathbb{N}}$ be a family of independent exponential random variable with mean 2.
- Let $(L_t^{(i,j)})_{t\geq 0}$ be the local time of $(B_t^{(i)} - B_t^{(j)})_{t\geq 0}$ at position 0 for $i,j\in\mathbb{N}$.
- Define $\zeta_1 := +\infty$, and inductively $\forall i>1$,

$$\zeta_i := \inf\Big\{t\geq 0 : \sum_{j<i} L^{(i,j)}_{t\wedge\zeta_j} \geq e^{(i)}\Big\}.$$ (Life time of ith particle)

- Define $Z_t^{(n)} := \sum_{i=1}^{n} \mathbb{1}_{t\in[0,\zeta_i)}\,\delta_{B_t^{(i)}}$, $t\geq 0$; $\quad Z_t := \sum_{i=1}^{+\infty} \mathbb{1}_{t\in[0,\zeta_i)}\,\delta_{B_t^{(i)}}$, $t\geq 0$.



space
0
time
1st particle
2nd particle
3rd particle
4th particle
⋮
$\zeta_1 = +\infty$



space
time
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1st particle
2nd particle
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4th particle
$\zeta_1 = +\infty$
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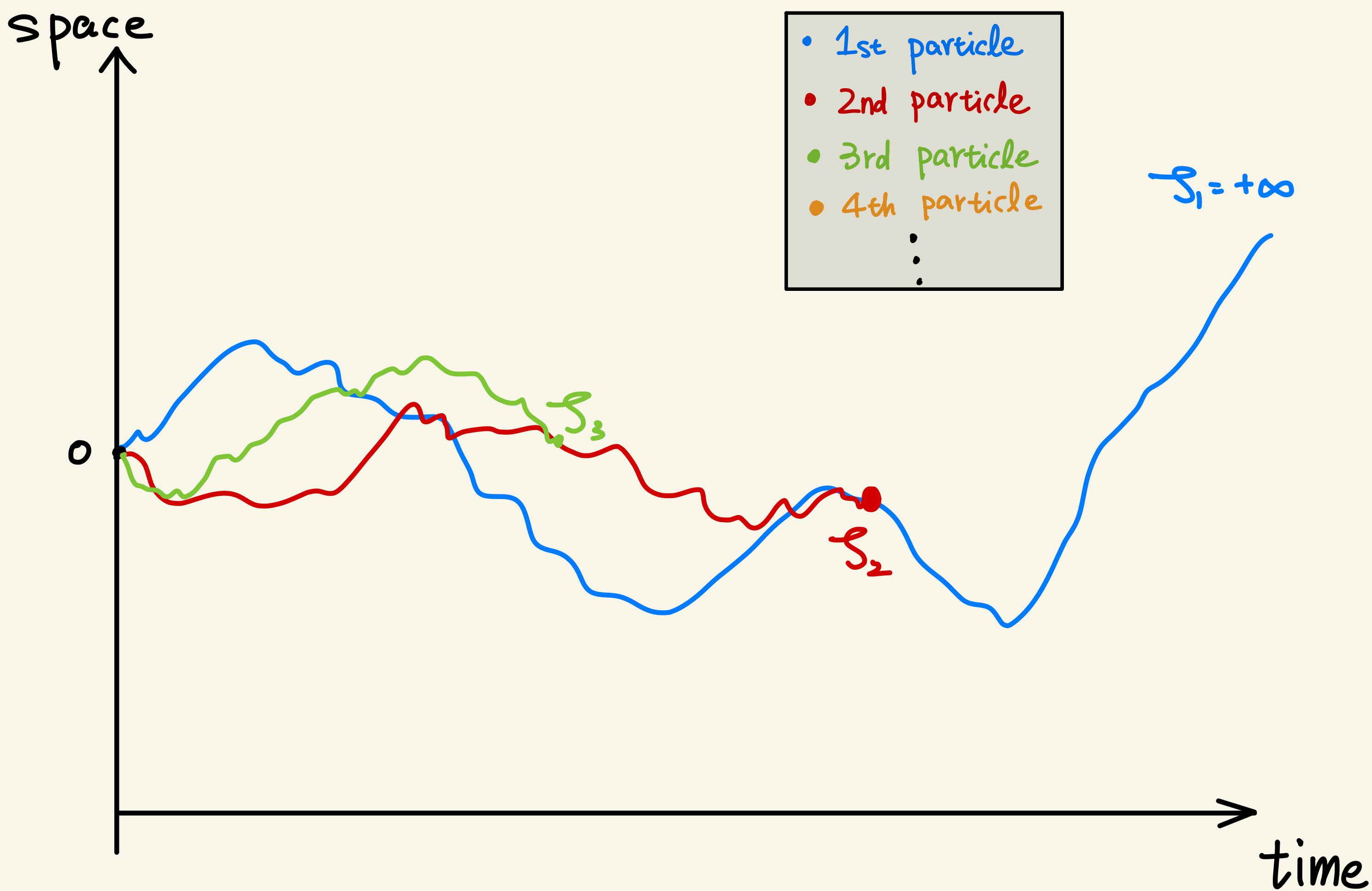

