SOLUTIONS TO THE SELECTED EXERCISES IN R. DURRETT'S PROBABILITY: THEORY AND EXAMPLES, II

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Exercise (6.5.1). To show that the convergence in (a) of Theorem 6.4.1. may occur arbitrarily slowly, let $X_{m,m+k} = f(k) \geq 0$, where $f(k)/k$ is decreasing, and check that $X_{m,m+k}$ is subaddditive.

Proof. Verify (i):

$$X_{0,m} + X_{m,n} = f(m) + f(n-m) = m\frac{f(m)}{m} + (n-m)\frac{f(n-m)}{n-m}$$
$$\geq m\frac{f(n)}{n} + (n-m)\frac{f(n)}{n} = f(n) = X_{0,n}.$$

Verify (ii): For each k, $(X_{nk,(n+1)k})_{n\geq 1} = (f(k))_{n\geq 1}$ is obviously a stationary sequence.

Verify (iii): The distribution of $(X_{m,m+k})_{k\geq 1} = (f(k))_{k\geq 1}$ obviously does not depend on m.

Verify (iv): Obviously $EX_{0,1}^+ = f(1) < \infty$. Denote by $\gamma_0 := \lim_{k\to\infty} f(k)/k \geq 0$, then we do have $EX_{0,n} = f(n) \geq \gamma_0 n$ and $\gamma_0 > -\infty$. □

Exercise (6.5.2.). Consider the longest common subsequence problem, Example 6.4.4. when $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are i.i.d. and take the values 0 and 1 with probability 1/2 each. (a) Compute EL_1 and $EL_2/2$ to get lower bounds on γ. (b) Show $\gamma < 1$ by computing the expected number of i and j sequence of length $K = an$ with the desired property.

Proof. (a) Since $L_{0,1} \in [0,1] \cap \mathbb{Z}$, we have

$$EL_{0,1} = P(L_{0,1} = 1) = P(X_1 = Y_1) = 1/2.$$

Similarly, since $L_{0,2} \in [0,2] \cap \mathbb{Z}$, we have

$$EL_{0,2} = P(L_{0,2} = 1) + 2P(L_{0,2} = 2) = (1 - P(L_{0,2} = 0) - P(L_{0,2} = 2)) + 2P(L_{0,2} = 2)$$
$$= 1 + P(L_{0,2} = 2) - P(L_{0,2} = 0) = 1 + P(X_1 = Y_1, X_2 = Y_2) - P(X_1 = X_2 \neq Y_1 = Y_2)$$
$$= 1 + 1/4 - 1/8 = 9/8$$

So we already know that

$$\gamma = \sup_{m\geq 1} E(L_{0,m})/m \geq 9/16.$$

(b) For each $k,n\in\mathbb{N}$ with $k\le n$, denote by $\mathcal{I}_{n,k}:=\{(i_1,\dots,i_k):1<i_1<\cdots<i_k\le n\}$ the collection of all increasing multi-index with length k in the index space $\{1,2,\dots,n\}$. For each $I=(i_1,\dots,i_k)\in\mathcal{I}_{n,k}$, denote by $X_I:=(X_{i_1},X_{i_2},\dots,X_{i_k})$ the I-subsequence of the process $(X_k)_{k\in\mathbb{N}}$. Similarly, we can define Y_I for each $I\in\mathcal{I}_{n,k}$. Note that there exists a constant $C>0$ such that for each integers $0<k<n$, we have

$$\begin{aligned}
P\left(L_{0,n}\ge k\right) &= P(\exists I,J\in\mathcal{I}_{n,k}\text{ s.t. } X_I=Y_J)\le \sum_{I,J\in\mathcal{I}_{n,k}} P(X_I=Y_J)\\
&= \#\{I,J\in\mathcal{I}_{n,k}\}\cdot 2^{-k}=\left(\frac{n!}{k!(n-k)!}\right)^2 2^{-k}\\
&\le C\left(\frac{n^{n+1/2}e^{-n}}{k^{k+1/2}e^{-k}(n-k)^{n-k+1/2}e^{-n-k}}2^{-k}\right)^2, \quad\text{by Stirling's formula}\\
&\overset{a:=n/k}{=} C\left(\frac{n^{n+1/2}}{(an)^{an+1/2}((1-a)n)^{(1-a)n+1/2}}2^{-na}\right)^2\\
&= C\frac{1}{a(1-a)n}\exp\left(-2n\ln\left(a^a(1-a)^{1-a}2^a\right)\right).
\end{aligned}$$

Denote by $g(a)=a^a(1-a)^{1-a}2^a$ for each $a\in(0,1)$, then it holds that $g(a)\underset{a\uparrow 1}{\longrightarrow}2$, which says that there exists an $0<a_0<1$ such that $g(a_0)>1$. Now taking $k_n=\lfloor a_0 n\rfloor$, according to $a_n:=k_n/n\to a_0$, we have

$$\sum_{n\in\mathbb{N}}P\left(\frac{L_{0,n}}{n}\ge a_0\right)\le\sum_{n\in\mathbb{N}}P\left(L_{0,n}\ge k_n\right)\le C\sum_{n\in\mathbb{N}}\frac{1}{a_n(1-a_n)n}\exp\left(-2n\ln g(a_n)\right)<\infty.$$

Therefore, B-C lemma says that almost surely

$$\gamma=\lim_{n\to\infty}\frac{L_{0,n}}{n}\le a_0<1.$$

□

Exercise (6.5.3.). Given a rate one Poisson process in $[0,\infty)\times[0,\infty)$, let X_1,Y_1 be the point that minimizes $x+y$. Let (X_2,Y_2) be the point in $[X_1,\infty)\times[Y_1,\infty)$ that minimizes $x+y$, and so on. Use this construction to show that in Example 6.5.2. $\gamma\ge(8/\pi)^{1/2}>1.59$.

