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Spine Decompositions and Limit Theorems for a Class of Critical Superprocesses

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Abstract In this paper we first establish a decomposition theorem for size-biased Poisson random measures. As consequences of this decomposition theorem, we get a spine decomposition theorem and a 2-spine decomposition theorem for some critical superprocesses. Then we use these spine decomposition theorems to give probabilistic proofs of the asymptotic behavior of the survival probability and Yaglom's exponential limit law for critical superprocesses.

Keywords Critical superprocess · Size-biased Poisson random measure · Spine decomposition · 2-Spine decomposition · Asymptotic behavior of the survival probability · Yaglom's exponential limit law · Martingale change of measure

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1 Introduction

1.1 Motivation

It is well known that for a critical Galton-Watson process $\{(Z_n)_{n\in\mathbb{N}}; P\}$, we have

$$nP(Z_n > 0) \xrightarrow[n\to\infty]{} \frac{2}{\sigma^2} \tag{1.1}$$

and

$$\left\{\frac{Z_n}{n}; P(\cdot|Z_n > 0)\right\} \xrightarrow[n\to\infty]{law} \frac{\sigma^2}{2}\mathbf{e}, \tag{1.2}$$

where σ^2 is the variance of the offspring distribution and $\mathbf{e}$ is an exponential random variable with mean 1. The result (1.1) was first proved by Kolmogorov in [26] under a third moment condition, and the result (1.2) is due to Yaglom [42]. For further references to these results, see [21, 24]. Ever since these pioneering papers of Kolmogorov and Yaglom, lots of analogous results have been obtained for more general critical branching processes. For continuous time critical branching processes, see [3]; for discrete time multitype critical branching processes, see [3, 22]; for continuous time multitype critical branching processes, see [4]; and for critical branching Markov processes, see [2]. We will call results like (1.1) Kolmogorov type results and results like (1.2) Yaglom type results. Similar results have also been obtained for some superprocesses. Evans and Perkins [16] obtained both Kolmogorov type and Yaglom type results for critical superprocesses when the branching mechanism is $(x, z) \mapsto z^2$ and the spatial motion satisfies some ergodicity conditions. Recently, Ren, Song and Zhang [38] obtained similar limit results for a class of critical superprocesses with general branching mechanisms and general spatial motions.

The proofs of the limit results in the papers mentioned above are all analytic in nature and thus not very transparent. More intuitive probabilistic proofs would be very helpful. This was first accomplished for critical Galton-Watson processes, see [17, 32] for probabilistic proofs of (1.1), and [18, 32, 35] for probabilistic proofs of (1.2). For more general models, Vatutin and Dyakonova [41] gave a probabilistic proof of a Kolmogorov type result for multitype critical branching processes. Recently, Powell [34] gave probabilistic proofs of both Kolmogorov type and Yaglom type results for a class of critical branching diffusions. As far as we know, there is no probabilistic proof of Yaglom type result for multitype critical branching processes, and there are no probabilistic proofs of both Kolmogorov type and Yaglom type results for critical superprocesses yet.

In this paper, we will use the spine method to give probabilistic proofs of both Kolmogorov type and Yaglom type results for a class of critical superprocesses. We will first establish a size-biased decomposition theorem for superprocesses (Theorem 1.2) which will serve as a general framework for the spine method. Then, we will establish a spine decomposition theorem for superprocesses (Theorem 1.5) which is more general than those previously considered in [12, 13, 31]. We will also establish a 2-spine decomposition theorem for a class of critical superprocesses (Theorem 1.9). Those spine decompositions are all special forms of the aforementioned size-biased decomposition. Finally, we use these tools to give probabilistic proofs of a Kolmogorov type result (Theorem 1.10) and a Yaglom type result (Theorem 1.11) for critical superprocesses under slightly weaker conditions than [38]. To develop our decomposition for critical superprocesses, we first prove a size-biased decomposition theorem for Poisson random measures (Theorem 1.3), which we think is of independent interest. Before we present our main results, we first give a brief review of earlier results on the spine method.

The spine method was first introduced in [32]. Roughly speaking, the spine decomposition theorem says that the size-biased transform of the branching process can be interpreted as an immigration branching process along with an immortal particle. This spine approach is generic in the sense that it can be adapted to a variety of general branching processes and is powerful in studying limit behaviors due to its relation with the size-biased transforms. In this paper, by the *size-biased transform of a stochastic process* we mean the following: Suppose that we are given, on some probability space $(\Omega, \mathscr{F}, P)$, a process $(X_t)_{t\in\Gamma}$, with Γ being an arbitrary index set, and a non-negative random variable G with $P[G] \in (0,\infty)$. We say a process $\{(\dot{X}_t)_{t\in\Gamma}; \dot{P}\}$ is a *G-transform* of the process $\{(X_t)_{t\in\Gamma}; P\}$ if $\{(\dot{X}_t)_{t\in\Gamma}; \dot{P}\} \overset{\text{f.d.d.}}{=} \{(X_t)_{t\in\Gamma}; P^G\}$, where P^G is a probability measure on Ω given by $dP^G := (G/P[G])\, dP$. (This also give the definition of a *size-biased transform of a random variable* since a random variable can be considered as a stochastic process whose index is a singleton.)

Using the spine decomposition theorem for the Galton-Watson process $(Z_n)_{n\geq 0}$, Lyons, Pemantle and Peres [32] investigated the Z_n-transform of the process $(Z_k)_{0\leq k\leq n}$, which is denoted by $(\dot{Z}_k)_{0\leq k\leq n}$. Their key observation in the critical case is that $U \cdot \dot{Z}_n$ is distributed approximately like Z_n conditioned on $\{Z_n > 0\}$, where U is an independent uniform random variable on [0, 1]. If one denotes by X the weak limit of $\frac{Z_n}{n}$ conditioned on $\{Z_n > 0\}$, and by $\dot{X}$ the weak limit of $\frac{\dot{Z}_n}{n}$, then [32] proved that $\dot{X}$ is the X-transform of the positive random variable X and $X \overset{law}{=} U \cdot \dot{X}$, which implies that X is an exponential random variable.

The spine method is also used by Powell [34] to establish results parallel to (1.1) and (1.2) for a class of critical branching diffusions $\{(Y_t)_{t\geq 0}; (P_x)_{x\in D}\}$ in a bounded smooth domain $D \subset \mathbb{R}^d$. As have been discussed in [34], a direct study of the partial differential equation satisfied by the survival probability $(t, x) \mapsto P_x(\|Y_t\| \neq 0)$ is tricky. Instead, by using a spine decomposition approach, Powell [34] showed that the survival probability decays like $a(t)\phi(x)$, where $\phi(x)$ is the principal eigenfunction of the mean semigroup of (Y_t) and $a(t)$ is a function capturing the uniform speed. In this paper, our proof of the Kolmogorov type result for critical superprocesses follows a similar argument.

The spine method for superprocesses was developed in [12, 13, 31] and is very useful in studying limit behaviors of supercritical superprocesses. Heuristically, the spine is the trajectory of an immortal moving particle and the spine decomposition theorem says that, after a martingale change of measure, the transformed superprocess can be decomposed in law as an immigration process along this spine. The spine decomposition theorem established in this paper is more general than those in [12, 13, 31]. We will say more about this in the next subsection.

Very recently, we developed a 2-spine decomposition technique in [35] for critical Galton-Watson processes and used it to give a new probabilistic proof of Yaglom's result (1.2). One of the facts we used in [35] is that, if X is a strictly positive random variable with finite second moment, then X is an exponential random variable if and only if

$$\ddot{X} \overset{law}{=} \dot{X} + U \cdot \dot{X}' \tag{1.3}$$

where $\dot{X}$ and $\dot{X}'$ are independent X-transforms of X; $\ddot{X}$ is the X^2-transform of X; and U is again an independent uniform random variable on [0, 1]. We then proved in [35] that the $Z_n(Z_n - 1)$-transform of the critical Galton-Watson process $(Z_k)_{0\leq k\leq n}$, which is denoted as $(\ddot{Z}_k^{(n)})_{0\leq k\leq n}$, can be interpreted as an immigration branching process along a 2-spine skeleton. One of those two spines is longer than the other. The spirit of our proof in [35] is to show that the immigration along the longer spine at generation n is distributed approximately like $\dot{Z}_n$, while the immigration along the shorter spine at generation n is distributed approximately like $\dot{Z}'_{[U\cdot n]}$. Here $\dot{Z}_n$ and $\dot{Z}'_n$ are independent Z_n-transforms of Z_n. Roughly speaking, we have $\ddot{Z}_n^{(n)} \overset{law}{\approx} \dot{Z}_n + \dot{Z}'_{[U\cdot n]}$, and therefore, if X is the weak limit of $\frac{Z_n}{n}$ conditioned on $\{Z_n > 0\}$, then X is a positive random variable satisfying (1.3). In this paper, we adapt the method of [35] to develop a 2-spine decomposition for critical superprocesses and then use this 2-spine decomposition to give probabilistic proofs of Kolmogorov type and Yaglom type results for superprocesses. The spirit of this paper is similar to that of [35], but the arguments are more complicated.

The idea of multi-spine decomposition is not new. It was first introduced by Harris and Roberts [19] in the context of branching processes. Our 2-spine methods for Galton-Watson trees [35] and for superprocesses in this paper are both inspired by [19]. An analogous k-spine decomposition theorem also appeared in [20] and [23] in the context of continuous time Galton-Watson processes. The k-th size-biased transform of Galton-Watson trees is also considered in [1]. A closely related infinite spine decomposition is also established in [1] for the supercritical Galton-Watson tree.

There is another decomposition theorem for supercritical Galton-Watson trees with infinite spines which is first introduced in [3, Sect. 12] and is now known as the skeleton decomposition. The infinite spines in [1] and the skeleton decomposition in [3, Sect. 12] are two different decomposition theorems. Our 2-spine methods for Galton-Watson trees [35] and for superprocesses in this paper are more relevant to [1].

We mention here that the analog of the skeleton decomposition in [3, Sect. 12] for supercritical superprocesses is also available and is very popular. Heuristically, the skeleton is the trajectories of all the prolific individuals, that is, individuals with infinite lines of descent. The skeleton decomposition says that the supercritical superprocess itself can be decomposed in law as an immigration process along this skeleton. For the skeleton methods and its applications under a variety of names, see [5, 6, 9, 12, 14, 15, 28, 29, 33, 36]. If we consider critical superprocesses conditioned to be never extinct, then we will get the transformed superprocesses (after a Doob's h-transformation) considered in [12, 13, 31] for the classical spine decomposition theorem. In this situation, there will be only one prolific individual which is exactly the spine particle. So the natural analog of the skeleton decomposition in the critical case is the classical spine decomposition. The skeleton decomposition will not be used in this paper.

1.2 Main Results

Let E be a locally compact separable metric space. We will use $b\mathscr{B}_E$ and $p\mathscr{B}_E$ to denote the collection of all bounded Borel functions and positive Borel functions on E respectively. We write $bp\mathscr{B}_E$ for $b\mathscr{B}_E \cap p\mathscr{B}_E$. For any functions f, g and measure μ on E, we write $\|f\|_\infty := \sup_{x\in E} |f(x)|$, $\mu(f) := \int_E f d\mu$, $\langle \mu, f\rangle := \int_E f d\mu$ and $\langle f, g\rangle_\mu := \int_E fg d\mu$ as long as they have meanings. We use $\mathbf{0}$ to denote the null measure and use $f \equiv 0$ to mean that f is the zero function. If $g(t,x)$ is a function on $[0,\infty)\times E$, we say g is *locally bounded* if $\sup_{t\in[0,T],x\in E} |g(t,x)| < \infty$ for every $T \geq 0$.

Let the *spatial motion* $\xi = \{(\xi_t)_{t\geq 0}; (\mathbb{P}_x)_{x\in E}\}$ be an E-valued Hunt process with its lifetime denoted by ζ and its transition semigroup denoted by $(P_t)_{t\geq 0}$. Let the *branching mechanism* ψ be defined as a function on $E\times[0,\infty)$ by

$$\psi(x,z) = -\beta(x)z + \alpha(x)z^2 + \int_0^\infty \big(e^{-zr} - 1 + zr\big)\pi(x,dr), \qquad x\in E,\ z\geq 0,$$

with $\beta \in b\mathscr{B}_E$, $\alpha \in bp\mathscr{B}_E$ and $\pi(x, dy)$ being a kernel from E to $(0, \infty)$ satisfying that

$$\sup_{x\in E}\int_{(0,\infty)}\big(y \wedge y^2\big)\pi(x, dy) < \infty.$$

Define an operator Ψ on $p\mathscr{B}_E$ by

$$(\Psi f)(x) := \psi\big(x, f(x)\big), \quad f \in p\mathscr{B}_E,\ x \in E.$$

Let $\mathscr{M}_f$ denote the space of all finite measures on E equipped with the weak topology. A (ξ, ψ)-*superprocess* is an $\mathscr{M}_f$-valued Hunt process $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathscr{M}_f}\}$ satisfying

$$\mathbf{P}_\mu\big[e^{-X_t(f)}\big] = e^{-\mu(V_t f)}, \quad t \geq 0,\ \mu \in \mathscr{M}_f,\ f \in bp\mathscr{B}_E, \tag{1.4}$$

where, for each $f \in bp\mathscr{B}_E$, the function $(t, x) \mapsto V_t f(x)$ on $[0, \infty) \times E$ is the unique locally bounded positive solution to the equation

$$V_t f(x) + \mathbb{P}_x\bigg[\int_0^t (\Psi V_{t-s} f)(\xi_s)ds\bigg] = \mathbb{P}_x\big[f(\xi_t)\big], \quad t \geq 0,\ x \in E. \tag{1.5}$$

We refer our readers to [8, 10] and [30, Sect. 2.3 & Theorem 5.11] for detailed discussions about the existence of such processes. Notice that we always have $\mathbf{P}_\mathbf{0}(X_t = \mathbf{0}) = 1$ for each $t \geq 0$, i.e. the null measure $\mathbf{0}$ is an absorption state of the superprocess.

We will always assume that our superprocess is *non-persistent*:

Assumption 1 $\mathbf{P}_{\delta_x}(X_t = \mathbf{0}) > 0$ for each $x \in E$ and $t > 0$.

By a *size-biased transform of a measure* we mean the following: For a non-negative measurable function g on a measure space $(D, \mathscr{F}_D, \mathbf{D})$ with $\mathbf{D}(g) \in (0, \infty)$, we define the *$g$-transform $\mathbf{D}^g$ of the measure* $\mathbf{D}$ by

$$d\mathbf{D}^g := \frac{g}{\mathbf{D}(g)}d\mathbf{D}.$$

Note that, the measure $\mathbf{D}$ is not necessarily a probability measure, but after the g-transform, $\mathbf{D}^g$ is always a probability measure.

Our first result is about a decomposition theorem of the size-biased transforms of superprocesses. To state it, we need to introduce the Kuznetsov measures $(\mathbb{N}_x)_{x\in E}$ (also known as the excursion measures or $\mathbb{N}$-measures) of the superprocess X.

Lemma 1.1 ([30, Sect. 8.4 & Theorem 8.24]) *Under Assumption 1, there exists an unique family of σ-finite measures $(\mathbb{N}_x)_{x\in E}$ defined on the Skorokhod space of measure-valued paths*

$$\mathscr{W} := \big\{w = (w_t)_{t\geq 0} : w \textit{ is an } \mathscr{M}_f\textit{-valued càdlàg function on } [0, \infty) \textit{ having } \mathbf{0} \textit{ as a trap}\big\}$$

such that

(1) $\mathbb{N}_x\{\forall t > 0, w_t = \mathbf{0}\} = 0$ *for each* $x \in E$;
(2) $\mathbb{N}_x\{w_0 \neq \mathbf{0}\} = 0$ *for each* $x \in E$;

(3) *for each $\mu \in \mathscr{M}_f$, if $\mathscr{N}(dw)$ is a Poisson random measure on $\mathscr{W}$ with mean measure*

$$\mathbb{N}_\mu(dw) := \int_E \mathbb{N}_x(dw)\mu(dx), \quad w \in \mathscr{W},$$

then the process defined by

$$\widetilde{X}_0 := \mu; \quad \widetilde{X}_t := \int_{\mathscr{W}} w_t \, \mathscr{N}(dw), \quad t > 0,$$

is a realization of the superprocess $\{X; \mathbf{P}_\mu\}$.

The measures $(\mathbb{N}_x)_{x\in E}$ are called the *Kuznetsov measures of the superprocess X*. Note that, the superprocess X itself can be considered as a $\mathscr{W}$-valued random element. Roughly speaking, the branching property of superprocess says that X can be considered as an "infinitely divisible" $\mathscr{W}$-valued random element. The Kuznetsov measure $\mathbb{N}_x$ can then be interpreted as the "Lévy measure" of X under $\mathbf{P}_{\delta_x}$. We refer our readers to [11] and [30, Sect. 8.4] for more details about such measures.

In the remainder of this paper, we will always use $(\mathbb{N}_x)_{x\in E}$ to denote the Kuznetsov measures of our superprocess X. We will always use $w = (w_t)_{t\geq 0}$ to denote a generic element in $\mathscr{W}$. With a slight abuse of notation, we always assume that our superprocess X is given by

$$X_0 := \mu; \quad X_t := \int_{\mathscr{W}} w_t \, \mathscr{N}(dw), \quad t > 0,$$

where, for each $\mu \in \mathscr{M}_f$, $\{\mathscr{N}; \mathbf{P}_\mu\}$ is a Poisson random measure on $\mathscr{W}$ with mean measure $\mathbb{N}_\mu$. Recall that, for any $w \in \mathscr{W}$ and $t \geq 0$, w_t is a finite measure on E, and thus $w_t(f) = \int_E f(x) w_t(dx)$ for any $f \in p\mathscr{B}_E$.

Our first result is about the $\mathscr{N}(F)$-transform of the superprocess X, where F is a non-negative measurable function on $\mathscr{W}$ with $\mathbb{N}_\mu[F] \in (0,\infty)$ for a given $\mu \in \mathscr{M}_f$. In this case, according to Campbell's formula, we have

$$\mathbf{P}_\mu\big[\mathscr{N}(F)\big] = \mathbb{N}_\mu[F] \in (0,\infty).$$

Therefore, both $\mathbb{N}_\mu^F$—the F-transform of $\mathbb{N}_\mu$, and $\mathbf{P}_\mu^{\mathscr{N}(F)}$—the $\mathscr{N}(F)$-transform of $\mathbf{P}_\mu$, are well defined probability measures.

