On the regularization of reaction-diffusion equations by the Wright-Fisher white noise

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Based on joint ongoing work with **Clayton Barnes** and **Leonid Mytnik**

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Regularization by Noise

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- The other solution $X_t = C_\alpha t^{\frac{1}{1-\alpha}}, t \geq 0$.

Pathwise Regularization by Additive Noise

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- Zvonkin's transform is not available for SPDE.

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where, with $\alpha > 0$ and $\beta > 0$,

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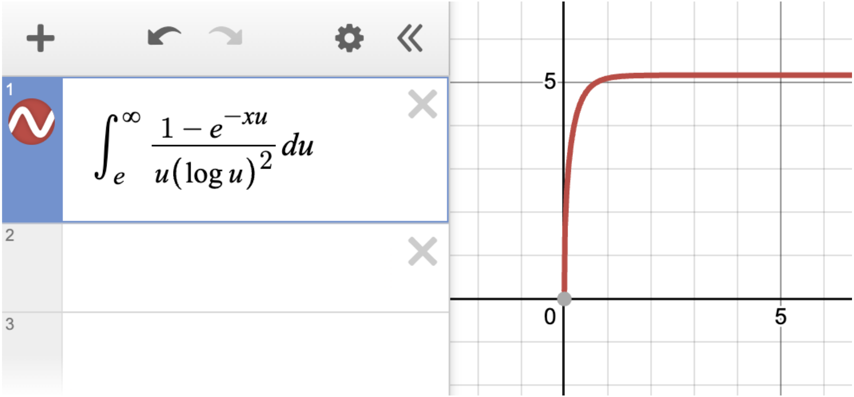
- The shape of the "critical" drift $b(x)$:



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- Reaction-diffusion equations with Wright-Fisher white noise

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  - the strong uniqueness?
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- Question: How strong is the regularization effect of the Wright-Fisher noise?

Motivation

- Shiga (1988): Wright-Fisher SPDE = scaling limit of "genetic stepping stone model."
  - $b(u) = c_1(1-u) - c_2 u + c_3 u(1-u)$.
  - $c_1 \geq 0$ and $c_2 \geq 0$ are mutation rates.
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- Mueller-Tribe (1995), Durrett-Fan (2016)...: Wright-Fisher SPDE = scaling limit of (biased) voter model.
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- Brunet-Derrida (1997), Mueller-Mytnik-Quastel (2011)...: The FKPP equation with Wright-Fisher white noise is related to the Brunet-Derrida particle systems.

Existence

Shiga (1994)

If $f \in \mathcal{C}(\mathbb{R}, [0,1])$, $b(\cdot)$ is continuous and $b(0) \geq 0 \geq b(1)$, then there exists a $\mathcal{C}(\mathbb{R}_+, \mathcal{C}(\mathbb{R}, [0,1]))$-valued weak solution to

$$\begin{cases} \partial_t u = \frac{\Delta}{2} u + b(u) + \sqrt{|u(1-u)|}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}. \end{cases}$$

- We assume the the red part throughout this talk.

Weak Uniqueness: Duality Method

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The weak uniqueness holds provided
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- Both Shiga (1988) and Athreya-Tribe (2000) used the duality argument.

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- Suppose that the random field u and the particle system X are independent.
- We say the moment duality holds between u and X if

$$\mathbb{E}\left[\prod_{i\in I_0} u_t(X_0^i)\right] = \mathbb{E}\left[\prod_{i\in I_t} u_0(X_t^i)\right], \quad t \geq 0.$$

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- The formula characterizes the one-dimensional distributions for both u and X.

Weak Uniqueness: The Girsanov transformation

Mueller-Mytnik-Ryzhik (2021)

The weak uniqueness holds provided

$$\sup_{u\in(0,1)} \frac{|b(u)|}{\sqrt{u(1-u)}} < \infty, \text{ and } f(x) = 1 - f(-x) = 0 \text{ for large enough } x.$$

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- When the blue part holds, we say the initial value f has a compact interface.
- The main tool is Girsanov transformation.

Quantification of the regularization effect

- Consider

$$(2) \qquad \begin{cases} \partial_t u = \frac{\Delta}{2} u + u^{\alpha}(1-u) + \sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}, \end{cases}$$

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- Mueller-Mytnik-Ryzhik (2021):
  $\alpha \in [\frac{1}{2}, 1]$ & f has compact interface $\implies$ weak uniqueness.
- **Question:** What happens when $\alpha \in (0, \frac{1}{2})$? What happens when f doesn't have compact interface?