Proof. The definition of rate one Poisson point process N is given by Example 3.7.7.. More precisely, $N(\omega,A)$ is a random measure on $[0,\infty)^2$ such that

- For each $w\in\Omega$, $N(w,\cdot)$ is a $\mathbb{N}\cap\{\infty\}$-valued measure on $[0,\infty)^2$.
- For each Borel subset $A\subset[0,\infty)^2$, $N(\cdot,A)$ is a Poisson distributed random variable with mean $\mu(A)$, the Lebesgue measure of A.

Note that N is a (random) atomic measure, therefore, it is valid to talk about the points of N. According to its definition, (X_1,Y_1) is a point of the atomic measure N in $[0,\infty)^2$ which minimizes $x+y$. And (X_2,Y_2) is the point in $[X_1,\infty)\times[Y_1,\infty)$ which minimizes $x+y$.

We claim the following fact: $((X_{k+1}, Y_{k+1}) - (X_k, Y_k))_{k\in\mathbb{N}}$ are i.i.d. random variables with the same distribution of (X_1, Y_1). This fact is crucial. Its proof relies on the strong Markov property of the Poisson point processes. Here we omit the details.

Note that for each $t \geq 0$, we have

$$P(X_1 + Y_1 > t) = P\left(N\left\{(x, y) : x \geq 0, y \geq 0, x + y \leq t\right\} = 0\right)$$
$$= e^{-\int_{(x,y):x\geq 0, y\geq 0, x+y\leq t} \mu(dx,dy)} = e^{-\frac{t^2}{2}}.$$

Therefore,

$$E[X_1 + Y_1] = \int_0^\infty P(X_1 + Y_1 > t)dt = \sqrt{\frac{\pi}{2}},$$

which, thanks to the symmetry, says that $EX_1 = EY_1 = \sqrt{\frac{\pi}{8}}$. Now, from Law of large numbers, we have almost surely

$$\frac{Z_n}{n} := \frac{\max(X_n, Y_n)}{n} \xrightarrow[n\to\infty]{} \sqrt{\frac{\pi}{8}}.$$

On the other hand, denoted by $L_{0,s}$ length of the longest increasing path lying in the square $R_{0,n}$ with vertices $(0,0)$, $(0,s)$, $(s,0)$ and (s,s). (Here the length of a path is simply the number of the 'Poisson' points on that path. So the length of a path is always an integer number.) It is now obvious that $(X_1, Y_2), \ldots, (X_n, Y_n)$ forms an increasing path in the square R_{0,Z_n}. Therefore, we have $L_{0,Z_n} \geq n$. Using the result in Example 6.5.2., we have

$$\gamma = \lim_{n\to\infty} \frac{L_{0,Z_n}}{Z_n} \geq \lim_{n\to\infty} \frac{n}{Z_n} = \sqrt{\frac{8}{\pi}}.$$

□

Exercise (6.5.4.). Let π_n be a random permutation of $\{1, \ldots, n\}$ and let J_k^n be the number of subsets of $\{1, \ldots, n\}$ of size k so that the associated $\pi_n(j)$ form an increasing subsequence. Compute EJ_k^n and take $k \sim \alpha n^{1/2}$ to conclude that in Example 6.5.2. $\gamma \leq e$.

Proof. For each $k, n \in \mathbb{N}$ with $k \leq n$, denote by $\mathcal{H}_{n,k}$ the collection of all subset of $\{1, \ldots, n\}$ with length k. For each $h \in \mathcal{H}_{n,k}$, we write $h = \{h_i : i = 1, \ldots, k\}$ such that $0 < h_1 < \cdots < h_k \leq n$. There exists a constant $C > 0$ such that for each $0 < k < n$, we have

$$EJ_k^n = \sum_{h\in\mathcal{H}_{n,k}} P\left((\pi_n(h_i))_{i=1}^k \text{ is increasing}\right) = \#\mathcal{H}_{n,k} \cdot \frac{C_n^k \cdot (n-k)!}{n!}$$
$$= \frac{n!}{k!(n-k)!k!} \leq \frac{n^k}{(k!)^2} \leq C\left(\frac{e\sqrt{n}}{k}\right)^{2k}, \quad \text{by Stirling formula.}$$

Therefore if $k \sim \alpha n^{1/2}$ with $\alpha > e$, we have

$$\sum_{n\in\mathbb{N}} EJ_k^n < \infty.$$

Now let $l(\pi_n)$ be the length of the longest increasing sequence in the random permutation π_n, then

$$\sum_{n\in\mathbb{N}} P\left(\frac{l(\pi_n)}{n^{1/2}} \geq \alpha\right) = \sum_{n\in\mathbb{N}} P\left(J^n_{\lceil \alpha n^{1/2}\rceil} \geq 1\right)$$
$$\leq \sum_{n\in\mathbb{N}} EJ^n_{\lceil \alpha n^{1/2}\rceil} < \infty$$

This, and B-C lemma says that

$$\gamma := \lim_{n\to\infty} \frac{l(\pi_n)}{n^{1/2}} \leq \alpha, \quad \text{almost surely.}$$

Finally, since α is chosen arbitrarily in (e,∞), we have that $\gamma \leq e$ almost surely. □

Exercise (6.5.5.). Let $\phi(\theta) = E\exp(-\theta t_i)$ and

$$Y_n = (\mu\phi(\theta))^{-n} \sum_{i=1}^{Z_n} \exp\left(-\theta T_n(i)\right)$$

where the sum is over individuals in generation n and $T_n(i)$ is the ith person's birth time. Show that Y_n is a nonnegative martingale and use this to conclude that if $\exp(-\theta a)/\mu\phi(\theta) > 1$, then $P(X_{0,n} \leq an) \to 0$. A little thought reveals that this bound is the same as the answer in the answer in the last exercise.