Theorem 1.2 *Suppose that Assumption 1 holds. Let $\mu \in \mathscr{M}_f$ and F be a non-negative measurable function on $\mathscr{W}$ with $\mathbb{N}_\mu(F) \in (0,\infty)$. Let $\{(Y_t)_{t\geq 0}; \mathbf{Q}_\mu\}$ be a $\mathscr{W}$-valued random element with law $\mathbb{N}_\mu^F$. Then we have $\{(X_t)_{t\geq 0}; \mathbf{P}_\mu^{\mathscr{N}(F)}\} \overset{f.d.d.}{=} \{(X_t + Y_t)_{t\geq 0}; \mathbf{P}_\mu \otimes \mathbf{Q}_\mu\}$.*

In order to prove Theorem 1.2, we develop a decomposition theorem for size-biased transforms of Poisson random measures which we think should be of independent interest:

Theorem 1.3 *Let $(S,\mathscr{S})$ be a measurable space with a σ-finite measure N. Let $\{\mathbf{N}; P\}$ be a Poisson random measure on $(S,\mathscr{S})$ with mean measure N. Let $g \in p\mathscr{S}$ satisfy $N(g) \in (0,\infty)$. Denote by N^g and $P^{\mathbf{N}(g)}$ the g-transform of N and the $\mathbf{N}(g)$-transform of P, respectively. Let $\{\vartheta; Q\}$ be an S-valued random element with law N^g. Then we have $\{\mathbf{N}; P^{\mathbf{N}(g)}\} \overset{law}{=} \{\mathbf{N} + \delta_\vartheta; P \otimes Q\}$.*

Define $(S_t)_{t\geq 0}$ the *mean semigroup* of the superprocess X by

$$S_t f(x) := \mathbb{P}_x\big[e^{\int_0^t \beta(\xi_s)\,ds} f(\xi_t)\big], \quad x \in E,\ t \geq 0,\ f \in p\mathscr{B}_E.$$

For each $\mu \in \mathscr{M}_f$, we define $(\mu\mathbb{P})(\cdot) := \int_E \mathbb{P}_x(\cdot)\mu(dx)$. Note that $\mu\mathbb{P}$ is not necessarily a probability measure. It is well known (see [30, Proposition 2.27] for example) that for each $\mu \in \mathscr{M}_f, t \geq 0$ and $f \in p\mathscr{B}_E$,

$$\mathbf{P}_\mu\big[X_t(f)\big] = \mathbb{N}_\mu\big[w_t(f)\big] = (\mu\mathbb{P})\big[e^{\int_0^T \beta(\xi_s)\,ds} f(\xi_T)\mathbf{1}_{T<\zeta}\big] = \mu(S_t f). \tag{1.6}$$

Thanks to Theorem 1.2, in order to study the size-biased transform of a superprocess we only have to study the corresponding size-biased transform of its Kuznetsov measures. We first consider the case when the function F in Theorem 1.2 takes the form of $F(w) = w_T(g)$ where $T > 0$ and $g \in p\mathscr{B}_E$ with $\mu(S_T g) \in (0, \infty)$ for a given $\mu \in \mathscr{M}_f$. In this case, according to (1.6), we have

$$\mathbf{P}_\mu\big[X_T(g)\big] = \mathbb{N}_\mu\big[w_T(g)\big] = (\mu\mathbb{P})\big[e^{\int_0^T \beta(\xi_s)\,ds} g(\xi_T)\mathbf{1}_{T<\zeta}\big] \in (0, \infty).$$

Therefore, $\mathbf{P}_\mu^{X_T(g)}$—the $X_T(g)$-transform of $\mathbf{P}_\mu$, $\mathbb{N}_\mu^{w_T(g)}$—the $w_T(g)$-transform of the Kuznetsov measure $\mathbb{N}_\mu$, and $\mathbb{P}_\mu^{(g,T)}$—the $(e^{\int_0^T \beta(\xi_s)\,ds} g(\xi_T)\mathbf{1}_{T<\zeta})$-transform of the measure $\mu\mathbb{P}$, are all well defined probability measures. Also note that, in this case, we have $X_T(g) = \mathscr{N}(F)$, therefore $\mathbf{P}_\mu^{X_T(g)} = \mathbf{P}_\mu^{\mathscr{N}(F)}$. Recall that the superprocess X itself can be considered as a $\mathscr{W}$-valued random element. Denote by $\mathbf{P}_\mu(X \in dw)$ the push-forward of $\mathbf{P}_\mu$ under X, i.e., the distribution of X under $\mathbf{P}_\mu$. Then, $\mathbf{P}_\mu(X \in dw)$ is a probability measure on $\mathscr{W}$. Recall that we always assume that Assumption 1 holds.

Definition 1.4 Suppose that $\mu \in \mathscr{M}_f$, $T > 0$ and $g \in p\mathscr{B}_E$ satisfy $\mu(S_T g) \in (0, \infty)$. We say $\{(\xi_t)_{0\leq t\leq T}, (Y_t)_{0\leq t\leq T}, \mathbf{n}_T; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is a *spine representation of* $\mathbb{N}_\mu^{w_T(g)}$ if the following are true:

(1) The *spine process* $\{(\xi_t)_{0\leq t\leq T}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is a copy of $\{(\xi_t)_{0\leq t\leq T}; \mathbb{P}_\mu^{(g,T)}\}$.
(2) Conditioned on $\sigma(\xi_t : 0 \leq t \leq T)$, the *immigration process* $\{(Y_t)_{0\leq t\leq T}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is an $\mathscr{M}_f$-valued process given by

$$Y_t := \int_{(0,t]\times\mathscr{W}} w_{t-s}\mathbf{n}_T(ds, dw), \quad 0 \leq t \leq T, \tag{1.7}$$

where, $\mathbf{n}_T$ is a Poisson random measure on $[0, T] \times \mathscr{W}$ with mean measure

$$\mathbf{m}_T^\xi(ds, dw) := 2\alpha(\xi_s)\mathbb{N}_{\xi_s}(dw)\cdot ds + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}(X \in dw)\pi(\xi_s, dy)\cdot ds. \tag{1.8}$$

We are now ready to present our theorem on the spine decomposition of superprocesses:

Theorem 1.5 *Suppose that Assumption 1 holds. Suppose that $\mu \in \mathscr{M}_f$, $T > 0$ and $g \in p\mathscr{B}_E$ satisfy $\mu(S_T g) \in (0, \infty)$. Let $\{(\xi_t)_{0\leq t\leq T}, (Y_t)_{0\leq t\leq T}, \mathbf{n}_T; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ be a spine representation of $\mathbb{N}_\mu^{w_T(g)}$. Then, $\{(Y_t)_{t\leq T}; \dot{\mathbf{P}}_\mu^{(g,T)}\} \overset{f.d.d.}{=} \{(w_t)_{t\leq T}; \mathbb{N}_\mu^{w_T(g)}\}$.*

As a simple consequence of Theorems 1.2 and 1.5, we have the following:

Corollary 1.6 *Suppose that Assumption 1 holds. Suppose that $\mu \in \mathscr{M}_f$, $T > 0$ and $g \in p\mathscr{B}_E$ satisfy $\mu(S_T g) \in (0,\infty)$. Let $\{(\xi_t)_{0\le t\le T}, (Y_t)_{0\le t\le T}, \mathbf{n}_T; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ be a spine representation of $\mathbb{N}_\mu^{w_T(g)}$. Then, $\{(X_t)_{t\ge 0}; \mathbf{P}_\mu^{X_T(g)}\} \overset{f.d.d.}{=} \{(X_t+Y_t)_{t\ge 0}; \mathbf{P}_\mu \otimes \dot{\mathbf{P}}_\mu^{(g,T)}\}$.*

Corollary 1.6 can be considered as a generalization of the classical spine decomposition theorem for superprocesses developed in [12, 13, 31]. In these earlier papers, the testing function g is chosen specifically to be the principal eigenfunction ϕ of the mean semigroup of the superprocess (which will be introduced shortly). In the classical case (i.e. $g=\phi$), the four families of probability measures $(\mathbf{P}_\mu^{X_T(g)})_{T\ge 0}$, $(\mathbb{P}_\mu^{(g,T)})_{T\ge 0}$, $(\dot{\mathbf{P}}_\mu^{(g,T)})_{T>0}$ and $(\mathbb{N}_\mu^{w_T(g)})_{T>0}$ are all consistent, but in the general case (i.e. $g\neq\phi$), they are typically not consistent. More details about these consistencies will be provided in Lemma 3.4 and Remark 3.6.

In the papers mentioned in the paragraph above, the Kuznetsov measures have already been used to describe infinitesimal immigrations along the spine. However, our Theorem 1.5 provides another relation between immigration and the Kuznetsov measures: the total immigration $\{(Y_t)_{t\ge 0}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ actually has the law of a size-biased transform of the Kuznetsov measures. It seems that this fact has not been exploited before, even in the classical case.

The study of the limit behavior of superprocesses X relies heavily on the spectral property of the mean semigroup. In this paper, we assume the following:

Assumption 2 There exist a σ-finite Borel measure m with full support on E and a family of strictly positive, bounded continuous functions $\{p(t,\cdot,\cdot): t>0\}$ on $E\times E$ such that,

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy), \qquad t>0,\ x\in E,\ f\in b\mathscr{B}_E, \tag{1.9}$$

$$\int_E p(t,x,y) m(dx) \le 1, \qquad t>0,\ y\in E, \tag{1.10}$$

$$\int_E\int_E p(t,x,y)^2 m(dx)m(dy) < \infty, \qquad t>0, \tag{1.11}$$

and that $x\mapsto \int_E p(t,x,y)^2 m(dy)$ and $y\mapsto \int_E p(t,x,y)^2 m(dx)$ are both continuous on E.

In the reminder of this paper, we will always use m to denote the reference measure in Assumption 2.

Assumption 2 is a pretty weak assumption. (1.10) implies that the adjoint operator P_t^* of P_t is also Markovian, and (1.11) implies that P_t and P_t^* are Hilbert-Schmidt operators. Under Assumption 2, it is proved in [38] and [39] that the semigroup $(P_t)_{t\ge 0}$ and its adjoint semigroup $(P_t^*)_{t\ge 0}$ are both strongly continuous semigroups of compact operators on $L^2(E,m)$. According to [38, Lemma 2.1], there exists a function $q(t,x,y)$ on $(0,\infty)\times E\times E$ which is continuous in (x,y) for each $t>0$ such that

$$e^{-\|\beta\|_\infty t} p(t,x,y) \le q(t,x,y) \le e^{\|\beta\|_\infty t} p(t,x,y), \quad t>0,\ x,\ y\in E,$$

and that for any $t>0, x\in E$ and $f\in b\mathscr{B}_E$,

$$S_t f(x) = \int_E q(t,x,y) f(y) m(dy). \tag{1.12}$$

(From (1.6), we see that $q(t,x,y)m(dy)$ can be roughly interpreted as the density of the expected mass of X_t at position y, under probability $\mathbf{P}_{\delta_x}$.) Define a family of transition kernels $(S_t^*)_{t\geq 0}$ on E by

$$S_0^* = I; \quad S_t^* f(y) := \int_E q(t,x,y) f(x) m(dx), \quad t > 0,\ y \in E,\ f \in b\mathscr{B}_E.$$

It is clear that $(S_t^*)_{t\geq 0}$ is the adjoint semigroup of $(S_t)_{t\geq 0}$ in $L^2(E,m)$. It is proved in [38] and [39] that $(S_t)_{t\geq 0}$ and $(S_t^*)_{t\geq 0}$ are also strongly continuous semigroups of compact operators in $L^2(E,m)$. Let L and L^* be the generators of the semigroups $(S_t)_{t\geq 0}$ and $(S_t^*)_{t\geq 0}$, respectively. Denote by $\sigma(L)$ and $\sigma(L^*)$ the spectra of L and L^*, respectively. According to [40, Theorem V.6.6.], $\lambda := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(L^*))$ is a common eigenvalue of multiplicity 1 for both L and L^*. Using the argument in [38], the eigenfunctions ϕ of L and ϕ^* of L^* associated with the eigenvalue λ can be chosen to be strictly positive and continuous everywhere on E. We further normalize ϕ and ϕ^* so that $\langle \phi, \phi \rangle_m = \langle \phi, \phi^* \rangle_m = 1$. Moreover, for each $t \geq 0, x \in E$, we have $S_t\phi(x) = e^{\lambda t}\phi(x)$ and $S_t^*\phi^*(x) = e^{\lambda t}\phi^*(x)$. We call ϕ the *principal eigenfunction* of the mean semigroup $(S_t)_{t\geq 0}$.

Remark 1.7 Note that we do not require the operators $(P_t)_{t\geq 0}$ to be self-adjoint in $L^2(E,m)$, i.e., we do not assume $p(t,x,y) = p(t,y,x)$ for each $x, y \in E$ and $t > 0$. In other word, the spatial motion ξ considered in this paper is not necessarily a symmetric Markov process with respect to the measure m. As a consequence, $(S_t)_{t\geq 0}$ are not necessarily self-adjoint either.

We will use the following function

$$A(x) := 2\alpha(x) + \int_{(0,\infty)} y^2 \pi(x,dy), \quad x \in E$$

in Assumption 3 below.

For all $t \geq 0$ and $x \in E$, it is now clear that $\mathbf{P}_{\delta_x}[X_t(\phi)] = S_t\phi(x) = e^{\lambda t}\phi(x)$. If $\lambda > 0$, the mean of $X_t(\phi)$ will increase exponentially; if $\lambda < 0$, the mean of $X_t(\phi)$ will decrease exponentially; and if $\lambda = 0$, the mean of $X_t(\phi)$ will be a constant. Because of this, we say X is *supercritical, critical* or *subcritical*, according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. In this paper, we are mainly interested in critical superprocesses with finite second moments. So, for the remainder of this paper, we always assume the following:

Assumption 3 (1) The superprocess X is critical, i.e., $\lambda = 0$.
(2) The function $\phi A : x \mapsto \phi(x)A(x)$ is bounded on E.

Assumption 3.(2) is satisfied, for example, when ϕ and A are bounded on E. These conditions appeared in the literature and was used by [38] in the proof of the Kolmogorov type and the Yaglom type results for critical superprocesses.

Denote by $\mathscr{M}_f^\phi$ the collection of all the measures $\mu \in \mathscr{M}_f$ such that $\mu(\phi) \in (0,\infty)$. It will be proved in Proposition 4.2 that $\mathbf{P}_\mu[X_t(\phi)^2] < \infty$ for each $\mu \in \mathscr{M}_f^\phi$ and $t > 0$ provided the function $\phi A : x \mapsto \phi(x)A(x)$ is bounded on E.

Taking $\mu \in \mathscr{M}_f^\phi$, $T \geq 0$ and $g = \phi$ in Definition 1.4.(1), it will be proved in Lemma 3.4 that the family of probability measures $(\mathbb{P}_\mu^{(\phi,T)})_{T\geq 0}$ is consistent, i.e., there exists an E-valued process $\{(\xi_t)_{t\geq 0}; \dot{\mathbb{P}}_\mu\}$ such that

$$\left\{(\xi_t)_{0\leq t\leq T}; \mathbb{P}_\mu^{(\phi,T)}\right\} \overset{\text{f.d.d}}{=} \left\{(\xi_t)_{0\leq t\leq T}; \dot{\mathbb{P}}_\mu\right\}, \quad T \geq 0.$$

The process $\{(\xi_t)_{t\geq 0}; \dot{\mathbb{P}}_\mu\}$ is exactly the spine process in the classical spine decomposition.

It will also be proved in Proposition 4.2 that, under Assumptions 1, 2 and 3, for all $\mu \in \mathscr{M}_f^\phi$ and $T > 0$, we have

$$\mathbb{N}_\mu\big[w_T(\phi)^2\big] = \langle \mu, \phi\rangle \dot{\mathbb{P}}_\mu\Big[\int_0^T (A\phi)(\xi_s)\,ds\Big] \in (0,\infty).$$

As a consequence, $\mathbb{N}_\mu^{w_T(\phi)^2}$—the $w_T(\phi)^2$-transform of $\mathbb{N}_\mu$, and $\ddot{\mathbb{P}}_\mu^{(T)}$—the $(\int_0^T (A\phi)(\xi_s)ds)$-transform of $\dot{\mathbb{P}}_\mu$, are both well defined probability measures. Recall that we always assume that Assumptions 1, 2 and 3 hold.

Definition 1.8 Let $\mu \in \mathscr{M}_f^\phi$ and $T > 0$. We say

$$\big\{(\xi_t)_{0\le t\le T}, \kappa, \big(\xi'_t\big)_{\kappa\le t\le T}, (Y_t)_{0\le t\le T}, \mathbf{n}_T, \big(Y'_t\big)_{\kappa\le t\le T}, \mathbf{n}'_T, \big(X'_t\big)_{\kappa\le t\le T}, (Z_t)_{0\le t\le T}; \ddot{\mathbf{P}}_\mu^{(T)}\big\}$$

is a *2-spine representation* of $\mathbb{N}_\mu^{w_T(\phi)^2}$ if the following are true:

(1) *The main spine* $\{(\xi_t)_{0\le t\le T}; \ddot{\mathbf{P}}_\mu^{(T)}\}$ is a copy of $\{(\xi_t)_{0\le t\le T}; \ddot{\mathbb{P}}_\mu^{(T)}\}$.

(2) Conditioned on $(\xi_t)_{0\le t\le T}$, the *splitting time* κ is a random variable taking values in $[0, T]$ with law

$$\ddot{\mathbf{P}}_\mu^{(T)}\big(\kappa \in ds | (\xi_t)_{0\le t\le T}\big) = \frac{\mathbf{1}_{0\le s\le T}(A\phi)(\xi_s)\,ds}{\int_0^T (A\phi)(\xi_r)\,dr}.$$

(3) Conditioned on $(\xi_t)_{t\le T}$ and κ, the *auxiliary spine* $(\xi'_t)_{\kappa\le t\le T}$ is defined such that

$$\big\{\big(\xi'_{\kappa+t}\big)_{0\le t\le T-\kappa}; \ddot{\mathbf{P}}_\mu^{(T)}(\cdot|\xi,\kappa)\big\} \overset{law}{=} \big\{(\xi_t)_{0\le t\le T-\kappa}; \dot{\mathbb{P}}_{\xi_\kappa}\big\}. \tag{1.13}$$

(4) Write $\mathscr{G} := \sigma\{(\xi_t)_{t\le T}, \kappa, (\xi'_t)_{\kappa\le t\le T}\}$. Conditioned on $\mathscr{G}$, *the main immigration* $(Y_t)_{0\le t\le T}$ is given by

$$Y_t := \int_{(0,t]\times\mathscr{W}} w_{t-s}\mathbf{n}_T(ds, dw), \quad t \in [0, T],$$

where $\mathbf{n}_T$ is a Poisson random measure on $[0, T] \times \mathscr{W}$ with mean measure

$$\mathbf{m}_T^\xi(ds, dw) := 2\alpha(\xi_s)\mathbb{N}_{\xi_s}(dw)\cdot ds + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}(X \in dw)\pi(\xi_s, dy)\cdot ds.$$

(5) Conditioned on $\mathscr{G}$, *the auxiliary immigration* $(Y'_t)_{\kappa\le t\le T}$ is given by

$$Y'_t := \int_{(\kappa,t]\times\mathscr{W}} w_{t-s}\mathbf{n}'_T(ds, dw), \quad t \in [\kappa, T],$$

where $\mathbf{n}'_T$ is a Poisson random measure on $[\kappa, T] \times \mathscr{W}$ with mean measure

$$\mathbf{m}_{\kappa,T}^{\xi'}(ds, dw) := 2\alpha\big(\xi'_s\big)\mathbb{N}_{\xi'_s}(dw)\cdot ds + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi'_s}}(X \in dw)\pi\big(\xi'_s, dy\big)\cdot ds.$$

(6) Conditioned on $\mathscr{G}$, *the splitting-time immigration* $(X'_t)_{\kappa\le t\le T}$ is defined by

$$\big\{\big(X'_{\kappa+t}\big)_{0\le t\le T-\kappa}; \ddot{\mathbf{P}}_\mu(\cdot|\mathscr{G})\big\} \overset{law}{=} \big\{(X_t)_{0\le t\le T-\kappa}; \widetilde{\mathbf{P}}_{\xi_\kappa}\big\},$$

where, for each $x \in E$, the probability measure $\widetilde{\mathbf{P}}_x$ is given by

$$\widetilde{\mathbf{P}}_x(\cdot) := \begin{cases} \frac{2\alpha(x)\mathbf{P}_{\mathbf{0}}(\cdot)+\int_{(0,\infty)} y^2 \mathbf{P}_{y\delta_x}(\cdot)\pi(x,dy)}{2\alpha(x)+\int_{(0,\infty)} y^2\pi(x,dy)}, & \text{if } A(x) > 0, \\ \mathbf{P}_{\mathbf{0}}(\cdot), & \text{if } A(x) = 0. \end{cases} \tag{1.14}$$

(7) Conditioned on $\mathscr{G}$, the main immigration $\{Y, \mathbf{n}_T\}$, the auxiliary immigration $\{Y', \mathbf{n}'_T\}$ and the splitting-time immigration X' are mutually independent. Setting $Y'_t = \mathbf{0}$ and $X'_t = \mathbf{0}$ for each $t \le \kappa$, the *total immigration* $(Z_t)_{0\le t\le T}$ is given by

$$Z_t := Y_t + Y'_t + X'_t, \quad 0 \le t \le T.$$

We are now ready to state our 2-spine decomposition theorem for critical superprocesses:

Theorem 1.9 *Suppose that Assumptions 1, 2 and 3 hold. Let $\mu \in \mathscr{M}_f^{\phi}$ and $T > 0$. Suppose that $\{(\xi_t)_{0\le t\le T}, \kappa, (\xi'_t)_{\kappa\le t\le T}, (Y_t)_{0\le t\le T}, \mathbf{n}_T, (Y'_t)_{\kappa\le t\le T}, \mathbf{n}'_T, (X'_t)_{\kappa\le t\le T}, (Z_t)_{0\le t\le T}; \ddot{\mathbf{P}}_\mu^{(T)}\}$ is a 2-spine representation of $\mathbb{N}_\mu^{w_T(\phi)^2}$. Then $\{(Z_t)_{t\le T}; \ddot{\mathbf{P}}_\mu^{(T)}\} \overset{f.d.d.}{=} \{(w_t)_{t\le T}; \mathbb{N}_\mu^{w_T(\phi)^2}\}$.*

As mentioned earlier in Sect. 1.1, this 2-spine decomposition theorem for superprocesses is an analog of the 2-spine decomposition theorem for Galton-Watson trees in [35], and is closely related to the multi-spine theory appeared in [19, 20, 23], and [1]. Of course, depend on the choice of F, there are many versions of Theorem 1.2. We only consider the cases when $F(w)$ takes the forms of $w_t(g)$ and $w_t(\phi)^2$, because they are sufficient for our purpose to give probabilistic proofs of the Kolmogorov type and Yaglom type results for critical superprocesses.