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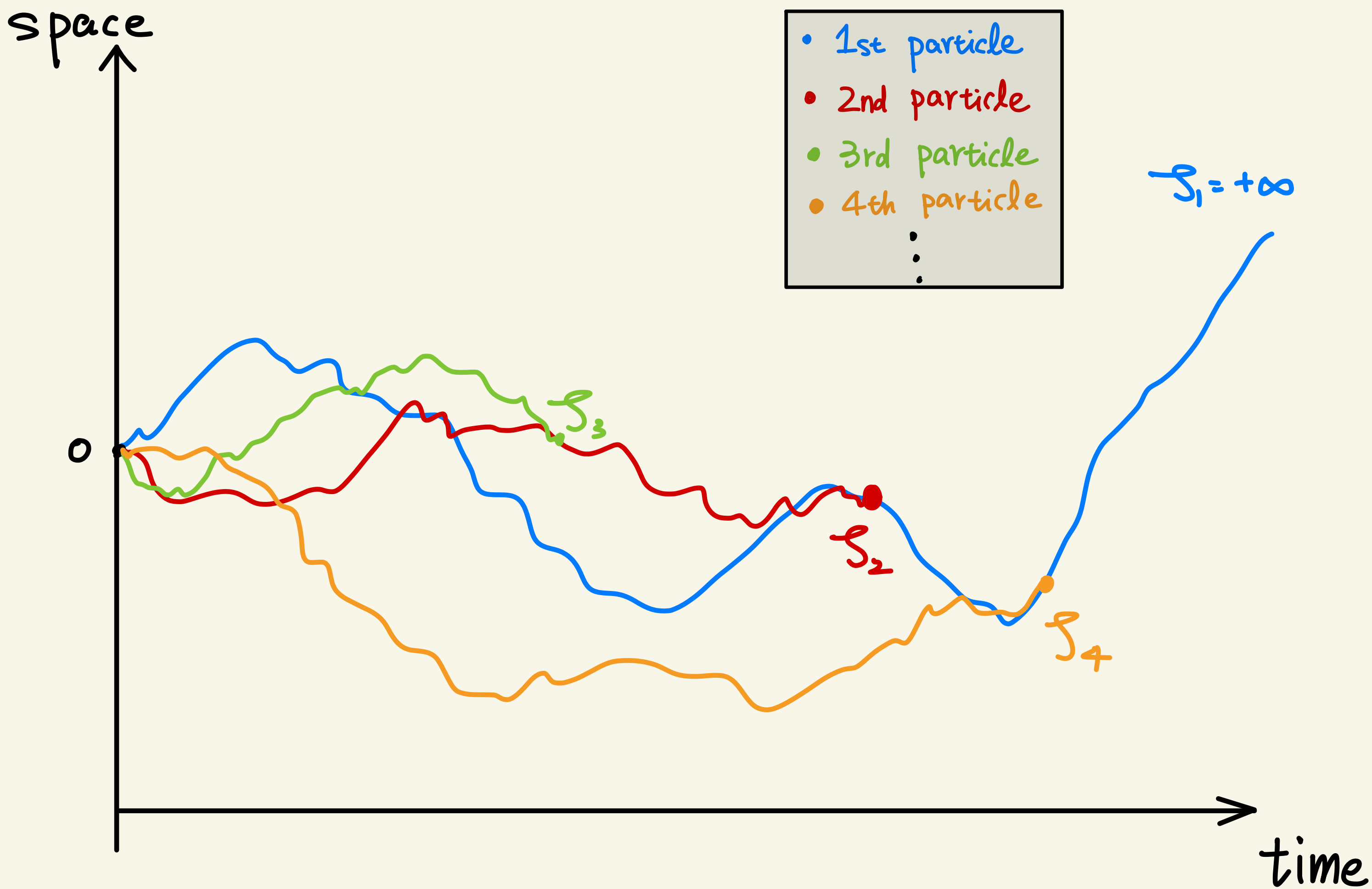

space
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1st particle
2nd particle
3rd particle
4th particle
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$\xi_1 = +\infty$
ξ_2
ξ_3
ξ_4

Does Shiga's coalescing Brownian motions have the property of coming down from infinity?