Proof. Let $\mathcal{F}_n$ be the filtration which contains all the information about the birth times of all the persons whose generations are smaller than or equal to n. Denote by $Z_1^{(n,i)}$ the number of children of i-th particle in generation n. Denote by $T_1^{(n,i)}(k)$ the birth time of the k-th child of the i-th particle in generation n. It can be verified from the independence of the birth of each particles that

$$E\left[Y_{n+1}|\mathcal{F}_n\right] = E\left[(\mu\phi(\theta))^{-(n+1)} \sum_{i=1}^{Z_n}\sum_{k=1}^{Z_1^{(n,i)}} \exp(-\theta T_1^{(n,i)}(k))\middle|\mathcal{F}_n\right]$$
$$= (\mu\phi(\theta))^{-(n+1)} \sum_{i=1}^{Z_n} \exp(-\theta T_n(i)) E\left[\sum_{k=1}^{Z_1^{(n,i)}} \exp\left(-\theta\left(T_1^{(n,i)}(k) - T_n(i)\right)\right)\middle|\mathcal{F}_n\right]$$
$$= (\mu\phi(\theta))^{-(n+1)} \sum_{i=1}^{Z_n} \exp(-\theta T_n(i)) E\left[\sum_{k=1}^{Z_1} \exp\left(-\theta T_1(k)\right)\right] = Y_n.$$

This says that Y_n is a non-negative martingale. Therefore, it has a finite almost sure limit, say Y_∞. Observe that, we always have

$$\frac{e^{-\theta X_{0,n}}}{(\mu\phi(\theta))^n} \leq \frac{1}{(\mu\phi(\theta))^n} \sum_{i=1}^{Z_n} \exp\left(-\theta T_n(i)\right).$$

Therefore, if $\exp(-\theta a)/\mu\phi(\theta) > 1$, then

$$P(X_{0,n} \le an) = P(e^{-\theta X_{0,n}} \ge e^{-\theta an}) \le \frac{Ee^{-\theta X_{0,n}}}{e^{-\theta an}}$$
$$\le \frac{E[\sum_{i=1}^{Z_n} e^{-\theta T_n(i)}]}{\mu^n \phi(\theta)^n} \left(\frac{\mu\phi(\theta)}{e^{-\theta a}}\right)^n = Y_n(1-\epsilon)^n \to 0.$$

□

Exercise (7.1.1.). Given $s < t$ fine $P\left(B(s) > 0, B(t) > 0\right)$.

Proof.

$$P(B(s) > 0, B(t) > 0) = P(B(s) > 0, B(t) - B(s) > -B(s))$$
$$= \int_0^\infty dx \int_{-x}^\infty \frac{1}{\sqrt{2\pi s}} \cdot e^{-\frac{x^2}{2s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{y^2}{2(t-s)}} dy$$
$$= \frac{1}{4} + \frac{1}{2\pi} \arcsin\sqrt{\frac{s}{t}}.$$

□

Exercise (7.1.2.). Find $E(B_1^2 B_2 B_3)$

$$E[B_1^2 B_2 B_3] = E\left[B_1^2 \left(B_1 + (B_2 - B_1)\right)\left(B_1 + (B_2 - B_1) + (B_3 - B_2)\right)\right]$$
$$= E[B_1^4] + E\left[B_1^2 \left(B_2 - B_1\right)^2\right]$$
$$= E[B_1^4] + E[B_1^2] E\left[(B_2 - B_1)^2\right]$$
$$= 4.$$

Exercise (7.1.4.). $A \in \mathcal{F}_o$ if and only if there is a sequence of times $t_1, t_2, \cdots$ in $[0, \infty)$ and a $B \in \mathcal{R}^{\{1,2,\cdots\}}$ so that $A = \{w : (w(t_1), w(t_2), \cdots) \in B\}$. In words, all events in $\mathcal{F}_o$ depend on only countably many coordinates.

Proof. Let $\Omega = \mathbb{R}^{[0,\infty)}$. Define coordinate process:

$$X^\omega(t) = \omega(t), \forall \omega \in \Omega, t \ge 0.$$

Denote by

$$\mathcal{I} = \{I = (t_k)_{k\in\mathbb{N}} : \forall k \in \mathbb{N}, t_k \in [0, \infty)\}$$

the collection of all the time sequence. For each time sequence $I = (t_k)_{k\in\mathbb{N}} \in \mathcal{I}$, define a map ψ_I from Ω to $\mathbb{R}^\mathbb{N}$ such that

$$\psi_I(\omega) := (X^\omega(t_k))_{k\in\mathbb{N}}.$$

Define $[I] := \{t_k : k \in \mathbb{N}\}$. Consider a σ-field on Ω given by $\mathcal{F}_{[I]} = \sigma(X(t) : t \in [I])$. Then by standard measure theory, we have

$$\mathcal{F}_{[I]} = \{\psi_I^{-1} B : B \in \mathcal{R}^\mathbb{N}\}. \tag{0.1}$$

Define the family of subsets of Ω by

$$\mathcal{G} := \{A \subset \Omega : \exists I \in \mathcal{I}, B \in \mathcal{R}^{\mathbb{N}} \ s.t. \ A = \psi_I^{-1} B\}.$$

What we needs to proof for this exercise is that $\mathcal{G} \subset \mathcal{F}_o$ and $\mathcal{F}_o \subset \mathcal{G}$.