We now turn our attention to the limit behavior of critical superprocesses. First, we want to consider the asymptotic behavior of $v_t(x) := -\log \mathbf{P}_{\delta_x}(X_t = \mathbf{0})$, where $t > 0$ and $x \in E$. (They are well defined thanks to Assumption 1.) From (1.4) and monotone convergence, we have

$$v_t(x) = \lim_{\theta\to\infty} V_t(\theta \mathbf{1}_E)(x), \quad t > 0,\ x \in E, \tag{1.15}$$

and

$$\mathbf{P}_\mu(X_t = \mathbf{0}) = e^{-\mu(v_t)}, \quad \mu \in \mathscr{M}_f,\ t \ge 0, \tag{1.16}$$

where the operators $(V_t)_{t\ge 0}$ are given by (1.4). We call $(V_t)_{t\ge 0}$ the *cumulant semigroup* of the superprocess X, because it satisfies the semigroup property in the sense that, for all $f \in p\mathscr{B}_E$, $t, s \ge 0$ and $x \in E$, it holds that $V_t V_s f(x) = V_{t+s} f(x)$ (see [30, Theorem 2.21]).

Let ψ_0 be a function on $E \times [0, \infty)$ defined by

$$\psi_0(x, z) := \psi(x, z) + \beta(x)z = \alpha(x)z^2 + \int_{(0,\infty)} \big(e^{-rz} - 1 + rz\big)\pi(x, dr), \quad x \in E,\ z \ge 0.$$

Let Ψ_0 be an operator on $p\mathscr{B}_E$ defined by

$$(\Psi_0 f)(x) := \psi_0\big(x, f(x)\big), \quad f \in p\mathscr{B}_E,\ x \in E.$$

It is known, see [30, Theorem 2.23] for example, that for each $f \in bp\mathscr{B}_E$, $(t, x) \mapsto V_t f(x)$ is the solution of the equation

$$V_t f(x) + \int_0^t (S_{t-s}\Psi_0 V_s f)(x)\, ds = S_t f(x), \quad t \ge 0,\ x \in E. \tag{1.17}$$

Indeed, (1.17) can be obtained from (1.5) using a Feynman–Kac type argument. It is also clear that

$$V_t v_s(x) = -\log \mathbf{P}_{\delta_x}\left[e^{-\langle X_t, \lim_{\theta\to\infty} V_s(\theta \mathbf{1}_E)\rangle}\right] = -\lim_{\theta\to\infty} \log \mathbf{P}_{\delta_x}\left[e^{-\langle X_t, V_s(\theta \mathbf{1}_E)\rangle}\right]$$
$$= -\lim_{\theta\to\infty} V_t V_s(\theta \mathbf{1}_E)(x) = v_{t+s}(x), \quad s,\ t > 0,\ x \in E. \tag{1.18}$$

So, if we allow extended values, it follows from (1.17) and (1.18) that we have the following equation for $(v_t)_{t\geq 0}$:

$$v_{t+s}(x) + \int_0^t (S_{t-r}\Psi_0 v_{r+s})(x)\,dr = S_t v_s(x), \quad x \in E,\ t \geq 0. \tag{1.19}$$

In order to study the asymptotic behavior of $(v_t)_{t\geq 0}$ using (1.19), we need to understand the asymptotic behavior of the mean semigroup $(S_t)_{t\geq 0}$. The following assumption is commonly used for this purpose:

Assumption 2′ In addition to Assumption 2, we further assume that the mean semigroup $(S_t)_{t\geq 0}$ is *intrinsically ultracontractive*, that is, for each $t > 0$ there exists $c_t > 0$ such that for all $x, y \in E$, we have $q(t,x,y) \leq c_t \phi(x)\phi^*(y)$.

The concept of intrinsic ultracontractivity was first introduced by Davies and Simon [7] in the symmetric setting and was extended to the non-symmetric setting in [25]. Assumption 2′ is a pretty strong condition on the mean semigroup $(S_t)_{t\geq 0}$. For instance, it excludes the case of super Brownian motions in the whole space. However, it is satisfied in a lot of cases. For a long list of (symmetric and non-symmetric) Markov processes satisfying Assumption 2′, see [37, 38].

A consequence of this assumption is that (see [25, Theorem 2.7]) there exist constants $c > 0$ and $\gamma > 0$ such that

$$\left|\frac{q(t,x,y)}{\phi(x)\phi^*(y)} - 1\right| \leq c e^{-\gamma t}, \quad x \in E,\ t > 1. \tag{1.20}$$

We will see in Sect. 3.2 that, under Assumption 2, the spine process $\{(\xi_t)_{t\geq 0}; (\dot{\mathbb{P}}_x)_{x\in E}\}$ in the classical spine decomposition is a time homogeneous Markov process with invariant measure $\phi(x)\phi^*(x)m(dx)$. It can be verified that its transition density with respect to measure $\phi(x)\phi^*(x)m(dx)$ is $\frac{q(t,x,y)}{\phi(x)\phi^*(y)}$. Therefore Assumption 2′ implies that the spine process in classical spine decomposition is exponentially ergodic.

Define $\nu(dy) := \phi^*(y)m(dy)$. Under Assumption 2′, $\nu(dy)$ is a finite measure on E. In fact, according to (1.20), for $t > 0$ large enough, there is a $c_t' > 0$ such that $\phi^*(y) \leq q(t,x,y)(c_t')^{-1}\phi^{-1}(x)$, and clearly, the right hand of this inequality is integrable in y with respect to measure m. Therefore, we can consider a superprocess X with initial configuration ν. Under Assumptions 1 and 2′, it will be proved in Lemma 5.2 that the following statements are equivalent:

- $S_t v_s(x) < \infty$ for some $s > 0, t > 0$ and some $x \in E$.
- $\mathbf{P}_\nu(X_t = \mathbf{0}) > 0$ for some $t > 0$.

Note that, in order to take advantage of (1.19), we need $S_t v_s(x)$ to be finite at least for some large $s, t > 0$ and some $x \in E$. Therefore, we also need the following assumption:

Assumption 1′ In addition to Assumption 1, we further assume that $\mathbf{P}_\nu(X_t = \mathbf{0}) > 0$ for some $t > 0$.

We are now ready to state our Kolmogorov type and Yaglom type limit results for superprocesses:

Theorem 1.10 *Suppose that Assumptions 1′, 2′ and 3 hold. Then,*

$$t\mathbf{P}_\mu(X_t \neq \mathbf{0}) \xrightarrow[t\to\infty]{} \frac{\langle \mu, \phi\rangle}{\frac{1}{2}\langle A\phi, \phi\phi^*\rangle_m}, \quad \mu \in \mathscr{M}_f^\phi,$$

where m is the reference measure appeared in Assumption 2.

Theorem 1.11 *Suppose that Assumptions 1′, 2′ and 3 hold. Let $f \in bp\mathscr{B}_E^\phi$ and $\mu \in \mathscr{M}_f^\phi$. Then,*

$$\left\{t^{-1}X_t(f); \mathbf{P}_\mu(\cdot|X_t \neq \mathbf{0})\right\} \xrightarrow[t\to\infty]{law} \frac{1}{2}\langle \phi^*, f\rangle_m \langle \phi A, \phi\phi^*\rangle_m \mathbf{e},$$

where $\mathbf{e}$ is an exponential random variable with mean 1, *and m is the reference measure in Assumption 2.*

As mentioned earlier, our Kolmogorov type and Yaglom type results for critical superprocesses are established under slightly weaker conditions than [38]. We now make this more precise. In [38], the authors considered a (ξ, ψ)-superprocess $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathscr{M}_f}\}$ which also satisfies Assumption 1, 2 and 3.(1) as the basic setting. In addition to that, [38] assumed the following

(a) the transition semigroup (P_t) of the spatial motion is intrinsically ultracontractive,
(b) the principal eigenfunction of (P_t) is bounded,
(c) the function A is bounded, and
(d) there exists $t_0 > 0$ such that $\inf_{x\in E} \mathbf{P}_{\delta_x}(X_{t_0} = \mathbf{0}) > 0$.

It is shown in [38] that, under conditions (a) and (b), the mean semigroup (S_t) is also intrinsically ultracontractive, and the principal eigenfunction ϕ of (S_t) is also bounded. Therefore, conditions (a), (b) and (c) combined together are stronger than our Assumption 1′ and 3. Condition (d) is stronger than our Assumption 2′ because according to (1.16), we always have the following:

$$\mathbf{P}_\nu(X_t = \mathbf{0}) = \exp\left\{-\langle v_t, \nu\rangle\right\} = \exp\left\{\langle \log \mathbf{P}_{\delta_\cdot}(X_t = \mathbf{0}), \nu\rangle\right\}, \quad t > 0.$$

2 Size-Biased Decomposition

2.1 Size-Biased Transform of Poisson Random Measures

In this subsection, we digress briefly from superprocesses and prove the size-biased decomposition theorem for Poisson random measures, i.e., Theorem 1.3. Let $(S, \mathscr{S})$ be a measurable space with a σ-finite measure N. Let $\{\mathbf{N}; P\}$ be a Poisson random measure on $(S, \mathscr{S})$ with mean measure N. Campbell's theorem, see [27, Proof of Theorem 2.7] for example, characterizes the law of $\{\mathbf{N}; P\}$ by its Laplace functionals:

$$P\left[e^{-\mathbf{N}(g)}\right] = e^{-N(1-e^{-g})}, \quad g \in p\mathscr{S}.$$

According to [27, Theorem 2.7], we also have that $P[\mathbf{N}(g)] = N(g)$ for each $g \in \mathscr{S}$ with $N(|g|) < \infty$. By monotonicity, one can verify that

$$P\big[\mathbf{N}(g)\big] = N(g), \quad g \in p\mathscr{S}.$$

Lemma 2.1 *If $g \in L^1(N)$ and $f \in p\mathscr{S}$, then $\mathbf{N}(g)e^{-\mathbf{N}(f)}$ is integrable and*

$$P\big[\mathbf{N}(g)e^{-\mathbf{N}(f)}\big] = P\big[e^{-\mathbf{N}(f)}\big]N\big[ge^{-f}\big]. \tag{2.1}$$

Furthermore, (2.1) *is true for each $g, f \in p\mathscr{S}$ if we allow extended values.*

Proof Since N is a σ-finite measure on $(S, \mathscr{S})$, there exists a strictly positive measurable function h on S such that $N(h) < \infty$. According to [27, Theorem 2.7.], $\mathbf{N}(h)$ has finite mean. For any $g \in bp\mathscr{S}^h := \{g \in p\mathscr{S} : \|h^{-1}g\|_\infty < \infty\}$ and $f \in p\mathscr{S}$, it is clear that $\mathbf{N}(g)$ and $\mathbf{N}(g)e^{-\mathbf{N}(f)}$ are integrable. Therefore, by the dominated convergence theorem, we deduce that

$$\begin{aligned}
P\big[\mathbf{N}(g)e^{-\mathbf{N}(f)}\big] &= P\big[-\partial_\theta|_{\theta=0}e^{-\mathbf{N}(f+\theta g)}\big] = -\partial_\theta|_{\theta=0}P\big[e^{-\mathbf{N}(f+\theta g)}\big] \\
&= -\partial_\theta|_{\theta=0}e^{-N(1-e^{-(f+\theta g)})} = e^{-N(1-e^{-f})}\partial_\theta|_{\theta=0}N\big(1-e^{-(f+\theta g)}\big) \\
&= P\big[e^{-\mathbf{N}(f)}\big]N\big[ge^{-f}\big].
\end{aligned}$$

For any $g \in p\mathscr{S}$ and $s \in S$, define $g^{(n)}(s) := h(s)\min\{h(s)^{-1}g(s), n\}$. Then $(g^{(n)})_{n\in\mathbb{N}}$ is a $bp\mathscr{S}^h$-sequence which increasingly converges to g pointwise. Note that (2.1) is true for each $g^{(n)}$ and f. Letting $n \to \infty$, by monotonicity, we see that if we allow extended values, then (2.1) is true for each $g, f \in p\mathscr{S}$. In the case when $g \in L^1(N)$, we simply consider its positive and negative parts. □

Proof of Theorem 1.3 By Lemma 2.1, it is easy to see that, for any $f \in p\mathscr{S}$,

$$\begin{aligned}
P^{\mathbf{N}(g)}\big[e^{-\mathbf{N}(f)}\big] &= N(g)^{-1}P\big[\mathbf{N}(g)e^{-\mathbf{N}(f)}\big] = N(g)^{-1}P\big[e^{-\mathbf{N}(f)}\big]N\big[ge^{-f}\big] \\
&= P\big[e^{-\mathbf{N}(f)}\big]N^g\big[e^{-f}\big] = (P\otimes Q)\big[e^{-\mathbf{N}(f)-f(\vartheta)}\big] = (P\otimes Q)\big[e^{-(\mathbf{N}+\delta_\vartheta)(f)}\big],
\end{aligned}$$

which completes the proof. □

Lemma 2.2 *For all $g, f \in L^1(N) \cap L^2(N)$, $\mathbf{N}(g)\mathbf{N}(f)$ is integrable and*

$$P\big[\mathbf{N}(g)\mathbf{N}(f)\big] = N(g)N(f) + N(gf). \tag{2.2}$$

Furthermore, (2.2) *is true for all $g, f \in p\mathscr{S}$ if we allow extended values.*

Proof Since N is a σ-finite measure on $(S, \mathscr{S})$, there exists a strictly positive measurable function $\tilde{h}$ on S such that $N(\tilde{h}) < \infty$. Define $h(s) := \min\{\tilde{h}(s), \tilde{h}(s)^{1/2}\}$ for each $s \in S$. It is clear that h is a strictly positive measurable function on S such that $N(h) < \infty$ and $N(h^2) < \infty$. According to [27, Theorem 2.7], $\mathbf{N}(h)$ has finite 1st and 2nd moments. For any $g, f \in bp\mathscr{S}^h := \{g \in p\mathscr{S} : \|h^{-1}g\|_\infty < \infty\}$, it is easy to see that $\mathbf{N}(g), \mathbf{N}(f), \mathbf{N}(f)\mathbf{N}(g)$

are integrable. Thus, using Lemma 2.1 and the dominated convergence theorem, we have

$$\begin{aligned}
P\big[\mathbf{N}(g)\mathbf{N}(f)\big] &= -P\big[\partial_\theta|_{\theta=0}\mathbf{N}(g)e^{-\mathbf{N}(\theta f)}\big] = -\partial_\theta|_{\theta=0}P\big[\mathbf{N}(g)e^{-\mathbf{N}(\theta f)}\big] \\
&= -\partial_\theta|_{\theta=0}P\big[e^{-\mathbf{N}(\theta f)}\big]N\big(ge^{-\theta f}\big) \\
&= -N[g]\partial_\theta|_{\theta=0}P\big[e^{-\mathbf{N}(\theta f)}\big] - \partial_\theta|_{\theta=0}N\big(ge^{-\theta f}\big) \\
&= -N(g)P\big[\partial_\theta|_{\theta=0}e^{-\mathbf{N}(\theta f)}\big] - N\big(\partial_\theta|_{\theta=0}ge^{-\theta f}\big) \\
&= N(g)N(f) + N(gf).
\end{aligned}$$

For any $g, f \in p\mathscr{S}$ and $s \in S$, define $g^{(n)}(s) := h(s)\min\{h(s)^{-1}g(s), n\}$. Then $(g^{(n)})_{n\in\mathbb{N}}$ is a $bp\mathscr{S}^h$-sequence which increasingly converges to g pointwise. Define $f^{(n)}$ similarly. Then from what we have proved, (2.2) is true for $g^{(n)}$ and $f^{(n)}$. Letting $n \to \infty$, by monotonicity, (2.2) is true for each $g, f \in p\mathscr{S}$ if we allow extended values. In the case when $g, f \in L^1(N) \cap L^2(N)$ we simply consider their positive and negative parts. □

2.2 Size-Biased Transform of the Superprocesses

Let $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathscr{M}_f}\}$ be the (ξ, ψ)-superprocess introduced in Sect. 1.2 which satisfies Assumption 1. In this subsection, we will give a proof of Theorem 1.2. Recall that, for any $\mu \in \mathscr{M}_f$, $\{\mathscr{N}; \mathbf{P}_\mu\}$ is a Poisson random measure with mean measure $\mathbb{N}_\mu$, and our (ξ, ψ)-superprocess $(X_t)_{t\geq 0}$ is given by

$$X_0 := \mu; \quad X_t(\cdot) := \mathscr{N}\big[w_t(\cdot)\big], \quad t > 0.$$

For any $T > 0$, we write $(K, f) \in \mathscr{K}_T$ if $f : (s, x) \mapsto f_s(x)$ is a bounded non-negative Borel function on $(0, T] \times E$ and K is an atomic measure on $(0, T]$ with finitely many atoms. For any $(K, f) \in \mathscr{K}_T$ and any $\mathscr{M}_f$-valued process $(Y_t)_{t>0}$, we define the random variable

$$K^f_{(s,T]}(Y) := \int_{(s,T]} Y_{r-s}(f_r)K(dr), \quad s \in [0, T].$$

It is clear that the two $\mathscr{M}_f$-valued processes $(Y_t)_{t>0}$ and $(X_t)_{t>0}$ have same finite-dimensional distributions if and only if

$$\mathbf{E}\big[e^{-K^f_{(0,T]}(X)}\big] = \mathbf{E}\big[e^{-K^f_{(0,T]}(Y)}\big], \quad (K, f) \in \mathscr{K}_T,\ T > 0.$$

Proof of Theorem 1.2 Since $\mathbb{N}_\mu(F) \in (0, \infty)$, it follows from Campbell's formula that $\mathbf{P}_\mu[\mathscr{N}(F)] = \mathbb{N}_\mu(F) \in (0, \infty)$. Therefore, $\mathbf{P}^{\mathscr{N}(F)}_\mu$—the $\mathscr{N}(F)$-transform of $\mathbb{P}_\mu$, and $\mathbb{N}^F_\mu$—the F-transform of $\mathbb{N}_\mu$, are both well defined probability measures. Notice that, under $\mathbf{P}^{\mathscr{N}(F)}_\mu$, $X_0 \overset{\text{a.s.}}{=} \mu$ is deterministic, and so is $X_0 + Y_0$ under $\mathbf{P}_\mu \otimes \mathbf{Q}_\mu$ since $X_0 + Y_0 \overset{\text{a.s.}}{=} \mu$. Therefore, we only have to show that,

$$\big\{(X_t)_{t>0}; \mathbf{P}^{\mathscr{N}(F)}_\mu\big\} \overset{\text{f.d.d.}}{=} \big\{(X_t + Y_t)_{t>0}; \mathbf{P}_\mu \otimes \mathbf{Q}_\mu\big\}.$$

It then immediately follows from Theorem 1.3 that

$$\big\{\mathscr{N}; \mathbf{P}^{\mathscr{N}(F)}_\mu\big\} \overset{law}{=} \{\mathscr{N} + \delta_Y; \mathbf{P}_\mu \otimes \mathbf{Q}_\mu\}.$$

This completes the proof since for any $T > 0$ and $(K, f) \in \mathcal{K}_T$,

$$\mathbf{P}_\mu^{\mathcal{N}(F)}\big[e^{-K_{(0,T]}^f(X)}\big] = \mathbf{P}_\mu^{\mathcal{N}(F)}\big[e^{-\mathcal{N}[K_{(0,T]}^f(w)]}\big] = (\mathbf{P}_\mu \otimes \mathbf{Q}_\mu)\big[e^{-(\mathcal{N}+\delta_Y)[K_{(0,T]}^f(w)]}\big]$$
$$= (\mathbf{P}_\mu \otimes \mathbf{Q}_\mu)\big[e^{-K_{(0,T]}^f(X+Y)}\big]. \qquad \square$$

3 Spine Decomposition of Superprocesses

The classical spine decomposition theorem characterizes the superprocess X after a martingale change of measure, and has been investigated in the literature in different situations, see [12, 13, 31] for example. The martingale that is used for the change of measure is defined by $M_t := e^{-\lambda t} X_t(\phi)$, where ϕ is the principal eigenfunction of the generator of the mean semigroup of X with λ being the corresponding eigenvalue. After this martingale change of measure, the transformed process preserves the Markov property, and thus, to prove the spine decomposition theorem, one only needs to focus on the one-dimensional distribution of the transformed process.