Propagation speed

Barnes-Mytnik-S. (2023a)

Suppose that $\alpha \in [\frac{1}{2}, 1]$ and that $f \in \mathcal{C}(\mathbb{R}_+, [0,1])$ has compact interface. Let u satisfy

$$\begin{cases} \partial_t u = \frac{\Delta}{2} u + u^{\alpha}(1-u) + \epsilon\sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}. \end{cases}$$

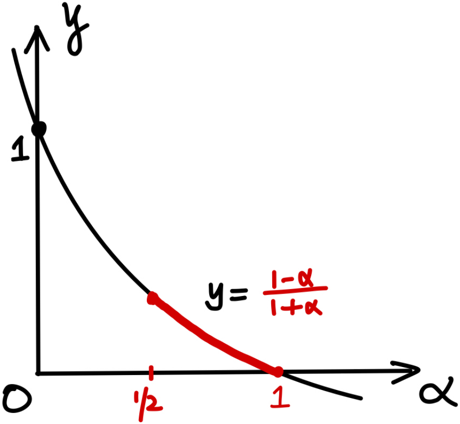
Then,

$$\text{the front of } u_t \ := \sup\{x : u_t(x) > 0\}$$

propagates with a deterministic speed $V(\epsilon) \approx \epsilon^{-2\frac{1-\alpha}{1+\alpha}}$ for small ϵ.

Propagation speed

- Here is an image of the exponent $\frac{1-\alpha}{1+\alpha}$:



Main Result

- Recall AT's condition:

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Barnes-Mytnik-S. (ongoing)

The weak existence and weak uniqueness holds for the 1-d Wright-Fisher type SPDE provided the initial value $f \in \mathcal{C}(\mathbb{R}, [0,1])$, and the drift term

$$b(u) = \sum_{k \in \{0, \infty\} \cup \mathbb{N}} b_k u^k$$

$$\text{with } b_1 \leq -\sum_{k \in \{0, \infty\} \cup \mathbb{N} \setminus \{1\}}^{\infty} |b_k| R^{k-1} \text{ for some } R \geq 1.$$

Conclusion

Corollary 1 (expected)

The weak uniqueness holds for the SPDE

$$\begin{cases} \partial_t u = \frac{\Delta}{2} u + u^{\alpha}(1-u) + \sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}, \end{cases}$$

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- Note that $u^{\alpha}(1-u) \uparrow \mathbf{1}_{\{u>0\}}(1-u)$ when $\alpha \downarrow 0$ for $u \in [0,1]$.
- The weak uniqueness **fails** for the SDE

$$dX_t = \mathbf{1}_{\{X_t>0\}}(1-X_t)\mathrm{d}t + \sqrt{X_t(1-X_t)}\mathrm{d}B_t; \quad X_0 = x \in [0,1].$$

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Corollary 2 (unexpected)

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- **Conjecture:** The weak existence and weak uniqueness holds for the 1d SPDE with Wright-Fisher white noise, arbitrary initial value $f \in \mathcal{C}(\mathbb{R}, [0,1])$, and arbitrary bounded measurable drift b satisfying $b(0) \geq 0 \geq b(1)$.

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- Three dynamics:
  - *Spatial movement:* Particle move as independent Brownian motions.

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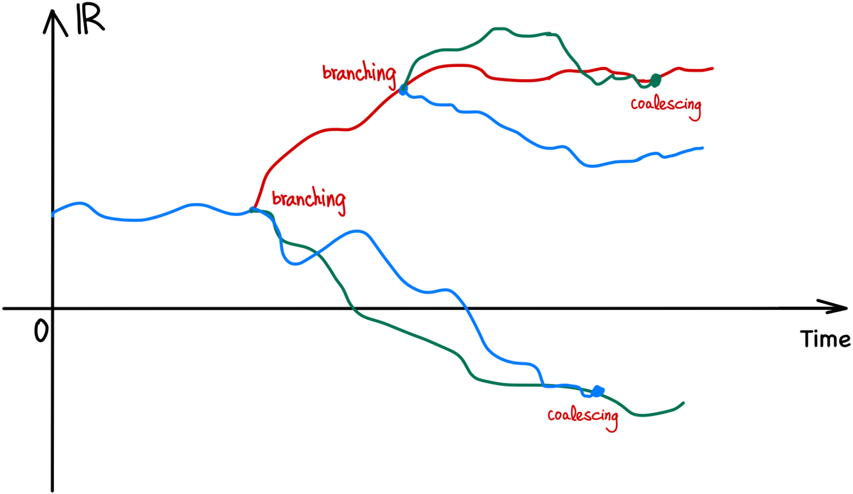
  - *Coalescing:* Each pair of particles coalesces as one particle with rate $1/2$ according to their intersection local time.

An illustration of the dual particle system



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- To build a duality relation between CBBMs and the Wright-Fisher SPDEs, we take
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- The definition of the particle system needs more justification!

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Moreover, in this case

$$\frac{Z_t(\mathbb{R})}{\int v_t(x)\mathrm{d}x} \xrightarrow[L^1]{} 1, \quad t \downarrow 0$$

where $v_t(x)$ is the unique non-negative solution to the 1d PDE

$$2\partial_t v = \Delta v - v^2; \quad v_0(x)\mathrm{d}x = Z_0(\mathrm{d}x).$$

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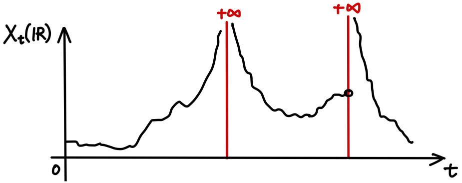
Barnes-Mytnik-S. (ongoing)

If $X_0(\mathbb{R}) < \infty$, then $X_t(\mathbb{R})$ is reflecting from ∞.



Thanks!