1. We claim that $\mathcal{G} \subset \mathcal{F}_o$. In fact for each $A \in \mathcal{G}$, there is an $I \in \mathcal{I}$ and $B \in \mathcal{R}^{\mathbb{N}}$ such that $A = \psi_I^{-1} B$. Therefore $A \in \mathcal{F}_{[I]} \subset \mathcal{F}_o$.

2. We claim that $\mathcal{G}$ is a σ-field. In fact, if $(A_k)_{k\in\mathbb{N}}$ is a sequence of elements in $\mathcal{G}$, then there exists a sequence of $\mathcal{I}$-elements $(I_k)_{k\in\mathbb{N}}$ and a sequence of $\mathcal{R}^{\mathbb{N}}$-elements $(B_k)_{k\in\mathbb{N}}$ such that

$$A_k = \psi_{I_k}^{-1} B_k, \quad k \in \mathbb{N}.$$

We also have that there exists a $J \in \mathcal{I}$ such that

$$[J] = \bigcup_{k\in\mathbb{N}} [I_k]$$

simply because the right hand side is countable. Now, from (0.1), we have $\bigcup_{k\in\mathbb{N}} A_k \in \sigma(\mathcal{F}_{[I_k]} : k \in \mathbb{N}) = \mathcal{F}_{[J]} \subset \mathcal{G}$. The rest of this claim is elementary.

3. We claim that $\mathcal{F}_o \subset \mathcal{G}$. In fact, for each $t \in [0, \infty)$, we have $X(t)$ is $\mathcal{G}$-measurable simply because $\mathcal{F}_t \subset \mathcal{G}$. □

Exercise (7.1.5.). Looking at the proof of Theorem 7.1.6. carefully shows that if $\gamma > 5/6$ then B_t is not Hölder continuous with exponent γ at any point in $[0, 1]$. Show, by considering k increments instead of 3, that the last conclusion is true for all $\gamma > 1/2 + 1/k$.

Proof. Fix a constant $C < \infty$. Let $A_n = \{w : \exists s \in [0, 1] \ s.t. \ \forall |t - s| \leq \frac{k}{n}, |B_t - B_s| \leq C|t - s|^\gamma\}$. For $1 \leq i \leq n - k + 1$, let

$$Y_{i,n} = \max\{|B(\frac{i + j}{n}) - B(\frac{i + j - 1}{n})| : j = 0, 1, \ldots, k - 1\}.$$

and

$$B_n = \{\text{at least one } Y_{k,n} \leq \frac{(2k - 1)C}{n^\gamma}\}.$$

Then it can be verified that $A_n \subset B_n$. Therefore

$$\begin{aligned} P(A_n) \leq P(B_n) &\leq nP\left(|B(\frac{1}{n})| \leq \frac{(2k - 1)C}{n^\gamma}\right)^k \\ &\leq nP(|B(1)| \leq \frac{(2k - 1)C}{n^{\gamma - \frac{1}{2}}})^k \\ &\leq n\left(\frac{2(2k - 1)C}{\sqrt{2\pi} n^{\gamma - \frac{1}{2}}}\right)^k \to 0. \end{aligned}$$

Rest is the same as Theorem 7.1.6. □

Exercise (7.1.6.). Fix t and let $\Delta_{m,n} = B(tm2^{-n}) - B(t(m - 1)2^{-n})$. Compute

$$E\left[\left(\sum_{m\leq 2^n} \Delta_{m,n}^2 - t\right)^2\right]$$

and use Borel-Cantelli to conclude that $\sup_{m\le 2^n}\Delta^2_{m,n}\to t$ a.s. as $n\to\infty$.

Proof.

$$E\left[\left(\sum_{m=1}^{2^n}\Delta^2_{m,n}-t\right)^2\right]=\sum_{m=1}^{2^n}E\left(\Delta^2_{m,n}-\frac{t}{2^n}\right)^2$$
$$=2^nE\left[\left(B(\frac{t}{2^n})-\frac{t}{2^n}\right)^2\right]=t^2 2^{-n+1}.$$

Therefore,

$$P\left(\left|\sum_{m=1}^{2^n}\Delta^2_{m,n}-t\right|>\frac{1}{n}\right)\le n^2t^2 2^{-n+1}.$$