In this section, we generalize this classical result by considering the $X_T(g)$-transform of the superprocess X, where g is a non-negative Borel function on E. If g is not equal to ϕ, the $X_T(g)$-transformed process is typically not a Markov process. So we have to use a different method to develop the theorem. Thanks to Theorem 1.2, we only have to consider the $w_T(g)$-transform of the Kuznetsov measures.

3.1 Spine Decomposition Theorem

Let $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ, ψ)-superprocess introduced in Sect. 1.2 which satisfies Assumption 1. In this subsection, we will give a proof of Theorem 1.5. Recall that $(\mathbb{N}_x)_{x\in E}$ are the Kuznetsov measures defined in Lemma 1.1. We now recall a result from [30] which is useful for calculations related to $(\mathbb{N}_x)_{x\in E}$.

Lemma 3.1 ([30, Theorems 5.15 and 8.23]) *Under Assumption 1, for all $T > 0$ and $(K, f) \in \mathcal{K}_T$, we have*

$$\mathbb{N}_\mu\big[1 - e^{-K_{(s,T]}^f(w)}\big] = \mu(u_s) = -\log \mathbf{P}_\mu\big[e^{-K_{(s,T]}^f(X)}\big], \quad s \in [0, T],\ \mu \in \mathcal{M}_f,$$

where the function $u : (s, x) \mapsto u_s(x)$ on $[0, T] \times E$ is the unique bounded positive solution to the following integral equation:

$$u_s(x) = \mathbb{P}_x\Big[\int_{(s,T]} f_r(\xi_{r-s}) K(dr) - \int_s^T (\Psi u_r)(\xi_{r-s})\, dr\Big], \quad s \in [0, T],\ x \in E.$$

We now prove the following lemmas:

Lemma 3.2 *For all $x \in E$, $T > 0$, $(K, f) \in \mathcal{K}_T$ and $g \in p\mathcal{B}_E$, we have*

$$\mathbb{N}_x\big[w_T(g) e^{-K_{(0,T]}^f(w)}\big] = \mathbb{P}_x\big[g(\xi_T) e^{-\int_0^T \psi'(\xi_s, u_s(\xi_s))\, ds}\big], \tag{3.1}$$

where

$$\psi'(x, z) := \partial_z \psi(x, z) = -\beta(x) + 2\alpha(x) z + \int_{(0,\infty)} \big(1 - e^{-yz}\big) y\pi(x, dy), \quad x \in E,\ z \geq 0,$$

and $u : (s, x) \mapsto u_s(x)$ on $[0, T] \times E$ is defined in Lemma 3.1.

Proof We first prove assertion (3.1) in the case when $g \in bp\mathscr{B}_E$. Throughout this proof, we fix $(K, f) \in \mathscr{K}_T$ and consider $0 \le \theta \le 1$. Define

$$u_s^\theta(x) := \mathbb{N}_x\big[1 - e^{-K^f_{(s,T]}(w) - w_{T-s}(\theta g)}\big], \quad s \ge 0,\ x \in E. \tag{3.2}$$

Let

$$\begin{aligned}\tilde{K}(dr) &:= \mathbf{1}_{0\le r<T} K(dr) + \delta_T(dr),\\ \tilde{f}_r &:= \mathbf{1}_{0\le r<T} f_r + \mathbf{1}_{r=T}\big(K\big(\{T\}\big) f_T + \theta g\big).\end{aligned}$$

Then $(\tilde{K}, \tilde{f}) \in \mathscr{K}_T$ and (3.2) can be rewritten as

$$u_s^\theta(x) := \mathbb{N}_x\big[1 - e^{-\tilde{K}^{\tilde{f}}_{(s,T]}(w)}\big], \quad s \ge 0,\ x \in E.$$

It follows from Lemma 3.1 that, for any $\theta \ge 0$, $(s, x) \mapsto u_s^\theta(x)$ is the unique bounded positive solution to the equation

$$u_s^\theta(x) = \mathbb{P}_x\bigg[\int_{(s,T]} \tilde{f}_r(\xi_{r-s})\tilde{K}(dr) - \int_s^T \big(\Psi u_r^\theta\big)(\xi_{r-s})\, dr\bigg], \quad s \in [0, T],\ x \in E,$$

which is equivalent to

$$u_s^\theta(x) = \mathbb{P}_x\bigg[\int_{(s,T]} f_r(\xi_{r-s})K(dr) + \theta g(\xi_{T-s}) - \int_s^T \big(\Psi u_r^\theta\big)(\xi_{r-s})\, dr\bigg]. \tag{3.3}$$

We claim that $u_s^\theta(x)$ is differentiable in θ at $\theta = 0$. In fact, since

$$\frac{|e^{-K^f_{(s,T]}(w) - w_{T-s}(\theta g)} - e^{-K^f_{(s,T]}(w)}|}{\theta} \le w_{T-s}(g), \quad 0 < \theta \le 1, \tag{3.4}$$

and

$$\mathbb{N}_x\big[w_{T-s}(g)\big] = S_{T-s}g(x) = \mathbb{P}_x\big[e^{\int_0^{T-s}\beta(\xi_r)dr} g(\xi_{T-s})\big] \le e^{T\|\beta\|_\infty}\|g\|_\infty, \tag{3.5}$$

it follows from (3.2) and the dominated convergence theorem that

$$\dot{u}_s(x) := \partial_\theta|_{\theta=0} u_s^\theta(x) = \mathbb{N}_x\big[w_{T-s}(g) e^{-K^f_{(s,T]}(w)}\big] \le e^{T\|\beta\|_\infty}\|g\|_\infty. \tag{3.6}$$

From (3.2), we also have the following upper bound for $u_s^\theta(x)$ with $0 \le \theta \le 1$:

$$\begin{aligned} u_s^\theta(x) &\le \mathbb{N}_x\bigg[\int_{(s,T]} w_{r-s}(f_r)K(dr) + w_{T-s}(\theta g)\bigg]\\ &= \int_{(s,T]} \mathbb{N}_x\big[w_{r-s}(f_r)\big]K(dr) + \mathbb{N}_x\big[w_{T-s}(\theta g)\big]\\ &\le e^{T\|\beta\|_\infty}\big(\|f\|_\infty K\big((0, T]\big) + \|g\|_\infty\big) =: L_0. \end{aligned} \tag{3.7}$$

By elementary analysis, one can verify that, for each $L > 0$, there exists a constant $C_{\psi,L} > 0$ such that for each $x \in E$ and $0 \le z, z_0 \le L$,

$$|\psi(x, z_0) - \psi(x, z)| \le C_{\psi,L}|z - z_0|. \tag{3.8}$$

In fact, one can choose $C_{\psi,L} := \|\beta\|_\infty + 2L\|\alpha\|_\infty + \max\{L, 1\} \sup_{x\in E} \int_{(0,\infty)}(y \wedge y^2)\pi(x, dy)$. This upper bound also implies that

$$\left|\psi'(x,z)\right| \le C_{\psi,L}, \quad x \in E, 0 \le z \le L.$$

Therefore, we can verify that $\mathbb{P}_x[\int_s^T (\Psi u_r^\theta)(\xi_{r-s})\,dr]$ is differentiable in θ at $\theta = 0$. In fact, by (3.8), (3.7), (3.2), (3.4) and (3.5), we have

$$\begin{aligned}\frac{|(\Psi u_r^\theta)(x) - (\Psi u_r^0)(x)|}{\theta} &\le C_{\psi,L_0} \frac{|u_r^\theta(x) - u_r^0(x)|}{\theta}\\ &\le C_{\psi,L_0} \cdot e^{T\|\beta\|_\infty} \|g\|_\infty, \quad 0 \le \theta \le 1.\end{aligned}$$

Therefore, by the bounded convergence theorem, we have

$$\partial_\theta|_{\theta=0}\mathbb{P}_x\left[\int_s^T \big(\Psi u_r^\theta\big)(\xi_{r-s})\,dr\right] = \mathbb{P}_x\left[\int_s^T \psi'\big(\xi_{r-s}, u_r^0(\xi_{r-s})\big)\dot{u}_r(\xi_{r-s})\,dr\right]. \tag{3.9}$$

Now, taking $\partial_\theta|_{\theta=0}$ on the both sides of (3.3), we obtain from (3.9) that

$$\dot{u}_s(x) = \mathbb{P}_x\left[g(\xi_{T-s}) - \int_s^T \psi'\big(\xi_{r-s}, u_r^0(\xi_{r-s})\big)\dot{u}_r(\xi_{r-s})\,dr\right], \quad s \in [0,T],\ x \in E. \tag{3.10}$$

Notice that the function $\dot{u} : (s,x) \mapsto \dot{u}_s(x)$ is bounded on $[0,T] \times E$ by $e^{T\|\beta\|_\infty}\|g\|_\infty$; g is bounded on E by $\|g\|_\infty$; and $\psi'(x, u_r^0(x))$ is bounded on E by C_{ψ,L_0}. These bounds allow us to apply the classical Feynman–Kac formula, see [10, Lemma A.1.5] for example, to equation (3.10) and get that

$$\dot{u}_0(x) = \mathbb{P}_x\big[g(\xi_T)e^{-\int_0^T \psi'(\xi_s, u_s(\xi_s))\,ds}\big]. \tag{3.11}$$

The desired result when $g \in bp\mathscr{B}_E$ then follows from (3.6) and (3.11).

In the case when $g \in p\mathscr{B}_E$, we write $g^{(n)}(x) := \min\{g(x), n\}$ for $x \in E$ and $n \in \mathbb{N}$. Then, from what we have proved, we know that

$$\mathbb{N}_x\big[w_T\big(g^{(n)}\big)e^{-K^f_{(0,T]}(w)}\big] = \mathbb{P}_x\big[g^{(n)}(\xi_T)e^{-\int_0^T \psi'(\xi_s, u_s(\xi_s))\,ds}\big], \quad n \in \mathbb{N}.$$

Letting $n \to \infty$ we complete the proof. □

Lemma 3.3 *Let $T > 0$, $k \in [0,T]$ and $(K, f) \in \mathscr{K}_T$. Let $\mu \in \mathscr{M}_f$ and $g \in p\mathscr{B}_E$ satisfy that $\mu(S_T g) \in (0,\infty)$. Suppose that $\{(\xi_t)_{0\le t\le T}, (Y_t)_{0\le t\le T}, \mathbf{n}_T; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is a spine representation of $\mathbb{N}_\mu^{w_T(g)}$. Then, we have*

$$-\log \dot{\mathbf{P}}_\mu^{(g,T)}\big[e^{-K^f_{(k,T]}(Y)}\big|\xi\big] = \int_k^T \psi_0'\big(\xi_{s-k}, u_s(\xi_{s-k})\big)\,ds, \tag{3.12}$$

where the function u is defined in Lemma 3.1.

Proof Throughout this proof, we denote by $\mathbf{n}_{T-k}$ and $\mathbf{m}^\xi_{T-k}$ the restriction of $\mathbf{n}_T$ and $\mathbf{m}^\xi_T$ on $[0, T-k] \times \mathscr{W}$ respectively. It follows from properties of Poisson random measures that, conditioned on ξ, $\mathbf{n}_{T-k}$ is a Poisson random measure with mean measure $\mathbf{m}^\xi_{T-k}$.

It follows from (1.7) and Fubini's theorem that

$$
\begin{aligned}
K^f_{(k,T]}(Y) &= \int_{(k,T]} Y_{r-k}(f_r)K(dr) \\
&= \int_{(k,T]} K(dr) \int_{(0,r-k]\times\mathscr{M}_f} w_{(r-k)-s}(f_r)\mathbf{n}_T(ds,dw) \\
&= \int_{(0,T-k]\times\mathscr{M}_f} \mathbf{n}_T(ds,dw) \int_{(k+s,T]} w_{r-(k+s)}(f_r)K(dr) \\
&= \int K^f_{(k+s,T]}(w)\mathbf{n}_{T-k}(ds,dw).
\end{aligned} \tag{3.13}
$$

Conditioned on ξ, it follows from Campbell's formula and Lemma 3.1 that

$$
\begin{aligned}
-\log \dot{\mathbf{P}}^{(g,T)}_\mu\big[e^{-K^f_{(k,T]}(Y)}\big|\xi\big] &= -\log \dot{\mathbf{P}}^{(g,T)}_\mu\big[e^{-\int K^f_{(k+s,T]}(w)\mathbf{n}_{T-k}(ds,dw)}\big|\xi\big] \\
&= \int \big(1-e^{-K^f_{(k+s,T]}(w)}\big)\mathbf{m}^\xi_{T-k}(ds,dw) \\
&= \int_0^{T-k}\bigg(2\alpha(\xi_s)\mathbb{N}_{\xi_s}\big[1-e^{-K^f_{(k+s,T]}(w)}\big] \\
&\quad + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}\big[1-e^{-K^f_{(k+s,T]}(X)}\big]\pi(\xi_s,dy)\bigg)\,ds \\
&= \int_0^{T-k}\bigg(2\alpha(\xi_s)u_{k+s}(\xi_s) + \int_{(0,\infty)}\big(1-e^{-yu_{k+s}(\xi_s)}\big)y\pi(\xi_s,dy)\bigg)\,ds \\
&= \int_0^{T-k} \psi_0'\big(\xi_s, u_{s+k}(\xi_s)\big)\,ds = \int_k^T \psi_0'\big(\xi_{s-k}, u_s(\xi_{s-k})\big)\,ds,
\end{aligned}
$$

as desired. □

Proof of Theorem 1.5 We only need to prove that

$$
\big\{(Y_t)_{0<t\le T}; \dot{\mathbf{P}}^{(g,T)}_\mu\big\} \overset{\text{f.d.d.}}{=} \big\{(w_t)_{0<t\le T}; \mathbb{N}^{w_T(g)}_\mu\big\},
$$

since both $\{Y_0; \dot{\mathbf{P}}^{(g,T)}_\mu\}$ and $\{w_0; \mathbb{N}^{w_T(g)}_\mu\}$ are deterministic with common value $\mathbf{0}$. By Lemma 3.2 and 3.3, we have

$$
\begin{aligned}
\mathbb{N}^{w_T(g)}_\mu\big[e^{-K^f_{(0,T]}(w)}\big] &= \mathbb{N}_\mu\big[w_T(g)\big]^{-1}\mathbb{N}_\mu\big[w_T(g)e^{-K^f_{(0,T]}(w)}\big] \\
&= \mu(S_T g)^{-1}\mathbb{P}_\mu\big[g(\xi_T)e^{-\int_0^T \psi'(\xi_s,u_s(\xi_s))\,ds}\big] \\
&= \mathbb{P}^{(g,T)}_\mu\big[e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))\,ds}\big] = \dot{\mathbf{P}}^{(g,T)}_\mu\big[\dot{\mathbf{P}}^{(g,T)}_\mu\big[e^{-K^f_{(0,T]}(Y)}|\xi\big]\big] \\
&= \dot{\mathbf{P}}^{(g,T)}_\mu\big[e^{-K^f_{(0,T]}(Y)}\big].
\end{aligned}
$$

The proof is complete. □

3.2 Classical Spine Decomposition Theorem

Let $X=\{(X_t)_{t\geq 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ,ψ)-superprocess introduced in Sect. 1.2 which satisfies Assumptions 1 and 2. In this subsection, we will recover the classical spine decomposition theorem for X which is developed previously in [12, 13, 31].

It is clear that $\{(e^{-\lambda t}\phi(\xi_t)e^{\int_0^t\beta(\xi_s)\,ds}\mathbf{1}_{t<\zeta})_{t\geq 0};(\mathbb{P}_x)_{x\in E}\}$ is a non-negative martingale. Denote by $\{(\xi_t)_{t\geq 0};(\dot{\mathbb{P}}_x)_{x\in E}\}$ the martingale transform (also known as Doob's h-transform) of $\{(\xi_t)_{t\geq 0};(\mathbb{P}_x)_{x\in E}\}$ via this martingale in the sense that

$$\frac{d\dot{\mathbb{P}}_x|_{\mathscr{F}_t^\xi}}{d\mathbb{P}_x|_{\mathscr{F}_t^\xi}}:=e^{-\lambda t}\frac{\phi(\xi_t)}{\phi(x)}e^{\int_0^t\beta(\xi_s)\,ds}\mathbf{1}_{t<\zeta},\quad x\in E,\ t\geq 0,$$

where $(\mathscr{F}_t^\xi)_{t\geq 0}$ is the natural filtration of the spatial motion ξ. It can be shown that (see [25] for example) $\{(\xi_t)_{t\geq 0};(\dot{\mathbb{P}}_x)_{x\in E}\}$ is a time homogeneous Markov process. Its semigroup is Doob's h-transform of $(S_t)_{t\geq 0}$ with $h=\phi$ and its transition density with respect to the measure m is

$$\dot{q}(t,x,y):=e^{-\lambda t}\frac{\phi(y)}{\phi(x)}q(t,x,y),\quad x,\ y\in E,\ t>0.$$

It can also be verified that $\phi(x)\phi^*(x)m(dx)$ is an invariant measure for $\{(\xi_t)_{t\geq 0};(\dot{\mathbb{P}}_x)_{x\in E}\}$.