Hobson & Tribe (2005)

Consider Shiga's coalescing Brownian motions on the unit length circle $\mathbb{S}$ with initial configuration $(x_i)_{i\in\mathbb{N}}$ sampled as i.i.d uniform r.v.

Then the total number of particles $Z_t(\mathbb{S})$ is finite for $\forall\, t>0$ a.s.

Moreover, $t Z_t(\mathbb{S}) \longrightarrow 2$ in probability when $t \downarrow 0$.

Does Shiga's coalescing Brownian motions on $\mathbb{R}$ have the property of coming down from infinity?

Barnes, Mytnik & S. (2023)

Consider Shiga's coalescing Brownian motions on $\mathbb{R}$ with arbitrary (deterministic) initial configuration $(x_i)_{i\in\mathbb{N}} \subset \mathbb{R}$.

Let $U \subset \mathbb{R}$ be an arbitrary open interval.

(1) If $\{x_i : i\in\mathbb{N}\} \cap U$ is compactly supported, then $\mathbb{P}(Z_t(U) < +\infty, \forall t>0) = 1$.

(2) If $\{x_i : i\in\mathbb{N}\} \cap U$ is not compactly supported, then $\mathbb{P}(Z_t(U) = +\infty, \forall t\geq 0) = 1$.

What are the rate of coming down from infinity for shiga's coalescing Brownian motions?

The initial trace of the solution v.

Le Gall (1996)

$\forall$ closed subset $\Lambda \subset \mathbb{R}$, and $\forall$ non-negative Radon measure μ on Λ^c, there exists a unique non-negative $v = v^{(\Lambda,\mu)} \in C^{1,2}((0,+\infty)\times\mathbb{R})$ s.t.

$$\begin{cases} \partial_t v = \frac{1}{2}\partial_x^2 v - \frac{1}{2}v^2, & \forall t>0,\ x\in\mathbb{R}, \\ \Lambda = \left\{ y\in\mathbb{R} : \forall r>0,\ \lim\limits_{t\downarrow 0} \int_{y-r}^{y+r} v_{t,x}\,dx = +\infty \right\}, \\ \int_{\Lambda^c} \phi(x)\mu(dx) = \lim\limits_{t\downarrow 0} \int_{\Lambda^c} \phi(x)\, v_{t,x}\,dx, & \forall \phi \in C_c(\Lambda^c). \end{cases}$$

Represented using the Brownian snake.

Let $(Z_t)_{t\geq 0}$ be Shiga's coalescing Brownian motions with initial configuration $(x_i)_{i\in\mathbb{N}}$.

Define $\Lambda := \{y\in\mathbb{R} : \forall r>0,\ Z_0((y-r,y+r)) = +\infty\}$ and $\mu := Z_0|_{\Lambda^c}$.

Barnes, Mytnik & S. (2023)

Let U be an arbtrary open interval.

If $\{x_i : i\in\mathbb{N}\}\cap U$ is compactly supported,

then $\mathbb{E}[Z_t(U)] < +\infty$, $\forall t>0$; Furthermore,

- if $\Lambda\cap\bar{U} = \phi$, then $\limsup_{t\downarrow 0} \mathbb{E}[Z_t(U)] < +\infty$;
- if $\Lambda\cap\bar{U} \neq \phi$, then $\lim_{t\downarrow 0} \mathbb{E}[Z_t(U)] = +\infty$, &

$$\frac{Z_t(U)}{\int_U v_{t,x}^{(\Lambda,\mu)}\,dx} \to 1 \quad \text{in } L^1 \text{ when } t\downarrow 0.$$

Corollary

If $\{x_i : i\in\mathbb{N}\}$ is unbounded, then $\mathbb{P}(Z_t(\mathbb{R})=+\infty, \forall t\geq 0)=1$.

If $\{x_i : i\in\mathbb{N}\}$ is bounded, then $\mathbb{E}[Z_t(\mathbb{R})]<+\infty, \forall t>0$; $\lim_{t\downarrow 0}\mathbb{E}[Z_t(\mathbb{R})]=+\infty$; &

$$\frac{Z_t(\mathbb{R})}{\int v_{t,x}^{(\Lambda,\mu)}\,dx} \to 1 \quad \text{in } L^1 \text{ when } t\downarrow 0.$$

What more exactly is the rate of coming down from infinity?

(What is the behavior of $\int_U V_{t,x}^{(\Lambda,\mu)}\,dx$ when $t \downarrow 0$?)