This says that

$$P\Big(\Big|\sum_{m=1}^{2^n}\Delta^2_{m,n}-t\Big|>\frac{1}{n}\ i.o.\Big)=0.$$

So we get the desired result by BC Lemma. □

Exercise (7.2.1.). Let $T_0=\inf\{s>0:B_s=0\}$ and let $R=\inf\{t>1:B_t=0\}$. R is for right or return. Use the Markov property at time 1 to get

$$P_x(R>1+t)=\int p_1(x,y)P_y(T_0>t)dy$$

Proof. Notice that $R=T_0\circ\theta_1+1$, therefore from Theorem 7.2.1. we have

$$E_x\left(\mathbf{1}_{R(\cdot)>1+t}\big|\mathcal{F}_1^+\right)=E_x\left(\mathbf{1}_{(T_0\circ\theta_1)>t}\big|\mathcal{F}_1^+\right)$$
$$=E_x\left[\left(\mathbf{1}_{T_0>t}\circ\theta_1\right)(\omega)\big|\mathcal{F}_1^+\right]$$
$$=E_{B_1}\left[\mathbf{1}_{T_0>t}\right]=\int p_1(x,y)P_y(T_0>t)dy.$$

□

Exercise (7.2.3.). Let $a<b$, then with probability one a is the limit of local maximum of B_t in (a,b). So the set of local maxima of B_t is almost surely a dense set. However, unlike the zero set it is countable.

Proof. From Theorem 7.2.5., we know that $T_0:=\inf\{t\in(a,b):B_t=B_a\}=a$ almost surely. This says that there exists a Ω_0 with probability 1 such that for any $\omega\in\Omega_0$, there exists a strictly decreasing sequence of t_n with $B_{t_n}=B_a$ and $t_n\downarrow a$.

From Theorem 7.2.4., we know that $T_1:=\inf\{t\in(a,b):B_t>B_a\}=a$ almost surely. This says that there exists a Ω_1 with probability 1 such that for any $\omega\in\Omega_1$, there exists a strictly decreasing sequence of s_n with $B_{s_n}>B_a$ and $s_n\downarrow a$.

Therefore, by chosen suitable subsequence, we know that for each $\omega\in\Omega_0\cap\Omega_1$, there exists a strictly decreasing sequence

$$t_1>s_1>t_2>s_2>\dots$$

such that $B_{t_k} = B_a$ and $B_{s_k} > B_a$ for each $k \in \mathbb{N}$ and that both $t_k \downarrow a$ and $s_k \downarrow a$ hold. Now since the Brownian path are continuous, there will be a sequence of $(r_k)_{k\in\mathbb{N}}$ such that

$$t_1 > r_1 > t_2 > r_2 > \dots$$

and each r_k is a local maxima. (Simply chose $r_k \in (t_k, t_{k+1})$ such that $B_{r_k} = \max\{B_r : r \in [t_k, t_{k+1}]\}$.)

To summarize, for each $a < b$, we have almost surely that a is the limit of local maximum of B_t in (a, b). Therefore, almost surely, for each $q \in \mathbb{Q}$, we have q is in the closure of the set of local maximum of the Brownian path. In another word, almost surely, the set of local maxima of Brownian path is a dense set. □

Exercise (7.2.4.). (i) Suppose $f(t) > 0$ for all $t > 0$. Use Theorem 7.2.3. to conclude that $\limsup_{t\downarrow 0} B(t)/f(t) = c$, P_0 a.s., where $c \in [0, \infty]$ is a constant. (ii) Show that if $f(t) = \sqrt{t}$ then $c = \infty$, so with probability one Brownian paths are not Hölder continuous of order $1/2$ at 0.

Proof. (i). Define $C = \limsup_{t\downarrow 0} B(t)/f(t)$, then C is a random variable which is $\mathcal{F}_0^+$-measurable. It can also be verified that C takes values in $[0, \infty]$ from Theorem 7.2.4., since there exists a sequence of strictly decreasing $t_0 \downarrow 0$ with $B(t_0) > 0$.

Now use Theorem 7.2.3. we know that C almost surely is a constant.

(ii). Define $X_t = tB(1/t)$, then by Theorem 7.2.6. and Theorem 7.2.8 we have

$$\begin{aligned}\limsup_{t\downarrow 0} \frac{B(t)}{\sqrt{t}} &= \limsup_{t\downarrow 0} \frac{tX(1/t)}{\sqrt{t}} \\ &= \limsup_{u\to\infty} \frac{X(u)}{\sqrt{u}} = \infty.\end{aligned}$$

□

Exercise (7.3.1.). Let A be an F_σ, that is, a countable union of closed sets. Show that $T_A = \inf\{t : B_t \in A\}$ is a stopping time.

Proof. Let

$$A = \bigcup_{n\in\mathbb{N}} K_n$$

where K_n are closed sets. Define closed sets $A_n = \bigcup_{k=1}^n K_k$, then we have

$$A = \bigcup_{n\in\mathbb{N}} A_n$$

Define $T_{(A_n)} = \inf\{t : B_t \in A_n\}$ which by Theorem 7.3.4. are stopping times. Now notice that

$$\bigcup_{n\in\mathbb{N}} \{t : B_t \in A_n\} = \{t : B_t \in A\}.$$

So we must have

$$T_A = \inf_{n\in\mathbb{N}} T_{(A_n)} = \lim_{n\to\infty} T_{(A_n)}.$$

Theorem 7.3.2. then says that T_A is a stopping time. □

Exercise (7.3.2.). If S and T are stopping times, then $S \wedge T = \min\{S,T\}$, $S \vee T = \max\{S,T\}$, and $S+T$ are also stopping times. In particular, if $t \geq 0$, then $S \wedge t, S \vee t$ and $S+t$ are stopping times.