Recall that, for each $T>0$, $\mathbb{P}_\mu^{(\phi,T)}$ is defined as the $(e^{\int_0^T\beta(\xi_s)\,ds}\phi(\xi_T)\mathbf{1}_{\zeta<T})$-transform of the measure $\mu\mathbb{P}(\cdot):=\int_E\mathbb{P}_x(\cdot)\mu(dx)$.

Lemma 3.4 *Let $\mu\in\mathcal{M}_f^\phi$. Define a probability measure $\dot{\mathbb{P}}_\mu(\cdot):=\mu(\phi)^{-1}\int_E\phi(x)\dot{\mathbb{P}}_x(\cdot)\times\mu(dx)$. Then, for each $T>0$, we have $\{(\xi_t)_{0\leq t\leq T};\mathbb{P}_\mu^{(\phi,T)}\}\overset{law}{=}\{(\xi_t)_{0\leq t\leq T};\dot{\mathbb{P}}_\mu\}$.*

Proof Let $A\in\mathscr{F}_T^\xi$. Then we have

$$\begin{aligned}\mathbb{P}_\mu^{(\phi,T)}(A)&=\frac{(\mu\mathbb{P})[\mathbf{1}_Ae^{\int_0^T\beta(\xi_s)\,ds}\phi(\xi_T)\mathbf{1}_{T<\zeta}]}{(\mu\mathbb{P})[e^{\int_0^T\beta(\xi_s)\,ds}\phi(\xi_T)\mathbf{1}_{T<\zeta}]}\\&=\mu(\phi)^{-1}(\mu\mathbb{P})\big[\mathbf{1}_Ae^{-\lambda T}e^{\int_0^T\beta(\xi_s)\,ds}\phi(\xi_T)\mathbf{1}_{T<\zeta}\big]\\&=\mu(\phi)^{-1}\int_E\mathbb{P}_x\big[\mathbf{1}_Ae^{-\lambda T}e^{\int_0^T\beta(\xi_s)\,ds}\phi(\xi_T)\mathbf{1}_{T<\zeta}\big]\,\mu(dx)\\&=\mu(\phi)^{-1}\int_E\phi(x)\dot{\mathbb{P}}_x(A)\,\mu(dx)=\dot{\mathbb{P}}_\mu(A).\end{aligned}$$

□

Fix a measure $\mu\in\mathcal{M}_f^\phi$. Define $M_t:=e^{-\lambda t}X_t(\phi)$ for each $t\geq 0$. It is clear that $\{(M_t)_{t\geq 0};\mathbf{P}_\mu\}$ is a non-negative martingale. Let $\{(X_t)_{t\geq 0};\mathbf{P}_\mu^M\}$ be the martingale transform of $\{(X_t)_{t\geq 0};\mathbf{P}_\mu\}$ via this martingale in the sense that

$$\frac{d\mathbf{P}_\mu^M|_{\mathscr{F}_t^X}}{d\mathbf{P}_\mu|_{\mathscr{F}_t^X}}:=\frac{M_t}{\mu(\phi)},\quad t\geq 0.$$

We now give the classical spine decomposition theorem:

Theorem 3.5 (Spine decomposition, [12, 13, 31]) *Suppose that Assumptions 1 and 2 hold. Let $\mu \in \mathscr{M}_f^\phi$. Let the spine immigration $\{(\xi_t)_{t\ge 0}, (Y_t)_{t\ge 0}, \mathbf{n}; \dot{\mathbf{P}}_\mu\}$ be defined as follows*:

(1) *The* spine process $\{(\xi_t)_{t\ge 0}; \dot{\mathbf{P}}_\mu\}$ *is a copy of* $\{(\xi_t)_{t\ge 0}; \dot{\mathbb{P}}_\mu\}$.
(2) *The* immigration process $\{(Y_t)_{t\ge 0}; \dot{\mathbf{P}}_\mu\}$ *is an $\mathscr{M}_f$-valued process given by*

$$Y_t := \int_{(0,t]\times\mathscr{W}} w_{t-s}\mathbf{n}(ds, dw), \quad t \ge 0,$$

where, conditioned on ξ, $\mathbf{n}$ is a Poisson random measure on $[0,\infty)\times\mathscr{W}$ with mean measure

$$\mathbf{m}^\xi(ds, dw) := 2\alpha(\xi_s)\mathbb{N}_{\xi_s}(dw)\cdot ds + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}(X\in dw)\pi(\xi_s, dy)\cdot ds.$$

Then, $\{(X_t)_{t\ge 0}; \mathbf{P}_\mu^M\} \overset{f.d.d.}{=} \{(X_t + Y_t)_{t\ge 0}; \mathbf{P}_\mu \otimes \dot{\mathbf{P}}_\mu\}$.

Proof Fix $T > 0$. We only need to show that

$$\left\{(X_t)_{t\le T}; \mathbf{P}_\mu^M\right\} \overset{\text{f.d.d.}}{=} \left\{(X_t + Y_t)_{t\le T}; \mathbf{P}_\mu \otimes \dot{\mathbf{P}}_\mu\right\}.$$

From Lemma 3.4, we can verify that

$$\left\{(Y_t)_{t\le T}; \dot{\mathbf{P}}_\mu\right\} \overset{\text{f.d.d.}}{=} \left\{(Y_t)_{t\le T}; \dot{\mathbf{P}}_\mu^{(\phi,T)}\right\}. \tag{3.14}$$

Also it follows easily from the definitions of $\mathbf{P}_\mu^M$ and $\mathbf{P}_\mu^{X_T(\phi)}$ that

$$\left\{(X_t)_{t\le T}; \mathbf{P}_\mu^M\right\} \overset{\text{f.d.d.}}{=} \left\{(X_t)_{t\le T}; \mathbf{P}_\mu^{X_T(\phi)}\right\}. \tag{3.15}$$

The desired result then follows from Corollary 1.6. $\square$

Remark 3.6 Lemma 3.4 indicates that $\{(\xi_t)_{0\le t\le T}; \mathbb{P}_\mu^{(\phi,T)}\}$ are consistent. From (3.15) we have that $\{(X_s)_{0\le s\le T}; \mathbf{P}_\mu^{X_T(\phi)}\}$ are consistent. From (3.14) we have that $\{(Y_t)_{t\le T}; \dot{\mathbf{P}}_\mu^{(\phi,T)}\}$ are consistent. According to Theorem 1.5, we have $\{(w_t)_{t\le T}; \mathbb{N}_\mu^{w_T(\phi)}\} \overset{\text{f.d.d}}{=} \{(Y_t)_{t\le T}; \dot{\mathbf{P}}_\mu^{(\phi,T)}\}$ which implies that $\{(w_t)_{t\le T}; \mathbb{N}_\mu^{w_T(\phi)}\}$ are also consistent.

4 2-Spine Decomposition of Critical Superprocesses

4.1 Second Moment Formula

Let $X = \{(X_t)_{t\ge 0}; (\mathbf{P}_\mu)_{\mu\in\mathscr{M}_f}\}$ be the (ξ,ψ)-superprocess introduced in Sect. 1.2 which satisfies Assumptions 1, 2 and 3. In this subsection, we give a second moment formula for superprocesses.

Lemma 4.1 *Suppose that Assumptions 1, 2 and 3 hold. Let $g, f \in bp\mathscr{B}_E^\phi$, $\mu \in \mathscr{M}_f^\phi$ and $t \ge 0$. Suppose that $\{(\xi_s)_{0\le s\le t}, (Y_s)_{0\le s\le t}, \mathbf{n}_t; \dot{\mathbf{P}}_\mu^{(g,t)}\}$ is the spine representation of $\mathbb{N}_\mu^{w_t(g)}$. Then*,

$$\dot{\mathbf{P}}_\mu^{(g,t)}\left[Y_t(f)|\xi\right] = \int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds \le t\|A\phi\|_\infty\|\phi^{-1}f\|_\infty, \quad \dot{\mathbf{P}}_\mu^{(g,t)}\text{-}a.s.$$

Proof Define $G(s,w) := \mathbf{1}_{s\le t} w_{t-s}(f)$ for all $s \ge 0$ and $w \in \mathscr{W}$. Under Assumption 3, it is clear from (1.8) that

$$\begin{aligned}\mathbf{m}_t^{\xi}(G) &= \int_0^t 2\alpha(\xi_s)\mathbb{N}_{\xi_s}\big[w_{t-s}(f)\big]\,ds + \int_0^t ds \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}\big[X_{t-s}(f)\big]\pi(\xi_s,dy)\\ &= \int_0^t 2\alpha(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds + \int_0^t ds \int_{(0,\infty)} y^2\cdot(S_{t-s}f)(\xi_s)\pi(\xi_s,dy)\\ &= \int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds.\end{aligned}$$

Since, conditioned on ξ, $\{\mathbf{n}_t;\dot{\mathbf{P}}_\mu^{(g,t)}\}$ is a Poisson random measure on $[0,t]\times\mathscr{W}$ with mean measure $\mathbf{m}_t^{\xi}$, we conclude from Campbell's theorem that

$$\dot{\mathbf{P}}_\mu^{(g,t)}\big[Y_t(f)|\xi\big] = \dot{\mathbf{P}}_\mu^{(g,t)}\big[\mathbf{n}_t(G)|\xi\big] = \mathbf{m}_t^{\xi}(G) = \int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds, \quad \dot{\mathbf{P}}_\mu^{(g,t)}\text{-a.s.}$$

Noticing that

$$\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds = \int_0^t \big[(A\phi)\phi^{-1}S_{t-s}\big(\phi\cdot\phi^{-1}f\big)\big](\xi_s)\,ds \le t\|A\phi\|_\infty\|\phi^{-1}f\|_\infty,$$

we have our result as desired. $\square$

Proposition 4.2 *Under Assumptions 1, 2 and 3, for all $g, f \in b\mathscr{B}_E^{\phi}$, $\mu\in\mathscr{M}_f^{\phi}$ and $t\ge 0$, we have that $X_t(g)X_t(f)$ is integrable with respect to $\mathbf{P}_\mu$ and*

$$\mathbf{P}_\mu\big[X_t(g)X_t(f)\big] = \langle\mu,S_tg\rangle\langle\mu,S_tf\rangle + \langle\mu,\phi\rangle\dot{\mathbb{P}}_\mu\left[\big(\phi^{-1}g\big)(\xi_t)\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds\right]. \tag{4.1}$$

Proof We first consider the case when $g, f\in bp\mathscr{B}_E^{\phi}$. In this case, the right hand of (4.1) is finite. Actually, by Lemma 4.1, the right side of (4.1) is less than or equal to

$$\begin{aligned}&\langle\mu,S_tg\rangle\langle\mu,S_tf\rangle + \langle\mu,\phi\rangle\dot{\mathbb{P}}_\mu\big[\big(\phi^{-1}g\big)(\xi_t)\big]t\big\|A\phi\|_\infty\|\phi^{-1}f\big\|_\infty\\ &\le \langle\mu,\phi\rangle^2 + \langle\mu,\phi\rangle t\|A\phi\|_\infty\big\|\phi^{-1}g\big\|_\infty\big\|\phi^{-1}f\big\|_\infty < \infty.\end{aligned}$$

We can also assume that $m(g)>0$. Since if $g\in bp\mathscr{B}_E$ with $m(g)=0$, then according to (1.12), (1.6) and Lemma 3.4, we have

$$\begin{aligned}S_tg(x) = \int_E q(t,x,y)g(y)m(dy) &= 0, \quad t>0,\ x\in E,\\ \mathbf{P}_\mu\big[X_t(g)\big] = \mu(S_tg) &= 0, \quad \mu\in\mathscr{M}_f,\ t>0,\\ \dot{\mathbb{P}}_\mu\big[\phi^{-1}g(\xi_t)\big] = \mathbb{P}_\mu^{(\phi,t)}\big[\phi^{-1}g(\xi_t)\big] = \frac{\mu(S_tg)}{\mu(\phi)} &= 0, \quad \mu\in\mathscr{M}_f,\ t>0.\end{aligned}$$

These imply that the both sides of (4.1) are 0.

Now in the case when $g, f \in bp\mathscr{B}_E^{\phi}$ and $m(g) > 0$, from Theorem 1.5 and Lemma 4.1 we know that, for each $x \in E$,

$$\begin{aligned}\mathbb{N}_x^{w_t(g)}\big[w_t(f)\big] &= \dot{\mathbf{P}}_{\delta_x}^{(g,t)}\big[Y_t(f)\big] = \dot{\mathbf{P}}_{\delta_x}^{(g,t)}\big[\dot{\mathbf{P}}_{\delta_x}^{(g,t)}\big[Y_t(f)\big|\xi\big]\big] \\ &= \dot{\mathbf{P}}_{\delta_x}^{(g,t)}\bigg[\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds\bigg] = \mathbb{P}_x^{(g,t)}\bigg[\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds\bigg] \\ &= S_t g(x)^{-1}\mathbb{P}_x\bigg[g(\xi_t)e^{\int_0^t \beta(\xi_s)\,ds}\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds\bigg].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{N}_x\big[w_t(g)w_t(f)\big] &= \mathbb{N}_x\big[w_t(g)\big]\mathbb{N}_x^{w_t(g)}\big[w_t(f)\big] \\ &= \mathbb{P}_x\bigg[g(\xi_t)e^{\int_0^t \beta(\xi_s)\,ds}\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds\bigg] \\ &= \phi(x)\dot{\mathbb{P}}_x\bigg[\big(\phi^{-1}g\big)(\xi_t)\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds\bigg].\end{aligned}$$

Integrating with $\mu \in \mathscr{M}_f^{\phi}$, we have

$$\mathbb{N}_\mu\big[w_t(g)w_t(f)\big] = \langle\mu,\phi\rangle\dot{\mathbb{P}}_\mu\bigg[\big(\phi^{-1}g\big)(\xi_t)\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)\,ds\bigg]. \tag{4.2}$$

It then follows from Lemmas 1.1 and 2.2 that

$$\begin{aligned}\mathbf{P}_\mu\big[X_t(g)X_t(f)\big] &= \mathbb{N}_\mu\big[w_t(g)\big]\mathbb{N}_\mu\big[w_t(f)\big] + \mathbb{N}_\mu\big[w_t(g)w_t(f)\big] \\ &= \langle\mu,S_t g\rangle\langle\mu,S_t f\rangle + \langle\mu,\phi\rangle\dot{\mathbb{P}}_\mu\bigg[\big(\phi^{-1}g\big)(\xi_t)\int_0^t (AS_{t-s}f)(\xi_s)\,ds\bigg]\end{aligned}$$

as desired. For the more general case when $g, f \in b\mathscr{B}_E^{\phi}$, we only need to consider their positive and negative parts. □

4.2 2-Spine Decomposition Theorem

Let $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathscr{M}_f}\}$ be the (ξ,ψ)-superprocess introduced in Sect. 1.2 which satisfies Assumptions 1, 2 and 3. In this subsection, we will prove the 2-spine decomposition theorem for superprocesses, i.e., Theorem 1.9.

First, we give a lemma which says that $\mathbb{N}_\mu^{w_T(\phi)^2}$—the $w_T(\phi)^2$-transform of $\mathbb{N}_\mu$, and $\ddot{\mathbb{P}}_\mu^{(T)}$—the $(\int_0^T (A\phi)(\xi_s)\,ds)$-transform of $\dot{\mathbb{P}}_\mu$, are both well defined probability measures.

Lemma 4.3 $\mathbb{N}_\mu[w_T(\phi)^2] = \mu(\phi)\dot{\mathbb{P}}_\mu[\int_0^T (A\phi)(\xi_s)\,ds] \in (0,\infty)$ *for all* $\mu \in \mathscr{M}_f^{\phi}$ *and* $T > 0$.

Proof According to (4.2), we have

$$\mathbb{N}_\mu\big[w_T(\phi)^2\big] = \mu(\phi)\dot{\mathbb{P}}_\mu\bigg[\int_0^T (A\phi)(\xi_s)\,ds\bigg] \leq \mu(\phi)T\|A\phi\|_\infty < \infty.$$

According to $\mathbb{N}_\mu[w_T(\phi)] = \mu(\phi) > 0$, we must have $\mathbb{N}_\mu[w_T(\phi)^2] > 0$. □

Remark 4.4 Note that $\mathbb{N}_\mu^{w_T(\phi)^2}$ is also the $w_T(\phi)$-transform of $\mathbb{N}_\mu^{w_T(\phi)}$. In fact, the size-biased transforms satisfy the following chain rule: If g, f are non-negative measurable functions on some measure space $(D, \mathscr{F}_D, \mathbf{D})$ with $\mathbf{D}(g) \in (0,\infty)$ and $\mathbf{D}(gf) \in (0,\infty)$. Denoted by $\mathbf{D}^g$ the g-transform of $\mathbf{D}$, then $(\mathbf{D}^g)^f = \mathbf{D}^{gf}$, i.e., the f-transform of $\mathbf{D}^g$ is the gf-transform of $\mathbf{D}$. This is true because it is easy to see that

$$\mathbf{D}^{gf}(ds) := \frac{g(s)f(s)\mathbf{D}(ds)}{\mathbf{D}[gf]} = \frac{f(s)\mathbf{D}^g(ds)}{\mathbf{D}^g[f]} = \left(\mathbf{D}^g\right)^f(ds), \quad s \in S.$$

For each $\mu \in \mathscr{M}_f^\phi$, let the spine immigration $\{(\xi_t)_{t\ge 0}, (Y_t)_{t\ge 0}, \mathbf{n}; \dot{\mathbf{P}}_\mu\}$ be given by Theorem 3.5. We first state a property of $\{Y; \dot{\mathbf{P}}_\mu\}$, which is needed later.

Lemma 4.5 $\dot{\mathbf{P}}_\mu(Y_t = \mathbf{0}) = 0$ *for all* $\mu \in \mathscr{M}_f^\phi$ *and* $t > 0$.

Proof According to Theorem 1.5, we have

$$\dot{\mathbf{P}}_\mu(Y_t = 0) = \mathbb{N}_\mu^{w_t(\phi)}\big(w_t(\phi) = 0\big) = \langle \mu, \phi\rangle^{-1}\mathbb{N}_\mu\big[w_t(\phi)\mathbf{1}_{w_t(\phi)=0}\big] = 0. \qquad \square$$

The proof of Theorem 1.9 relies on the following lemma:

Lemma 4.6 *For any* $\mu \in \mathscr{M}_f^\phi$, $T > 0$ *and* $(K, f) \in \mathscr{K}_T$, *we have*

$$\dot{\mathbf{P}}_\mu\big[Y_T(\phi)e^{-K^f_{(0,T]}(Y)}|\xi\big] = \dot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Y)}|\xi\big]\int_0^T (A\phi)(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}\big[e^{-K^f_{(s,T]}(Y)}\big]\widetilde{\mathbf{P}}_{\xi_s}\big[e^{-K^f_{(s,T]}(X)}\big]\,ds,$$

where $\widetilde{\mathbf{P}}_x$ *is defined by* (1.14) *for each* $x \in E$.