Example 1

If $x_i = 0$ for every $i \in \mathbb{N}$, then

$$\sqrt{t}\, Z_t(\mathbb{R}) \to C := \int V_{1,x}^{(\{0\},\,\mathrm{null})}\,dx \text{ in } L^1 \text{ when } t \downarrow 0.$$

Example 2

If $\{x_i : i \in \mathbb{N}\}$ is a dense subset of $[0,1]$, then

$$t\, Z_t(\mathbb{R}) \to 2 \text{ in } L^1 \text{ when } t \downarrow 0.$$

parallel result of Hobson & Tribe (2005)

? For $\frac{1}{2} < \alpha < 1$, does there exist initial configuration $(x_i)_{i \in \mathbb{N}}$ so that the total population $Z_t(\mathbb{R})$ behaves like $t^{-\alpha}$ as $t \downarrow 0$?

$\forall\ A \subset \mathbb{R}$,

- define A's **r-neighborhood**

$$A_r := \{ y \in \mathbb{R} : \exists x \in A, |y - x| < r \} \text{ for every } r > 0;$$

- we say A has **Minkowski dimension** $\delta \in [0,1]$, if

$$\frac{\log \mathrm{Leb}(A_r)}{\log r} \to 1 - \delta \quad \text{as} \quad r \downarrow 0;$$

- when A has Minkowski dimension $\delta \in [0,1]$, we say it is **Minkowski measurable** with **Minkowski content** $k \in (0, +\infty)$, if

$$\frac{\mathrm{Leb}(A_r)}{r^{1-\delta}} \to k \quad \text{as} \quad r \downarrow 0.$$

Barnes, Mytnik & S. (2023)

Suppose that $(x_i)_{i\in\mathbb{N}}$ is bounded, without isolated points $(\mu=0)$.

Suppose that Λ has Minkowski dimension $\delta\in[0,1]$. Then

$$\frac{\log Z_t(\mathbb{R})}{\log t} \to -\frac{1+\delta}{2} \quad \text{in probability as } t\downarrow 0.$$

Conjecture

Further suppose that Λ is Minkowski measurable

with Minkowski content $K\in(0,+\infty)$, then

$\exists\, C(\delta)>0$, depending only on δ, s.t.

$$t^{\frac{1+\delta}{2}} Z_t(\mathbb{R}) \to C(\delta)K \quad \text{in } L^1 \text{ as } t\downarrow 0.$$

Shiga's coalescing Brownian motions.

Shiga's duality

Shiga (1988)

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \sqrt{u(1-u)}\,\dot{W}$$

"Similarity"

$$\partial_t \tilde{u} = \frac{1}{2}\partial_x^2 \tilde{u} + \sqrt{\tilde{u}}\,\dot{W}$$

Rate of

Coming down from infinity

Konno & Shiga (1988)

Reimers (1989)

density

super-Brownian motion

Le Gall (1994)

genealogical structure

$$\begin{cases} \partial_t V = \frac{1}{2}\partial_x^2 V - \frac{1}{2}V^2 \\ \text{Singular points of } V_0 = \Lambda \\ V_0|_{\Lambda^c} = \mu \end{cases}$$

Probabilistic representation

Le Gall (1996)

Brownian snake

? Similarity between u & $\tilde{u}$?

Consider the case when $x_i = 0$ for every $i \in \mathbb{N}$.

Let u & $\tilde{u}$ be the weak solution to SPDEs $(0 < \varepsilon < 1)$

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \sqrt{u(1-u)}\,\dot{W}, \qquad u_0 \equiv \varepsilon,$$

$$\partial_t \tilde{u} = \frac{1}{2}\partial_x^2 \tilde{u} + \sqrt{\tilde{u}}\,\dot{W}, \qquad \tilde{u}_0 \equiv \varepsilon.$$

Shiga's duality $\Rightarrow$ $\mathbb{E}\left[(1-\varepsilon)^{Z_t(\mathbb{R})}\right] = \mathbb{P}(u_{t,0} = 0)$.

Standard result for Super-Brownian motion

$$\Rightarrow \exp\left(-\varepsilon \int V_{t,x}^{(\{0\},0)}\,dx\right) = \mathbb{P}(\tilde{u}_{t,0} = 0).$$

We can argue using SPDE tools that

$$\left|\mathbb{P}(u_{t,0} = 0) - \mathbb{P}(\tilde{u}_{t,0} = 0)\right| \lesssim \varepsilon$$

for small ε and t.

This is based on my joint work with

&

Clayton Barnes

Leonid Mytnik

Thanks !!