Proof. It can be verified that for each $t > 0$,

$$\{S \wedge T > t\} = \{S > t\} \cap \{T > t\} \in \mathcal{F}_t,$$

and

$$\{S \vee T > t\} = \{S > t\} \cup \{T > t\} \in \mathcal{F}_t,$$

and

$$\{S+T < t\} = \{\exists q_1, q_2 \in \mathbb{Q},\ \text{s.t. } S \leq q_1, T \leq q_2, q_1+q_2 < t\}$$
$$= \bigcup_{q_1,q_2\in\mathbb{Q}:0\leq q_1+q_2\leq t} \{S \leq q_1, T \leq q_2\} \in \mathcal{F}_t.$$

□

Exercise (7.3.4.). Let S be a stopping time, let $A \in \mathcal{F}_S$, and let $R = S$ on A and $R = \infty$ on A^c. Show that R is a stopping time.

Proof. It can be verified that for each $t \in [0,\infty)$

$$\{R \leq t\} = \{A, S \leq t\} \in \mathcal{F}_t.$$

□

Exercise (7.3.5.). Let S and T be stopping times. $\{S < T\}$, $\{S > T\}$, and $\{S = T\}$ are in $\mathcal{F}_S$ (and in $\mathcal{F}_T$).

Proof. It can be verified that, for each $s > 0$,

$$\{S < T\} \cap \{S < s\} = \{\exists q \in \mathbb{Q} \cap (0,s),\ \text{s.t. } S < q < T\}$$
$$= \bigcup_{q\in\mathbb{Q}\cap(0,s)} \{S < q, T > q\} \in \mathcal{F}_s.$$

This shows that $\{S < T\} \in \mathcal{F}_S$. It can also be verified that, for each $s > 0$,

$$\{S > T\} \cap \{S < s\} = \{\exists q \in \mathbb{Q},\ \text{s.t. } T < q < S < s\}$$
$$= \bigcup_{q\in\mathbb{Q}:0<q<s} (\{T < q\} \cup \{q < S < s\}) \in \mathcal{F}_s.$$

This says that $\{S > T\} \in \mathcal{F}_S$. Finally

$$\{S = T\} = (\{S > T\} \cup \{S < T\})^c \in \mathcal{F}_S.$$

□

Exercise. Let $B_t = (B_t^1, B_t^2)$ be a two-dimensional Brownian motion starting from 0. Let $T_a = \inf\{t : B_t^2 = a\}$. Show that (1) $B^1(T_a), a \geq 0$ has independent (stationary) increments and (2) $B^1(T_a) =_d aB^1(T_1)$. Use this to conclude that $B^1(T_a)$ has a Cauchy distribution.

Proof. (1) Suppose that $0 = a_0 < a_1 < \dots a_n = u < a_{n+1} = v$. Thanks to induction, in order to show that increments $\{B^1(T_{(a_{(k+1)})}) - B^1(T_{(a_k)}) : k = 0, \dots, n\}$ are mutually independent with each other, we only have to show that the last increment $B^1(T_v) - B^1(T_u)$ is independent of the previous increments $\{B^1(T_{(a_{(k+1)})}) - B^1(T_{(a_k)}) : k = 0, \dots, n-1\}$. Notice that all $(T_a)_{a\geq 0}$ are stopping times of the two-dimensional Brownian motion $(B_t)_{t\geq 0}$ whose corresponding filtration will be denote by $(\mathcal{F}_t)_{t\geq 0}$. Then it is easy to see that the previous increments $\{B^1(T_{(a_{k+1})}) - B^1(T_{(a_k)}) : k = 0, \dots, n-1\}$ are $\mathcal{F}_{(T_u)}$-measurable. So we only have to show that $B^1(T_v) - B^1(T_u)$ is independent of the σ-field $\mathcal{F}_{(T_u)}$. Now from the strong Markov property: for any bounded function f,

$$\begin{aligned} &E\left[f\left(B^1(T_v) - B^1(T_u)\right) | \mathcal{F}_{(T_u)}\right] = E_{(B^1(T_u),u)}\left[f\left(B^1(T_v) - B^1(0)\right)\right] \\ &= E_{(0,0)}\left[f\left(B^1(T_{v-u})\right)\right]. \end{aligned}$$

The RHS is non-random, so $B^1(T_v) - B^1(T_u)$ is independent of the σ-field $\mathcal{F}_{(T_u)}$. The RHS also says that the distribution of $B^1(T_u) - B^1(T_v)$ is only dependent on $v - u$. This is the stationary part.

(2) Note that from (7.1.1.) we have for each $a > 0$

$$(X_s)_{s\geq 0} := (a^{-1}B_{a^2 s})_{s\geq 0} \overset{d}{=} (B_s)_{s\geq 0}$$

Therefore,

$$\begin{aligned} B^1(T_a) &= B^1(\inf\{t : B^2(t) = a\}) = B^1(a^2 \inf\{s : a^{-1}B^2(a^2 s) = 1\}) \\ &= a \cdot X^1(\inf\{s : X_s^2 = 1\}) \overset{d}{=} aB^1(T_1) \end{aligned}$$

(3). From (1) we have that

$$B^1(T_n) \overset{d}{=} \sum_{k=0}^{n} Y_k$$

where $(Y_k)_{k\in\mathbb{Z}_+} \overset{d}{=} (B(T_{k+1}) - B(T_k))_{k\in\mathbb{Z}_+}$ are i.i.d. random variables. So now, we have that

$$B^1(T_1) \overset{d}{=} n^{-1}B^1(T_n) \overset{d}{=} n^{-1}\sum_{k=1}^{n} Y_k \xrightarrow[n\to\infty]{} B^1(T_1).$$