Proof Define $G(s,w) := \mathbf{1}_{s\le T}w_{T-s}(\phi)$ for all $s \ge 0$ and $w \in \mathscr{W}$. Notice that from (3.13), under the probability $\dot{\mathbf{P}}_\mu$, we have $Y_T(\phi) = \mathbf{n}(G)$ and $K^f_{(0,T]}(Y) = \mathbf{n}(K^f_{(s,T]}(w))$. From Lemmas 4.1 and 4.5 we know that

$$0 < \dot{\mathbf{P}}_\mu\big[Y_T(\phi)|\xi\big] < \infty, \quad \dot{\mathbf{P}}_\mu\text{-a.s.}$$

Therefore, we can apply Lemma 2.1 to the conditioned Poisson random measure $\mathbf{n}$, and get

$$\dot{\mathbf{P}}_\mu\big[\mathbf{n}(G)e^{-\mathbf{n}(K^f_{(s,T]}(w))}|\xi\big] = \dot{\mathbf{P}}_\mu\big[e^{-\mathbf{n}(K^f_{(s,T]}(w))}|\xi\big]\mathbf{m}^\xi\big[Ge^{-K^f_{(s,T]}(w)}\big]. \tag{4.3}$$

It is clear from the definitions of $\mathbf{m}^\xi$, $\mathbb{N}^{w_t(\phi)}$ and $\mathbf{P}^M$ that

$$\begin{aligned}\mathbf{m}^\xi\big[Ge^{-K^f_{(s,T]}(w)}\big] &= \int_0^T \Big(2\alpha(\xi_s)\mathbb{N}_{\xi_s}\big[w_{T-s}(\phi)e^{-K^f_{(s,T]}(w)}\big] \\ &\quad + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}\big[X_{T-s}(\phi)e^{-K^f_{(s,T]}(X)}\big]\pi(\xi_s, dy)\Big)\,ds \\ &= \int_0^T \Big(2(\alpha\phi)(\xi_s)\mathbb{N}_{\xi_s}^{w_{T-s}(\phi)}\big[e^{-K^f_{(s,T]}(w)}\big] \\ &\quad + \int_{(0,\infty)} y^2\phi(\xi_s)\mathbf{P}^M_{y\delta_{\xi_s}}\big[e^{-K^f_{(s,T]}(X)}\big]\pi(\xi_s, dy)\Big)\,ds.\end{aligned} \tag{4.4}$$

According to Theorem 1.5, we have

$$\mathbb{N}_x^{w_{T-s}(\phi)}\big[e^{-K^f_{(s,T]}(w)}\big]=\dot{\mathbf{P}}_{\delta_x}\big[e^{-K^f_{(s,T]}(Y)}\big]=\dot{\mathbf{P}}_{\delta_x}\big[e^{-K^f_{(s,T]}(Y)}\big]\mathbf{P}_{\mathbf{0}}\big[e^{-K^f_{(s,T]}(X)}\big], \tag{4.5}$$

where we used the fact that $\mathbf{P}_{\mathbf{0}}(X_t=\mathbf{0},\ \text{for any } t\geq 0)=1$. It follows from Theorem 3.5 that for any $s\in[0,T]$, $x\in E$ and $y\in(0,\infty)$,

$$\mathbf{P}^M_{y\delta_x}\big[e^{-K^f_{(s,T]}(X)}\big]=\dot{\mathbf{P}}_{y\delta_x}\big[e^{-K^f_{(s,T]}(X+Y)}\big]=\dot{\mathbf{P}}_{\delta_x}\big[e^{-K^f_{(s,T]}(Y)}\big]\mathbf{P}_{y\delta_x}\big[e^{-K^f_{(s,T]}(X)}\big]. \tag{4.6}$$

Plugging (4.5) and (4.6) back into (4.4) and rearranging terms, we have that

$$\begin{aligned}
\mathbf{m}^{\xi}\big[Ge^{-K^f_{(s,T]}(w)}\big]&=\int_0^T\Big(2(\alpha\phi)(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}\big[e^{-K^f_{(s,T]}(Y)}\big]\mathbf{P}_{\mathbf{0}}\big[e^{-K^f_{(s,T]}(X)}\big]\\
&\quad+\int_{(0,\infty)}y^2\phi(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}\big[e^{-K^f_s(Y)}\big]\mathbf{P}_{y\delta_{\xi_s}}\big[e^{-K^f_{(s,T]}(X)}\big]\pi(\xi_s,dy)\Big)\,ds.\\
&=\int_0^T\phi(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}\big[e^{-K^f_{(s,T]}(Y)}\big]\\
&\quad\times\Big(2\alpha(\xi_s)\mathbf{P}_{\mathbf{0}}\big[e^{-K^f_{(s,T]}(X)}\big]+\int_{(0,\infty)}y^2\mathbf{P}_{y\delta_{\xi_s}}\big[e^{-K^f_{(s,T]}(X)}\big]\pi(\xi_s,dy)\Big)\,ds\\
&=\int_0^T(A\phi)(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}\big[e^{-K^f_{(s,T]}(Y)}\big]\widetilde{\mathbf{P}}_{\xi_s}\big[e^{-K^f_{(s,T]}(X)}\big]\,ds.
\end{aligned} \tag{4.7}$$

Plugging (4.7) back into (4.3), we get the desired result. □

Proof of Theorem 1.9 Note that $\{Z_0;\ddot{\mathbf{P}}^{(T)}_\mu\}$ and $\{w_0;\mathbb{N}_\mu^{w_T(\phi)^2}\}$ are both deterministic with common value $\mathbf{0}$. So we only have to proof $\{(Z_t)_{0<t\leq T};\ddot{\mathbf{P}}^{(T)}_\mu\}\overset{\text{f.d.d.}}{=}\{(w_t)_{0<t\leq T};\mathbb{N}_\mu^{w_T(\phi)^2}\}$. In order to show this, according to Theorem 1.5 and Remark 4.4, we only need to show that $\{(Z_t)_{0<t\leq T};\ddot{\mathbf{P}}^{(T)}_\mu\}$ is the $Y_T(\phi)$-transform of process $\{(Y_t)_{0<t\leq T};\dot{\mathbf{P}}_\mu\}$.

Let $(K,f)\in\mathscr{K}_T$. Similar to (3.13), we have $K^f_{(r,T]}(Y)=\mathbf{n}_T[K^f_{(r+\cdot,T]}]$ and $K^f_{(r,T]}(Y')=\mathbf{n}'_T[K^f_{(r+\cdot,T]}]$ for each $r\leq T$. Therefore, using Campbell's theorem and an argument similar to that used in the proof of Lemma 3.3, one can verify that

$$-\log\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Y)}\big|\mathscr{G}\big]=\int_0^T\psi_0'\big(\xi_s,u_s(\xi_s)\big)\,ds \tag{4.8}$$

and

$$-\log\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Y')}\big|\mathscr{G}\big]=\int_\kappa^T\psi_0'\big(\xi'_s,u_s(\xi'_s)\big)\,ds, \tag{4.9}$$

where $u:(s,x)\mapsto u_s(x)$ is the function on $[0,T]\times E$ defined in Lemma 3.1. It is then clear from (4.9), (1.13) and Lemma 3.3 that

$$\begin{aligned}
\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Y')}\big|\xi,\kappa\big]&=\ddot{\mathbf{P}}_\mu\big[e^{-\int_\kappa^T\psi_0'(\xi'_s,u_s(\xi'_s))\,ds}\big|\xi,\kappa\big]\\
&=\mathbb{P}_{\xi_r}\big[e^{-\int_r^T\psi_0'(\xi_{s-r},u_s(\xi_{s-r}))\,ds}\big]\big|_{r=\kappa}=\dot{\mathbf{P}}_{\delta_{\xi_r}}\big[e^{-K^f_{(r,T]}(Y)}\big]\big|_{r=\kappa}.
\end{aligned} \tag{4.10}$$

By the construction of the splitting immigration X' at time κ, we also have

$$\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(X')}\big|\mathscr{G}\big] = \widetilde{\mathbf{P}}_{\xi_r}\big[e^{-K^f_{(r,T]}(X)}\big]\big|_{r=\kappa}. \tag{4.11}$$

Using (4.8), (4.10), (4.11) and the construction of the 2-spine immigration, we deduce that

$$\begin{aligned}\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Z)}\big|\xi,\kappa\big] &= \ddot{\mathbf{P}}_\mu\big[\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Z)}\big|\mathscr{G}\big]\big|\xi,\kappa\big]\\ &= \ddot{\mathbf{P}}_\mu\big[\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Y)}\big|\mathscr{G}\big]\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Y')}\big|\mathscr{G}\big]\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(X')}\big|\mathscr{G}\big]\big|\xi,\kappa\big]\\ &= e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))\,ds}\dot{\mathbf{P}}_{\delta_{\xi_r}}\big[e^{-K^f_{(r,T]}(Y)}\big]\widetilde{\mathbf{P}}_{\xi_r}\big[e^{-K^f_{(r,T]}(X)}\big]\big|_{r=\kappa}.\end{aligned}$$

Therefore, from the conditioned law of κ given ξ, we have

$$\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Z)}|\xi\big] = \frac{e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))\,ds}}{\int_0^T (A\phi)(\xi_r)\,dr}\int_0^T (A\phi)(\xi_r)\dot{\mathbf{P}}_{\delta_{\xi_r}}\big[e^{-K^f_{(r,T]}(Y)}\big]\widetilde{\mathbf{P}}_{\xi_r}\big[e^{-K^f_{(r,T]}(X)}\big]\,dr. \tag{4.12}$$

Taking expectation, we get that

$$\begin{aligned}\ddot{\mathbf{P}}_\mu\big[e^{-K^f_{(0,T]}(Z)}\big] &\overset{(4.12)}{=} \ddot{\mathbb{P}}^{(T)}_\mu\left\{\frac{e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))\,ds}}{\int_0^T (A\phi)(\xi_r)\,dr}\int_0^T (A\phi)(\xi_r)\dot{\mathbf{P}}_{\delta_{\xi_r}}\big[e^{-K^f_{(r,T]}(Y)}\big]\widetilde{\mathbf{P}}_{\xi_r}\big[e^{-K^f_{(r,T]}(X)}\big]\,dr\right\}\\ &= \dot{\mathbb{P}}_\mu\left\{\frac{e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))\,ds}}{\dot{\mathbb{P}}_\mu[\int_0^T (A\phi)(\xi_r)\,dr]}\int_0^T (A\phi)(\xi_r)\dot{\mathbf{P}}_{\delta_{\xi_r}}\big[e^{K^f_{(r,T]}(Y)}\big]\widetilde{\mathbf{P}}_{\xi_r}\big[e^{-K^f_{(r,T]}(X)}\big]\,dr\right\}\\ &\overset{(3.12)}{=} \dot{\mathbf{P}}_\mu\left\{\frac{\dot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Y)}|\xi]}{\dot{\mathbf{P}}_\mu[\int_0^T (A\phi)(\xi_r)\,dr]}\int_0^T (A\phi)(\xi_r)\dot{\mathbf{P}}_{\delta_{\xi_r}}\big[e^{-K^f_{(r,T]}(Y)}\big]\widetilde{\mathbf{P}}_{\xi_r}\big[e^{-K^f_{(r,T]}(X)}\big]\,dr\right\}\\ &\overset{\text{Lemma 4.6}}{=} \dot{\mathbf{P}}_\mu\left\{\frac{\dot{\mathbf{P}}_\mu[Y_T(\phi)e^{-K^f_{(0,T]}(Y)}|\xi]}{\dot{\mathbf{P}}_\mu[Y_T(\phi)]}\right\} = \frac{\dot{\mathbf{P}}_\mu[Y_T(\phi)e^{-K^f_{(0,T]}(Y)}]}{\dot{\mathbf{P}}_\mu[Y_T(\phi)]},\end{aligned}$$

where in the second equality we used the definition of $\ddot{\mathbb{P}}^{(T)}_\mu$. The display above says that $(Z_t)_{0<t\le T}$ is the $Y_T(\phi)$-transform of the process $\{(Y_t)_{0<t\le T};\dot{\mathbf{P}}_\mu\}$, as desired. □

5 The Asymptotic Behavior of Critical Superprocesses

5.1 Intrinsic Ultracontractivity

Let $\{(X_t)_{t\ge 0};(\mathbf{P}_\mu)_{\mu\in\mathscr{M}_f}\}$ be the (ξ,ψ)-superprocess introduced in Sect. 1.2 which satisfies Assumptions 1 and 2′. In this subsection, we give some more results related to intrinsic ultracontractivity.

Lemma 5.1 *Suppose that $F(x,u,t)$ is a bounded Borel function on $E\times[0,1]\times[0,\infty)$ such that $F(x,u) := \lim_{t\to\infty} F(x,u,t)$ exists for all $x\in E$ and $u\in[0,1]$. Then we have,*

$$\int_0^1 F(\xi_{ut},u,t)\,du \xrightarrow[t\to\infty]{L^2(\dot{\mathbb{P}}_x)} \int_0^1 \big\langle F(\cdot,u),\phi\phi^*\big\rangle_m\,du, \quad x\in E.$$

Proof We first show that

$$\dot{\mathbb{P}}_x\big[F(\xi_{ut},u,t)\big] \xrightarrow[t\to\infty]{} \big\langle F(\cdot,u),\phi\phi^*\big\rangle_m, \quad x\in E,\ u\in(0,1). \tag{5.1}$$

In fact,

$$\dot{\mathbb{P}}_x\big[F(\xi_{ut},u,t)\big] = \int_E \frac{\dot{q}(ut,x,y)}{(\phi\phi^*)(y)} F(y,u,t)\big(\phi\phi^*\big)(y)m(dy).$$

Note that $\int_\cdot (\phi\phi^*)(y)m(dy)$ is a finite measure, $(y,t)\mapsto \frac{\dot{q}(ut,x,y)}{(\phi\phi^*)(y)}F(y,u,t)$ is bounded by $(1+ce^{-\gamma ut})\|F\|_\infty$ for $t>u^{-1}$, and $\frac{\dot{q}(ut,x,y)}{(\phi\phi^*)(y)}F(y,u,t) \xrightarrow[t\to\infty]{} F(y,u)$. Using the bounded convergence theorem, we get (5.1). By Fubini's theorem,

$$\dot{\mathbb{P}}_x\bigg[\int_0^1 F(\xi_{ut},u,t)\,du\bigg] = \int_0^1 \dot{\mathbb{P}}_x\big[F(\xi_{ut},u,t)\big]\,du, \quad x\in E.$$

Since $\dot{\mathbb{P}}_x[F(\xi_{ut},u,t)]$ is bounded by $\|F\|_\infty$ and $\dot{\mathbb{P}}_x[F(\xi_{ut},u,t)] \xrightarrow[t\to\infty]{} \langle F(\cdot,u),\phi\phi^*\rangle_m$, by the bounded convergence theorem, we get

$$\dot{\mathbb{P}}_x\bigg[\int_0^1 F(\xi_{ut},u,t)\,du\bigg] \xrightarrow[t\to\infty]{} c_F := \int_0^1 \big\langle F(\cdot,u),\phi\phi^*\big\rangle_m\,du.$$

Using (1.20) and a similar argument, one can verify that for any $0<u<v\le 1$,

$$\begin{aligned}
&\dot{\mathbb{P}}_x\big[F(\xi_{ut},u,t)F(\xi_{vt},v,t)\big]\\
&= \int_E\int_E \dot{q}(ut,x,y)\dot{q}\big((v-u)t,y,z\big)F(y,u,t)F(z,v,t)m(dy)m(dz)\\
&\xrightarrow[t\to\infty]{} \big\langle F(\cdot,u),\phi\phi^*\big\rangle_m\big\langle F(\cdot,v),\phi\phi^*\big\rangle_m.
\end{aligned}$$

The above convergence is also true for $0<v<u\le 1$ since the limit is symmetric in u and v. We have again, by Fubini's theorem and the bounded convergence theorem,

$$\dot{\mathbb{P}}_x\bigg[\bigg(\int_0^1 F(\xi_{ut},u,t)\,du\bigg)^2\bigg] = \int_0^1 du\int_0^1 \dot{\mathbb{P}}_x\big[F(\xi_{ut},u,t)F(\xi_{vt},v,t)\big]dv \xrightarrow[t\to\infty]{} c_F^2.$$

Finally, we have

$$\begin{aligned}
&\dot{\mathbb{P}}_x\bigg[\bigg(\int_0^1 F(\xi_{ut},u,t)\,du - c_F\bigg)^2\bigg]\\
&= \dot{\mathbb{P}}_x\bigg[\bigg(\int_0^1 F(\xi_{ut},u,t)\,du\bigg)^2\bigg] - 2c_F\dot{\mathbb{P}}_x\bigg[\int_0^1 F(\xi_{ut},u,t)\,du\bigg] + c_F^2\\
&\xrightarrow[t\to\infty]{} 0,
\end{aligned}$$

as desired. □

As mentioned earlier in Sect. 1.2, in order to study the asymptotic behavior of $(v_t)_{t\ge 0}$ and take advantage of (1.19), we need $S_t v_s(x)$ to be finite at least for some large $s,t>0$ and for some $x\in E$. The following lemma addresses this need.

Lemma 5.2 *Under Assumption 1 and 2′, the following statements are equivalent*:

(1) $S_t v_s(x) < \infty$ *for some* $s > 0, t > 0$ *and* $x \in E$.
(1′) *There is an* $s_0 > 0$ *such that for any* $s \ge s_0$, $t > 0$ *and* $x \in E$, *we have* $S_t v_s(x) < \infty$.
(2) $\langle v_s, \phi^* \rangle_m < \infty$ *for some* $s > 0$.
(2′) *There is an* $s_0 > 0$ *such that for any* $s \ge s_0$, *we have* $\langle v_s, \phi^* \rangle_m < \infty$.
(3) *There is an* $s_0 > 0$ *such that for any* $s \ge s_0$, *we have* $v_s \in bp\mathscr{B}_E^{\phi}$.
(4) $\mathbf{P}_\nu(X_t = \mathbf{0}) > 0$ *for some* $t > 0$.
(5) $\phi^{-1} v_t$ *converges to* 0 *uniformly when* $t \to \infty$.
(6) *For any* $\mu \in \mathscr{M}_f^{\phi}$, $\mathbf{P}_\mu(\exists t > 0,\ s.t.\ X_t = \mathbf{0}) = 1$.

Proof We first give some estimates. In this proof, we allow the extended value $+\infty$. According to (1.16) and the fact that $\mathbf{0}$ is an absorption state of the superprocess X, we have

$$\begin{aligned}\langle v_{s_0}, \phi^* \rangle_m &= -\log \mathbf{P}_\nu(X_{s_0} = \mathbf{0}) \\ &\ge -\log \mathbf{P}_\nu(X_s = \mathbf{0}) = \langle v_s, \phi^* \rangle_m, \quad 0 < s_0 \le s.\end{aligned} \tag{5.2}$$

According to Assumption 2′, we have for each $t \ge 0$, there is a $c_t > 0$ such that $q(t,x,y) \le c_t \phi(x)\phi^*(y)$. Using an argument similar to that of [25, Proposition 2.5], we have for each $t \ge 0$, there is a $c_t' < 0$ such that $q(t,x,y) \ge c_t' \phi(x)\phi^*(y)$. Therefore, we have

$$\phi(x)\langle v_s, \phi^* \rangle_m c_t' \le S_t v_s(x) \le \phi(x)\langle v_s, \phi^* \rangle_m c_t, \quad s > 0, t > 0, x \in E. \tag{5.3}$$

Let $c, \gamma > 0$ be the constants in (1.20). Notice that ϕ is strictly positive, using (1.17), one can verify that

$$\frac{V_t f(x)}{\phi(x)} \le \frac{S_t f(x)}{\phi(x)} \le \left(1 + ce^{-\gamma t}\right)\langle f, \phi^* \rangle, \quad f \in bp\mathscr{B}_E, x \in E, t > 1. \tag{5.4}$$

Taking $f = V_s(\theta \mathbf{1}_E)$ in (5.4) and letting $\theta \to \infty$, by (1.15) and (1.18), we have that,

$$\frac{v_{t+s}(x)}{\phi(x)} \le \left(1 + ce^{-\gamma t}\right)\langle v_s, \phi^* \rangle_m, \quad x \in E, s > 0, t > 1. \tag{5.5}$$

We can also verify that

$$S_t v_s(x) \le \|\phi^{-1} v_s\|_\infty S_t \phi(x) = \|\phi^{-1} v_s\|_\infty \phi(x) \quad s, t > 0, x \in E. \tag{5.6}$$

Now, we are ready to give the proof of this lemma using the following steps: (1′) ⇒ (1) ⇒ (2) ⇒ (2′) ⇒ (3) ⇒ (1′) and (2) ⇒ (5) ⇒ (6) ⇒ (4) ⇒ (2). In fact, it is obvious that (1′) ⇒ (1). For (1) ⇒ (2) we use (5.3). For (2) ⇒ (2′) we use (5.2). For (2′) ⇒ (3) we use (5.5). For (3) ⇒ (1′) we use (5.6).