Simply notice that $(B_t^1, B_t^2)_{t\geq 0} \overset{d}{=} (-B_t^1, B_t^2)$, one can verify that $(Y_k)_{k\in\mathbb{Z}_+}$ actually have symmetric distribution. So $B^1(T_1)$ must have symmetric α-stable distribution with $\alpha = 1$, which by its definition, is Cauchy distribution. $\square$

Exercise (7.4.2.). Use (7.2.3) to show that $R := \inf\{t > 1 : B_t = 0\}$ has probability density

$$P_0(R - 1 \in dt) = \frac{1}{(\pi t^{1/2}(1+t))} dt.$$

Proof. According to (7.2.3.) and (7.4.6), we have

$$\begin{aligned} P_0(R - 1 \in dt) &= \int_{y \in \mathbb{R}} p_1(0, y) P_y(T_0 \in dt) dy \\ &= \int_{y \in \mathbb{R}} p_1(0, y) P_0(T_y \in dt) dy = \int_{y \in \mathbb{R}} p_1(0, y) \frac{1}{\sqrt{2\pi s^3}} y e^{-\frac{y^2}{2t}} \, dt \, dy \\ &= \frac{1}{\pi t^{1/2}(1+t)} dt. \end{aligned}$$

□

Exercise (7.4.4.). Let $A_{s,t}$ be the event Brownian motion has at least one zero in $[s, t]$. Show that $P_0(A_{s,t}) = \frac{2}{\pi} \arccos(\sqrt{s/t})$.

Proof. According to 7.4.6., note that

$$\begin{aligned} P_0(A_{s,t}) &= 2 \int_0^\infty p_s(0, x) P_x(T_0 \le t - s) dx \\ &= 2 \int_0^\infty (2\pi s)^{-1/2s} e^{-x^2/2} \int_0^{t-s} (2\pi u^3)^{-1/2} x e^{-x^2/2u} \, du \, dx \\ &= \frac{2}{\pi} arccos(\sqrt{s/t}). \end{aligned}$$

□

Exercise (7.5.1.). Let $T = \inf\{B_t \notin (-a, a)\}$. Show that

$$E_0 \exp(-\lambda T) = 1/\cosh(a\sqrt{2\lambda})$$

Proof. According to the fact that $e^{\theta B_t - \frac{1}{2}\theta^2 t}$ is a martingale, and the fact (from symmetry) that conditionally given T, $B_T = a$ or $B_T = -a$ with $1/2$ probability, we have that

$$\begin{aligned} 1 &= E_0 \left[e^{\theta B_T - \frac{1}{2}\theta^2 T} \right] \\ &= E_0 \left[e^{-\frac{1}{2}\theta^2 T} \right] \cdot \frac{e^{\theta a} + e^{-\theta a}}{2}. \end{aligned}$$

This implies the desired result. □

Exercise (7.5.3). Let $\sigma = \inf\{t : B_t \notin (a, b)\}$ and let $\lambda > 0$.

(1) Use the strong Markov property to show

$$E_x \exp -\lambda T_a = E_x \left(e^{-\lambda\sigma}; T_a < T_b \right) + E_x \left(e^{-\lambda\sigma}; T_b < T_a \right) E_b \exp\{-\lambda T_a\}.$$

(2) Interchange the roles of a and b to get a second equation, use Theorem 7.5.7. to get

$$E_x\left[e^{-\lambda\sigma}; T_a < T_b\right] = \sinh(\sqrt{2\lambda}(b-x))/\sinh(\sqrt{2\lambda}(b-a)).$$

Proof. (1). We have that $E_x[e^{-\lambda(T_a-\sigma)}|\mathcal{F}_\sigma] = E_{B_\sigma}[e^{-\lambda T_a}]$. Therefore,

$$\begin{aligned} E_x e^{-\lambda T_a} &= E_x\left[e^{-\lambda\sigma}E\left[e^{-\lambda(T_a-\sigma)}\middle|\mathcal{F}_\sigma\right]\right] \\ &= E_x\left[e^{-\lambda\sigma}, T_a < T_b\right] + E_x[e^{-\lambda\sigma}, T_a > T_b]E_b e^{-\lambda T_a}. \end{aligned}$$

(2). Change a and b in the above, we have

$$E_x\left[e^{-\lambda T_b}\right] = E_x\left[e^{-\lambda\sigma}; T_b < T_a\right] + E_x\left[e^{-\lambda\sigma}; T_a < T_b\right]E_a[e^{-\lambda T_b}].$$

According to Theorem 7.5.7., we have

$$\begin{aligned} E_x\left[e^{-\lambda T_a}\right] &= \exp(-(a-x)\sqrt{2\lambda}); \\ E_x\left[e^{-\lambda T_b}\right] &= \exp(-(x-b)\sqrt{2\lambda}); \\ E_a[e^{-\lambda T_b}] = E_b[e^{-\lambda T_a}] &= \exp(-(b-a)\sqrt{2\lambda}). \end{aligned}$$

From those, we can obtain the desired result by solving a linear equation. □

Exercise (7.5.5.). Find a martingale of the form $B_t^6 - c_1 t B_t^4 + c_2 t^2 B_t^2 - c_3 t^3$ and use it to compute the third moment of $T = \inf\{t : B_t \neq (-a, a)\}$.