For (2) ⇒ (5), we follow the argument in [38, Lemma 3.3]. Note that, from what we have proved, (2) is equivalent to (1), (1′), (2′) and (3). Integrating (1.17) with respect to the measure ν, by Fubini's theorem and monotonicity, we have that, for any $f \in p\mathscr{B}_E$ and $t \ge 0$,

$$\begin{aligned}\langle f, \phi^* \rangle_m &= \langle f, S_t^* \phi^* \rangle_m = \langle S_t f, \phi^* \rangle_m \\ &= \langle V_t f, \phi^* \rangle_m + \int_0^t \langle S_{t-r} \Psi_0 V_r f, \phi^* \rangle_m \, dr \\ &= \langle V_t f, \phi^* \rangle_m + \int_0^t \langle \Psi_0 V_r f, \phi^* \rangle_m \, dr.\end{aligned} \tag{5.7}$$

Define

$$v(x) := \lim_{t\to\infty} v_t(x) = \lim_{t\to\infty}\left(-\log \mathbf{P}_{\delta_x}(X_t = \mathbf{0})\right) = -\log \mathbf{P}_{\delta_x}(\exists t > 0,\ \text{s.t. } X_t = \mathbf{0}).$$

Since $v_t(x) = -\log \mathbf{P}_{\delta_x}(X_t = \mathbf{0})$ is non-increasing in t, and by (3), we know that $v_t \in bp\mathscr{B}_E^{\phi}$ for t large enough. Therefore, we have $v \in bp\mathscr{B}_E^{\phi} \subset L^2(E, m)$. Taking $f = V_s(\theta \mathbf{1}_E)$ in (5.7) and letting $\theta \to \infty$, by monotonicity and (2′), we have that, there is an $s_0 > 0$ such that

$$\int_0^t \left\langle \Psi_0 v_{r+s}, \phi^*\right\rangle_m dr = \left\langle v_s, \phi^*\right\rangle_m - \left\langle v_{t+s}, \phi^*\right\rangle_m, \quad s \ge s_0,\ t \ge 0. \tag{5.8}$$

Letting $s \to \infty$, by monotonicity, we have

$$\int_0^t \left\langle \Psi_0 v, \phi^*\right\rangle_m dr = t\left\langle \Psi_0 v, \phi^*\right\rangle_m = \left\langle v, \phi^*\right\rangle_m - \left\langle v, \phi^*\right\rangle_m = 0.$$

Since ϕ^* is strictly positive on E, we must have $\Psi_0(v) = 0$, m-a.e.. This, with (1.9), implies that $S_t\Psi_0(v) \equiv 0$ for any $t > 0$. By (1′), we know that $S_t v_s(x)$ take finite value for s large enough. Letting $s \to \infty$ in the (1.19), by monotonicity, we have

$$v(x) = S_t v(x) - \int_0^t S_{t-r}\Psi_0(v)(x)\, dr = S_t v(x), \quad x \in E, t \ge 0,$$

which says that the non-negative function v, if not identically 0, is an eigenfunction of L corresponding to $\lambda = 0$, where L is the generator of the semigroups $(S_t)_{t\ge 0}$. Since $v \in L^2(E, m)$, by the uniqueness of the eigenfunction in $L^2(E, m)$ corresponding to $\lambda = 0$, there is a constant $c \in \mathbb{R}$, such that $v(x) = c\phi(x)$ for all $x \in E$. So with $\Psi_0(v) \equiv 0, m$-a.e., we must have $v \equiv 0$. Using the fact that $v_t(x)$ converges to 0 pointwise, by monotonicity and (5.5), we can verify the desired result (5).

For (5) ⇒ (6), note that, by the definition of v_t, for any $\mu \in \mathscr{M}_f^{\phi}$, we have

$$-\log \mathbf{P}_\mu\{\exists t > 0,\ \text{s.t. } X_t = 0\} = \lim_{t\to\infty}\left(-\log \mathbf{P}_\mu(X_t = \mathbf{0})\right) = \lim_{t\to\infty}\langle \mu, v_t\rangle = 0.$$

Finally, note that (6) ⇒ (4) and (4) ⇒ (2) are obvious. □

5.2 Kolmogorov Type Result

Let $\{(X_t)_{t\ge 0}; (\mathbf{P}_\mu)_{\mu\in\mathscr{M}_f}\}$ be the (ξ, ψ)-superprocess introduced in Sect. 1.2 which satisfies Assumptions 1′ and 2′ and 3. In this subsection, we will give a proof of Theorem 1.10. Thanks to Lemma 5.2, we know that each of the statements in 5.2 is true. In particular, $v_t(x)/\phi(x)$ converges to 0 uniformly in $x \in E$.

Lemma 5.3 *Under Assumptions* 1′, 2′ *and* 3, *we have*

$$\sup_{x\in E}\left|\frac{v_t(x)}{\langle v_t, \phi^*\rangle_m \phi(x)} - 1\right| \xrightarrow[t\to\infty]{} 0.$$

Proof We use an argument similar to that used in [34] for critical branching diffusions. Fix a non-trivial $\mu \in \mathscr{M}_f^{\phi}$, and let the spine immigration $\{(\xi_t)_{t\ge 0}, (Y_t)_{t\ge 0}, \mathbf{n}; \dot{\mathbf{P}}_\mu\}$ be given by

Theorem 3.5. For any $t>0$, we have

$$\begin{aligned}
&\langle\mu,\phi\rangle\dot{\mathbf{P}}_\mu\big[\big(Y_t(\phi)\big)^{-1}\big]\\
&\overset{(3.14)}{=}\langle\mu,\phi\rangle\mathbf{P}_\mu^{(\phi,T)}\big[\big(Y_t(\phi)\big)^{-1}\big]\\
&\overset{\text{Theorem 1.5}}{=}\langle\mu,\phi\rangle\mathbb{N}_\mu^{w_t(\phi)}\big[\big(w_t(\phi)\big)^{-1}\big]=\mathbb{N}_\mu\big\{w_t(\phi)>0\big\}=\lim_{\lambda\to\infty}\mathbb{N}_\mu\big[1-e^{-\lambda w_t(\phi)}\big]\\
&\overset{\text{Campbell's formula}}{=}\lim_{\lambda\to\infty}\big(-\log\mathbf{P}_\mu\big[e^{-\lambda X_t(\phi)}\big]\big)=-\log\mathbf{P}_\mu\{X_t=\mathbf{0}\}\\
&\overset{(1.16)}{=}\langle\mu,v_t\rangle.
\end{aligned}\tag{5.9}$$

Taking $\mu=\delta_x$ in (5.9), we get $v_t(x)/\phi(x)=\dot{\mathbf{P}}_{\delta_x}[(Y_t(\phi))^{-1}]$. Taking $\mu=\nu$, we get $\langle v_t,\phi^*\rangle_m=\dot{\mathbf{P}}_\nu[(Y_t(\phi))^{-1}]$. Therefore, to complete the proof, we only need to show that

$$\sup_{x\in E}\left|\frac{\dot{\mathbf{P}}_{\delta_x}[(Y_t(\phi))^{-1}]}{\dot{\mathbf{P}}_\nu[(Y_t(\phi))^{-1}]}-1\right|\xrightarrow[t\to\infty]{}0.$$

For any Borel subset $G\subset(0,t]$, define

$$Y_t^G:=\int_{G\times\mathscr{W}}w_{t-s}\mathbf{n}(ds,dw).$$

Then we have the following decomposition of Y:

$$Y_t=Y_t^{(0,t_0]}+Y_t^{(t_0,t]},\quad 0<t_0<t<\infty.\tag{5.10}$$

It is easy to see, from the construction and the Markov property of the spine immigration $\{Y,\xi;\dot{\mathbf{P}}\}$, that for any $0<t_0<t<\infty$,

$$\dot{\mathbf{P}}\big[\big(Y_t^{(t_0,t]}(\phi)\big)^{-1}\big|\mathscr{F}_{t_0}^\xi\big]=\dot{\mathbf{P}}_{\delta_{\xi_{t_0}}}\big[\big(Y_{t-t_0}(\phi)\big)^{-1}\big]=\big(\phi^{-1}v_{t-t_0}\big)(\xi_{t_0}).$$

Therefore, we have

$$\dot{\mathbf{P}}_\nu\big[\big(Y_t^{(t_0,t]}(\phi)\big)^{-1}\big]=\dot{\mathbb{P}}_\nu\big[\big(\phi^{-1}v_{t-t_0}\big)(\xi_{t_0})\big]=\big\langle v_{t-t_0},\phi^*\big\rangle_m$$

and

$$\dot{\mathbf{P}}_{\delta_x}\big[\big(Y_t^{(t_0,t]}(\phi)\big)^{-1}\big]=\dot{\mathbb{P}}_x\big[\big(\phi^{-1}v_{t-t_0}\big)(\xi_{t_0})\big]=\int_E\dot{q}(t_0,x,y)\big(\phi^{-1}v_{t-t_0}\big)(y)m(dy).\tag{5.11}$$

By the decomposition (5.10), we have

$$\begin{aligned}
\phi^{-1}v_t(x)=\dot{\mathbf{P}}_{\delta_x}\big[\big(Y_t(\phi)\big)^{-1}\big]&\\
&=\dot{\mathbf{P}}_\nu\big[\big(Y_t^{(t_0,t]}(\phi)\big)^{-1}\big]+\big(\dot{\mathbf{P}}_{\delta_x}\big[\big(Y_t^{(t_0,t]}(\phi)\big)^{-1}\big]-\dot{\mathbf{P}}_\nu\big[\big(Y_t^{(t_0,t]}(\phi)\big)^{-1}\big]\big)\\
&\quad+\big(\dot{\mathbf{P}}_{\delta_x}\big[\big(Y_t(\phi)\big)^{-1}-\big(Y_t^{(t_0,t]}(\phi)\big)^{-1}\big]\big)\\
&=:\big\langle v_{t-t_0},\phi^*\big\rangle_m+\epsilon_x^1(t_0,t)+\epsilon_x^2(t_0,t).
\end{aligned}\tag{5.12}$$

Suppose that $t_0 > 1$, and let $c, \gamma > 0$ be the constants in (1.20), we have

$$
\begin{aligned}
\left|\epsilon_x^1(t_0,t)\right| &= \left|\dot{\mathbf{P}}_{\delta_x}\left[\left(Y_t^{(t_0,t]}(\phi)\right)^{-1}\right] - \dot{\mathbf{P}}_\nu\left[\left(Y_t^{(t_0,t]}(\phi)\right)^{-1}\right]\right| \\
&= \left|\int_E \dot{q}(t_0,x,y)\left(\phi^{-1}v_{t-t_0}\right)(y)m(dy) - \left\langle v_{t-t_0},\phi^*\right\rangle_m\right| \\
&\le \int_{y\in E}\left|\dot{q}(t_0,x,y) - \left(\phi\phi^*\right)(y)\right|\left(\phi^{-1}v_{t-t_0}\right)(y)m(dy) \\
&\le ce^{-\gamma t_0}\left\langle v_{t-t_0},\phi^*\right\rangle_m.
\end{aligned}
\tag{5.13}
$$

We also have

$$
\begin{aligned}
\left|\epsilon_x^2(t_0,t)\right| &= \left|\dot{\mathbf{P}}_{\delta_x}\left[\left(Y_t(\phi)\right)^{-1} - \left(Y_t^{(t_0,t]}(\phi)\right)^{-1}\right]\right| \\
&= \dot{\mathbf{P}}_{\delta_x}\left[Y_t^{(0,t_0]}(\phi)\cdot\left(Y_t(\phi)\right)^{-1}\cdot\left(Y_t^{(t_0,t]}(\phi)\right)^{-1}\right] \\
&\le \dot{\mathbf{P}}_{\delta_x}\left[\mathbf{1}_{Y_t^{(0,t_0]}(\phi)>0}\cdot\left(Y_t^{(t_0,t]}(\phi)\right)^{-1}\right] \\
&= \dot{\mathbf{P}}_{\delta_x}\left[\dot{\mathbf{P}}_{\delta_x}\left[\mathbf{1}_{Y_t^{(0,t_0]}(\phi)>0}|\mathscr{F}_{t_0}^\xi\right]\cdot\dot{\mathbf{P}}_{\delta_x}\left[\left(Y_t^{(t_0,t]}(\phi)\right)^{-1}|\mathscr{F}_{t_0}^\xi\right]\right].
\end{aligned}
\tag{5.14}
$$

Notice that, by Campbell's formula, one can verify that

$$
\dot{\mathbf{P}}_{\delta_x}\left[e^{-\langle Y_t^{(0,t_0]},\theta\mathbf{1}_E\rangle}|\mathscr{F}_{t_0}^\xi\right] = e^{-\int_0^{t_0}\psi_0'(\xi_s,V_{t-s}(\theta\mathbf{1}_E)(\xi_s))\,ds}.
$$

Letting $\theta\to\infty$ we have

$$
\dot{\mathbf{P}}_{\delta_x}\left[\mathbf{1}_{Y_t^{(0,t_0]}=\mathbf{0}}|\mathscr{F}_{t_0}^\xi\right] = e^{-\int_0^{t_0}\psi_0'(\xi_s,v_{t-s}(\xi_s))\,ds}.
$$

We also have

$$
\begin{aligned}
\psi_0'\left(x,v_{t-s}(x)\right) &= 2\alpha(x)v_{t-s}(x) + \int_{(0,\infty)}\left(1-e^{-yv_{t-s}(x)}\right)y\pi(x,dy) \\
&\le \left(2\alpha(x) + \int_{(0,\infty)}y^2\pi(x,dy)\right)v_{t-s}(x) \\
&= A(x)v_{t-s}(x) \le \|A\phi\|_\infty\left\|\phi^{-1}v_{t-s}\right\|_\infty.
\end{aligned}
$$

Therefore

$$
\dot{\mathbf{P}}_{\delta_x}\left[\mathbf{1}_{Y_t^{(0,t_0]}\neq\mathbf{0}}\Big|\mathscr{F}_{t_0}^\xi\right] = 1 - e^{-\int_0^{t_0}\psi_0'(\xi_s,v_{t-s}(\xi_s))\,ds} \le t_0\|A\phi\|_\infty\left\|\phi^{-1}v_{t-t_0}\right\|_\infty. \tag{5.15}
$$

Plugging (5.15) into (5.14), using (5.11) and letting $c, \gamma > 0$ be the constants in (1.20), we have that

$$
\begin{aligned}
\left|\epsilon_x^2(t_0,t)\right| &\le t_0\|A\phi\|_\infty\left\|\left(\phi^{-1}v_{t-t_0}\right)\right\|_\infty\dot{\mathbf{P}}_{\delta_x}\left[\left(Y_t^{(t_0,t]}(\phi)\right)^{-1}\Big|\mathscr{F}_{t_0}^\xi\right] \\
&\le t_0\|A\phi\|_\infty\left\|\left(\phi^{-1}v_{t-t_0}\right)\right\|_\infty\int_E\dot{q}(t_0,x,y)\left(\phi^{-1}v_{t-t_0}\right)(y)m(dy) \\
&\le t_0\|A\phi\|_\infty\left\|\phi^{-1}v_{t-t_0}\right\|_\infty\left(1+ce^{-\gamma t_0}\right)\left\langle v_{t-t_0},\phi^*\right\rangle_m.
\end{aligned}
\tag{5.16}
$$

Combining (5.12), (5.13) and (5.16), we have that

$$\left|\frac{\phi^{-1}v_t(x)}{\langle v_{t-t_0},\phi^*\rangle_m}-1\right|\le\frac{|\epsilon_x^1(t_0,t)|}{\langle v_{t-t_0},\phi^*\rangle_m}+\frac{|\epsilon_x^2(t_0,t)|}{\langle v_{t-t_0},\phi^*\rangle_m}$$
$$\le ce^{-\gamma t_0}+t_0\|A\phi\|_\infty\left\|\phi^{-1}v_{t-t_0}\right\|_\infty\left(1+ce^{-\gamma t_0}\right). \tag{5.17}$$

Since we know from Lemma 5.2(5) that $\|\phi^{-1}v_t\|_\infty\to 0$ when $t\to\infty$, there exists a map $t\mapsto t_0(t)$ such that,

$$t_0(t)\xrightarrow[t\to\infty]{}\infty;\quad t_0(t)\left\|\phi^{-1}v_{t-t_0(t)}\right\|_\infty\xrightarrow[t\to\infty]{}0.$$

Plugging this choice of $t_0(t)$ back into (5.17), we have that

$$\sup_{x\in E}\left|\frac{\phi^{-1}v_t(x)}{\langle v_{t-t_0(t)},\phi^*\rangle_m}-1\right|\xrightarrow[t\to\infty]{}0. \tag{5.18}$$

Now notice that

$$\left|\frac{\langle v_t,\phi^*\rangle_m}{\langle v_{t-t_0(t)},\phi^*\rangle_m}-1\right|\le\int\left|\frac{\phi^{-1}v_t(x)}{\langle v_{t-t_0(t)},\phi^*\rangle}-1\right|\phi\phi^*(x)m(dx)$$
$$\le\sup_{x\in E}\left|\frac{\phi^{-1}v_t(x)}{\langle v_{t-t_0(t)},\phi^*\rangle_m}-1\right|\xrightarrow[t\to\infty]{}0. \tag{5.19}$$

Finally, by (5.18), (5.19) and the property of uniform convergence,

$$\sup_{x\in E}\left|\frac{\phi^{-1}v_t(x)}{\langle v_t,\phi^*\rangle_m}-1\right|\xrightarrow[t\to\infty]{}0,$$

as desired. □

Lemma 5.4 *Under Assumptions 1′, 2′ and 3, we have*

$$\frac{1}{t\langle v_t,\phi^*\rangle_m}\xrightarrow[t\to\infty]{}\frac{1}{2}\langle A\phi,\phi\phi^*\rangle_m.$$

Proof We use an argument similar to that used in [34] for critical branching diffusions. According to [38], we have that, for any $x\in E$ and $z\ge 0$,

$$R(x,z):=\psi_0(x,z)-\frac{1}{2}A(x)z^2\le e(x,z)z^2,$$

where

$$e(x,z):=\int_{(0,\infty)}y^2\left(1\wedge\frac{1}{6}yz\right)\pi(x,dy)\le A(x).$$

By monotonicity, we have that

$$e(x,z)\xrightarrow[z\to 0]{}0,\quad x\in E. \tag{5.20}$$

Taking $b(t) := \langle v_t, \phi^* \rangle_m$ and writing $l_t(x) := v_t(x) - b(t)\phi(x)$, Lemma 5.3 says that,

$$\sup_{x \in E} \left| \frac{l_t(x)}{b(t)\phi(x)} \right| \xrightarrow[t \to \infty]{} 0. \tag{5.21}$$

Now, taking $s_0 > 0$ as in (5.8), we have that $t \mapsto b(t)$ is differentiable on the set

$$C = \left\{ t > s_0 : \text{the function } t \mapsto \left\langle \Psi_0(v_t), \phi^* \right\rangle_m \text{ is continuous at } t \right\}$$

and that

$$\begin{aligned} \frac{d}{dt} b(t) &= -\left\langle \Psi_0(v_t), \phi^* \right\rangle_m = -\left\langle \frac{1}{2} A \cdot v_t^2 + R\big(\cdot, v_t(\cdot)\big), \phi^* \right\rangle_m \\ &= -\left\langle \frac{1}{2} A \cdot \big(b(t)\phi + l_t\big)^2 + R\big(\cdot, v_t(\cdot)\big), \phi^* \right\rangle_m \\ &= -b(t)^2 \left[\frac{1}{2} \left\langle A\phi, \phi\phi^* \right\rangle_m + g(t) \right], \quad t \in C, \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} g(t) &= \left\langle \frac{l_t}{b(t)\phi}, A\phi^2\phi^* \right\rangle_m + \frac{1}{2} \left\langle \left(\frac{l_t}{b(t)\phi} \right)^2, A\phi^2\phi^* \right\rangle_m + \left\langle \frac{R(\cdot, v_t(\cdot))}{b(t)^2\phi^2}, \phi^2\phi^* \right\rangle_m \\ &=: g_1(t) + g_2(t) + g_3(t). \end{aligned}$$

From (5.21), we have $g_1(t) \to 0$ and $g_2(t) \to 0$ as $t \to \infty$. From

$$\frac{R(x, v_t(x))}{b(t)^2\phi(x)^2} \le \frac{e(x, v_t(x)) \cdot v_t(x)^2}{b(t)^2\phi(x)^2} = e\big(x, v_t(x)\big) \left(1 + \frac{l_t(x)}{b(t)\phi(x)} \right)^2,$$

using (5.21), (5.20), Lemma 5.2 (5) and the dominated convergence theorem ($e(x, v_t(x))$ is dominated by $A(x)$), we conclude that $g_3(t) \to 0$ as $t \to \infty$.