Proof. It can be verified that when $c_1 = 15, c_2 = 45, c_3 = 15$, function

$$u(x,t) := x^6 - c_1 t x^4 + c_2 t^2 x^2 - c_3 t^3$$

solves equations

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0.$$

Therefore $B_t^6 - c_1 t B_t^4 + c_2 t^2 B_t^2 - c_3 t^3$ is a martingale. Now it can be calculated from bounded convergence and monotone convergence that

$$\begin{aligned} E\left[T^3\right] &= \lim_{t\to\infty} E\left[(T\wedge t)^3\right] \\ &= \lim_{t\to\infty} c_3^{-1}E\left[B_{T\wedge t}^6 - c_1(T\wedge t)B_{T\wedge t}^4 + c_2(T\wedge t)^2 B_{T\wedge t}^2\right] \\ &= c_3^{-1}E\left[a^6 - c_1 T a^4 + c_2 T^2 a^2\right] = \frac{61}{15}a^6. \end{aligned}$$

□

Exercise (7.6.5.). Show that $B_t^3 - \int_0^t 3B_s ds$ is a martingale.

Proof. According to Ito's formula,

$$d\left(B_t^3 - \int_0^t 3B_s ds\right) = 3B_t^2 dB_t + \frac{1}{2}\cdot 6B_t dt - 3B_t dt = 3B_t^2 dB_t.$$

This and Theorem 7.6.4. imply that $B_t^3 - \int_0^t 3B_s ds$ is a martingale. □

Exercise (7.6.3.). Let $\beta_{2k}(t) = E_0 B_t^{2k}$. Use Ito's formula to relate $\beta_{2k}(t)$ to $\beta_{2k-2}(t)$ and use this relationship to derive a formula for $\beta_{2k}(t)$.

Proof. According to Ito's formula:

$$dB_t^{2k} = 2kB_t^{2k-1}dB_t + \frac{1}{2}2k(2k-1)B_t^{2k-2}dt.$$

Therefore,

$$\beta_{2k}(t) = E\left[B_t^{2k}\right] = E\left[\int_0^t 2kB_s^{2k-1}dB_s + \frac{2k(2k-1)}{2}\int_0^t B_s^{2k-2}ds\right]$$
$$= \frac{2k(2k-1)}{2}\int_0^t E\left[B_s^{2k-2}\right]ds = \frac{2k(2k-1)}{2}\int_0^t E\left[B_s^{2k-2}\right]ds$$
$$= \frac{2k(2k-1)}{2}\int_0^t \beta_{2(k-1)}(s)ds.$$

Now, it is trivial to verify that

$$\beta_{2k}(t) = (2k-1)!! \cdot t^k.$$

□

Exercise (7.6.5.). Apply Ito's formula to (d-dimensional) $|B_t|^2$. Use this to conclude that $E_0|B_t|^2 = td$.

Proof. According to Ito's formula Theorem 7.6.7., let $f(x) = |x|^2, x \in \mathbb{R}^d$, we have

$$df(B_t) = \sum_{k=1}^d D_k f(B_s)dB_s^{(k)} + \frac{1}{2}\sum_{k=1}^d D_{kk}f(B_s)ds$$
$$= \sum_{k=1}^d 2B_s^{(k)}dB_s^{(k)} + \sum_{k=1}^d ds$$

Therefore, thanks to Theorem 7.6.4.,

$$E_0\left[|B_t|^2\right] = E_0\left[2\sum_{k=1}^d \int_0^t B_s^{(k)}dB_s^{(k)} + \sum_{k=1}^d \int_0^t ds\right] = td.$$

□

Exercise (8.1.1.). (In the context of the proof of Theorem 8.1.1.,) use Exercise 7.5.4. to conclude that $E(T_{U,V}^2) \le 4EX^4$.

Proof. Recall the Exercise 7.5.4. which says that if $T_{a,b} = \inf\{t : B_t \notin (a,b)\}$ where $a < 0 < b$, then $ET^2 \le 4E(B_T^4)$ and $EB_T^4 \le 36ET^2$. Now, noting that for each $a < 0 < b$ $T_{a,b}, B_{(T_{a,b})} \in \mathcal{F}^B$ is independent of U and V, and that from Theorem 8.1.1 $B_{(T_{U,V})} \overset{d}{=} X$, we have

$$E\left[T_{U,V}^2\right] = \int E_0\left[T_{u,v}^2\right]P\left(U \in du, V \in dv\right)$$

$$\leq 4 \int E_0 \left[B^4_{T_{u,v}} \right] P\left(U \in du, V \in dv\right) = 4E_0 \left[B^4_{(T_{U,V})} \right] = 4EX^4.$$

□

Exercise (8.1.2.). Suppose S_n is one-dimensional simple random walk and let

$$R_n = 1 + \max_{m \leq n} S_m - \min_{m \leq n} S_m$$

be the number of points visited by time n. Show that

$$R_n / \sqrt{n} \xrightarrow[t \to \infty]{d} \text{something}.$$

Proof. For each continuous function f on $[0, 1]$, consider functional $Rf := \sup_{t \in [0,1]} f(t) - \inf_{t \in [0,1]} f(t)$. Then R is a continuous map from $C[0, 1]$ to $\mathbb{R}$. Now Theorem 8.1.5. says that

$$R\left(S(n\cdot)/\sqrt{n}\right) \xrightarrow[n \to \infty]{d} RB,$$

where B is a Brownian motion on $[0, 1]$, and therefore RB is a random variable. Finally, note that

$$(R_n - 1)/\sqrt{n} = R(S(n\cdot)/\sqrt{n}), \quad n \geq 0.$$

□

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