Finally, from (5.22) we can write

$$\frac{d}{dt} \left(\frac{1}{b(t)} \right) = -\frac{db(t)}{b(t)^2 dt} = \frac{1}{2} \left\langle A\phi, \phi\phi^* \right\rangle_m + g(t), \quad t \in C. \tag{5.23}$$

Notice that, since the function $t \mapsto \langle \Psi_0(v_t), \phi^* \rangle_m$ is non-increasing in t, the complement of C has at most countably many elements. Therefore, using (5.8) and (5.23), one can verify that $t \mapsto \frac{1}{b(t)}$ is absolutely continuous on the interval $[s_0, t_0]$ as long as s_0 and t_0 are large enough. This allows us to integrate (5.23) on the interval $[s_0, t_0]$ with respect to the Lebesgue measure, and get that

$$\frac{1}{b(t_0)} = \frac{1}{b(s_0)} + \frac{1}{2} \left\langle A\phi, \phi\phi^* \right\rangle_m (t_0 - s_0) + \int_{s_0}^{t_0} g(s)\, ds, \quad \text{for } 0 \le s_0 \le t_0 \text{ large enough.}$$

Dividing by t_0 and letting $t_0 \to \infty$ in the above equation, we have

$$\frac{1}{b(t)t} \xrightarrow[t \to \infty]{} \frac{1}{2} \left\langle A\phi, \phi\phi^* \right\rangle_m$$

as desired. □

Proof of Theorem 1.10 For $\mu \in \mathcal{M}_f^\phi$, from Lemma 5.2.(5) we know that

$$\langle \mu, v_t \rangle = \int_E v_t(x)\mu(dx) = \int_E \frac{v_t(x)}{\phi(x)}\phi(x)\mu(dx) \xrightarrow[t\to\infty]{} 0. \tag{5.24}$$

From Lemma 5.3 we know that

$$\frac{\langle \mu, v_t \rangle}{\langle v_t, \phi^* \rangle_m} = \int_E \frac{v_t(x)}{\langle v_t, \phi^* \rangle_m \phi(x)}\phi(x)\mu(dx) \xrightarrow[t\to\infty]{} \langle \mu, \phi \rangle. \tag{5.25}$$

It then follows from (5.24), (5.25) and Lemma 5.4 that

$$\begin{aligned} t\mathbf{P}_\mu(X_t \neq \mathbf{0}) &= t\big(1 - e^{-\langle \mu, v_t \rangle}\big) = t\big\langle v_t, \phi^* \big\rangle \frac{\langle \mu, v_t \rangle}{\langle v_t, \phi^* \rangle_m} \frac{1 - e^{-\langle \mu, v_t \rangle}}{\langle \mu, v_t \rangle} \\ &\xrightarrow[t\to\infty]{} \frac{\langle \mu, \phi \rangle}{\frac{1}{2}\langle A\phi, \phi\phi^* \rangle_m}, \quad x \in E. \end{aligned}$$

□

5.3 Yaglom Type Result

Let $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ, ψ)-superprocess introduced in Sect. 1.2 which satisfies Assumptions $1'$ and $2'$ and 3. In this subsection, we will give a proof of Theorem 1.11.

Slutsky's theorem is used quite often to prove convergence in law of two components, in which one contributes to the limit, and the other one is negligible. The following proposition says that under $\dot{\mathbf{P}}_\mu$, the weighted mass $Y_t(\phi)$ coming off spine, normalized by t, converges to a Gamma distribution as $t \to \infty$.

Proposition 5.5 *Suppose that Assumptions $1'$, $2'$ and 3 hold. Suppose that $\mu \in \mathcal{M}_f^\phi$. Let $\{(\xi_t)_{t\geq 0}, (Y_t)_{t\geq 0}, \mathbf{n}; \dot{\mathbf{P}}_\mu\}$ be the spine immigration given by Theorem 3.5. Then $W_t := \frac{Y_t(\phi)}{t}$ converges weakly to a Gamma distribution $\Gamma(2, c_0^{-1})$ with $c_0 := \frac{1}{2}\langle \phi A, \phi\phi^* \rangle_m$.*

Proof We only have to prove that

$$\dot{\mathbf{P}}_\mu\big[e^{-\theta W_t}\big] \xrightarrow[t\to\infty]{} \frac{1}{(1 + c_0\theta)^2}, \quad \theta \geq 0,\ \mu \in \mathcal{M}_f^\phi.$$

First we consider the case when $\mu = \delta_x$ for an arbitrary $x \in E$. To simplify notation, for all $x \in E, \theta \geq 0$ and $t \geq 0$, we write

$$\begin{aligned} J(x, \theta, t) &:= (\phi A)(x)\dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big]\widetilde{\mathbf{P}}_x\big[e^{-X_t(\frac{\theta\phi}{t})}\big], \\ J_0(x, \theta, t) &:= (\phi A)(x)\dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big] \end{aligned}$$

and

$$M(x, \theta, t) := \left|\frac{1}{(1 + c_0\theta)^2} - \dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big]\right|.$$

Step 1. We will show that

$$\dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big] = \dot{\mathbf{P}}_{\delta_x}\big[e^{-\int_0^1 du \int_0^\theta d\rho \cdot J(\xi_{ut}, \rho(1-u), t(1-u))}\big]. \tag{5.26}$$

In fact, we have

$$\frac{\partial}{\partial\theta}\dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big|\xi\big] = -\dot{\mathbf{P}}_{\delta_x}\big[W_t e^{-\theta W_t}\big|\xi\big], \quad t\geq 0, \theta\geq 0.$$

Applying Lemma 4.6 with $K(dr)=\delta_t(dr)$ and $f_t=\frac{\theta\phi}{t}$, for each $\theta\geq 0$, we have

$$\begin{aligned}
-\frac{\partial}{\partial\theta}\log\dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big|\xi\big] &= \frac{\dot{\mathbf{P}}_{\delta_x}[W_t e^{-\theta W_t}|\xi]}{\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}|\xi]}\\
&= \frac{1}{t}\int_0^t (A\phi)(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}\big[e^{-(\theta\frac{t-s}{t})W_{t-s}}\big]\widetilde{\mathbf{P}}_{\xi_s}\big[e^{-X_{t-s}(\frac{\theta\phi}{t})}\big]ds\\
&= \int_0^1 J\big(\xi_{ut},\theta(1-u),t(1-u)\big)\,du.
\end{aligned}$$

Integrating both sides of the above equation yields that

$$-\log\dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big|\xi\big] = \int_0^1 du\int_0^\theta J\big(\xi_{ut},\rho(1-u),t(1-u)\big)\,d\rho,$$

which implies (5.26).

Step 2. We will show that

$$\int_0^1 du\int_0^\theta (J_0-J)\big(\xi_{ut},\rho(1-u),t(1-u)\big)\,d\rho \xrightarrow[t\to\infty]{L^2(\dot{\mathbf{P}}_{\delta_x})} 0, \quad \theta\geq 0. \tag{5.27}$$

To get this result, we will apply Lemma 5.1 with

$$\begin{aligned}
F(x,u,t) &:= \int_0^\theta d\rho\cdot(J_0-J)\big(x,\rho(1-u),t(1-u)\big)\\
&= \int_0^\theta d\rho\cdot(A\phi)(x)\dot{\mathbf{P}}_{\delta_x}\big[e^{-\rho(1-u)W_{t(1-u)}}\big]\widetilde{\mathbf{P}}_x\big[1-e^{-X_{t(1-u)}(\frac{\rho\phi}{t})}\big].
\end{aligned} \tag{5.28}$$

Firstly note that $F(x,u,t)$ is bounded by $\theta\|\phi A\|_\infty$ on $E\times[0,1]\times[0,\infty)$. Secondly note that $F(x,u,t)\xrightarrow[t\to\infty]{} 0$ for each $x\in E$ and $u\in[0,1]$, since $|J_0-J|$ is bounded by $\|\phi A\|_\infty$ and

$$\begin{aligned}
\big|(J_0-J)(x,\theta,t)\big| &= (A\phi)(x)\dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big]\widetilde{\mathbf{P}}_x\big[1-e^{-X_t(\frac{\theta\phi}{t})}\big]\\
&\leq (A\phi)(x)\widetilde{\mathbf{P}}_x(X_t\neq\mathbf{0})\\
&= (A\phi)(x)\frac{2\alpha(x)\mathbf{P}_{\mathbf{0}}(X_t\neq\mathbf{0})+\int_{(0,\infty)}y^2\mathbf{P}_{y\delta_x}(X_t\neq\mathbf{0})\pi(x,dy)}{2\alpha(x)+\int_{(0,\infty)}y^2\pi(x,dy)}\\
&\xrightarrow[t\to\infty]{} 0, \quad x\in E,\theta\geq 0.
\end{aligned}$$

Therefore, we can apply Lemma 5.1 with $F(x,u,t)$ given by (5.28), and get (5.27).

Step 3. We will show that

$$\frac{1}{(1+c_0\theta)^2} = \lim_{t\to\infty}\dot{\mathbf{P}}_{\delta_x}\big[e^{-\int_0^1 du\int_0^\theta d\rho\frac{(A\phi)(\xi_{ut})}{(1+c_0\rho(1-u))^2}}\big], \quad \theta\geq 0. \tag{5.29}$$

By elementary calculus, the following map

$$(x,u)\mapsto \int_0^\theta \frac{(A\phi)(x)}{(1+c_0\rho(1-u))^2}\,d\rho=\frac{(A\phi)(x)\theta}{1+c_0\theta(1-u)}$$

is bounded by $\theta\|A\phi\|_\infty$ on $E\times[0,1]$. According to Lemma 5.1, we have that

$$\begin{aligned}\int_0^1 du\int_0^\theta \frac{(A\phi)(\xi_{ut})}{(1+c_0\rho(1-u))^2}\,d\rho \xrightarrow[t\to\infty]{L^2(\dot{\mathbf{P}}_{\delta_x})} &\int_0^1\left\langle \frac{\theta A\phi}{1+c_0\theta(1-u)},\phi\phi^*\right\rangle_m du\\ &=\langle A\phi,\phi\phi^*\rangle_m\int_0^1\frac{\theta}{1+c_0\theta(1-u)}\,du\\ &=2\log(1+c_0\theta).\end{aligned}$$

Therefore, by the bounded convergence theorem, we get (5.29).

Step 4. We will show that

$$M(x,\theta):=\limsup_{t\to\infty} M(x,\theta,t)=0,\quad x\in E,\theta\ge 0. \tag{5.30}$$

In fact,

$$M(x,\theta,t)\le I_1+I_2+I_3, \tag{5.31}$$

where

$$\begin{aligned}I_1&:=\left|\frac{1}{(1+c_0\theta)^2}-\dot{\mathbf{P}}_{\delta_x}\left[e^{-\int_0^1 du\int_0^\theta \frac{(A\phi)(\xi_{ut})}{[1+c_0\rho(1-u)]^2}\,d\rho}\right]\right| \xrightarrow[t\to\infty]{\text{by (5.29)}} 0,\\ I_2&:=\left|\dot{\mathbf{P}}_{\delta_x}\left[e^{-\int_0^1 du\int_0^\theta \frac{(A\phi)(\xi_{ut})}{(1+c_0\rho(1-u))^2}\,d\rho}\right]-\dot{\mathbf{P}}_{\delta_x}\left[e^{-\int_0^1 du\int_0^\theta J_0(\xi_{ut},\rho(1-u),t(1-u))\,d\rho}\right]\right|\\ &\le \dot{\mathbf{P}}_{\delta_x}\left[\int_0^1 du\int_0^\theta (A\phi)(\xi_{ut})M\big(\xi_{ut},\rho(1-u),t(1-u)\big)\,d\rho\right]\\ &=\int_0^1 du\int_0^\theta d\rho\int_E \dot q(ut,x,y)(A\phi)(y)M\big(y,\rho(1-u),t(1-u)\big)m(dy),\end{aligned}$$

and by (5.26) and (5.27),

$$\begin{aligned}I_3&:=\left|\dot{\mathbf{P}}_{\delta_x}\left[e^{-\int_0^1 du\int_0^\theta J_0(\xi_{ut},\rho(1-u),t(1-u))\,d\rho}\right]-\dot{\mathbf{P}}_{\delta_x}\left[e^{-\theta W_t}\right]\right|\\ &=\left|\dot{\mathbf{P}}_{\delta_x}\left[e^{-\int_0^1 du\int_0^\theta J_0(\xi_{ut},\rho(1-u),t(1-u))\,d\rho}\right]-\dot{\mathbf{P}}_{\delta_x}\left[e^{-\int_0^1 du\int_0^\theta J(\xi_{ut},\rho(1-u),t(1-u))\,d\rho}\right]\right|\\ &\le \dot{\mathbf{P}}_{\delta_x}\left[\left|\int_0^1 du\int_0^\theta (J_0-J)\big(\xi_{ut},\rho(1-u),t(1-u)\big)\,d\rho\right|\right]\xrightarrow[t\to\infty]{} 0.\end{aligned}$$

Therefore, taking $\limsup_{t\to\infty}$ in (5.31), by the reverse Fatou's lemma, we get

$$M(x,\theta)\le\int_0^1 du\int_0^\theta \big\langle A\phi M\big(\cdot,\rho(1-u)\big),\phi\phi^*\big\rangle_m\,d\rho,\quad x\in E,\theta\ge 0. \tag{5.32}$$

Integrating with respect to the finite measure $(A\phi\phi\phi^*)(x)m(dx)$ yields that

$$\langle A\phi M(\cdot,\theta),\phi\phi^*\rangle_m\le\langle A\phi,\phi\phi^*\rangle_m\int_0^1 du\int_0^\theta\big\langle A\phi M\big(\cdot,\rho(1-u)\big),\phi\phi^*\big\rangle_m\,d\rho,\quad \theta\ge 0.$$

According to [35, Lemma 3.1], this inequality implies that $\langle A\phi M(\cdot,\theta), \phi\phi^*\rangle_m = 0$ for each $\theta \geq 0$. This and (5.32) imply (5.30), which completes the proof when $\mu = \delta_x$.

Finally, for any $\mu \in \mathscr{M}_f^\phi$, since

$$\langle \mu, \phi\rangle \dot{\mathbf{P}}_\mu\big[e^{-\theta W_t}\big] = \langle \mu, \phi\rangle \mathbb{N}_\mu^{w_t(\phi)}\big[e^{-\theta \frac{w_t(\phi)}{t}}\big] = \mathbb{N}_\mu\big[w_t(\phi) e^{-\theta \frac{w_t(\phi)}{t}}\big]$$
$$= \int_E \mu(dx)\mathbb{N}_x\big[w_t(\phi)e^{-\theta \frac{w_t(\phi)}{t}}\big] = \int_E \mu(dx)\phi(x)\dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big],$$

we have that, by the bounded convergence theorem,

$$\left|\dot{\mathbf{P}}_\mu\big[e^{-\theta W_t}\big] - \frac{1}{(1+c_0\theta)^2}\right| \leq \int_E \left|\dot{\mathbf{P}}_{\delta_x}\big[e^{-\theta W_t}\big] - \frac{1}{(1+c_0\theta)^2}\right| \frac{\phi(x)\mu(dx)}{\langle \mu, \phi\rangle} \xrightarrow[t\to\infty]{} 0,$$

as desired. □

The following lemma says that, conditional on survival up to time t, the weighted and normalized mass $t^{-1}X_t(\phi)$ (weighted by ϕ, and normalized by t) has a limit distribution which is exponential with explicit parameter. Later we will consider limit of $t^{-1}X_t(f)$ with a general $f \in bp\mathscr{B}_E^\phi$.

Lemma 5.6 *Suppose that Assumptions 1′, 2′ and 3 hold. Let $\mu \in \mathscr{M}_f^\phi$. Then it holds that $\{t^{-1}X_t(\phi); \mathbf{P}_\mu(\cdot|X_t \neq \mathbf{0})\}$ converges weakly to an exponential distribution* $\mathrm{Exp}(c_0^{-1})$ *with* $c_0 := \frac{1}{2}\langle \phi A, \phi\phi^*\rangle_m$.

Proof We only have to show that

$$\mathbf{P}_\mu\big[e^{-\theta t^{-1}X_t(\phi)}\big|X_t \neq \mathbf{0}\big] \xrightarrow[t\to\infty]{} \frac{1}{1+c_0\theta}, \quad \theta \geq 0,\ \mu \in \mathscr{M}_f^\phi.$$

Notice that, by Lemma 5.2(6), we have

$$\big\{t^{-1}X_t(\phi); \mathbf{P}_\mu\big\} \xrightarrow[t\to\infty]{law} 0.$$

Therefore, by Theorem 3.5 and Proposition 5.5, we have

$$\mathbf{P}_\mu^M\big[e^{-\theta t^{-1}X_t(\phi)}\big] = (\mathbf{P}_\mu \otimes \dot{\mathbf{P}}_\mu)\big[e^{-\theta t^{-1}(X_t+Y_t)(\phi)}\big] \xrightarrow[t\to\infty]{} \frac{1}{(1+c_0\theta)^2}.$$

Also notice that, by elementary calculus

$$\frac{1-e^{-\theta u}}{u} = \int_0^\theta e^{-\rho u}\, d\rho, \quad u > 0.$$

From Theorem 3.5 and Lemma 4.5 we know that $\mathbf{P}_\mu^M(X_t = \mathbf{0}) = 0$. Therefore by the bounded convergence theorem, we have

$$\mathbf{P}_\mu^M\left[\frac{1-e^{-\theta t^{-1}X_t(\phi)}}{t^{-1}X_t(\phi)}\right] = \mathbf{P}_\mu^M\left[\int_0^\theta e^{-\rho t^{-1}X_t(\phi)}\, d\rho\right] = \int_0^\theta \mathbf{P}_\mu^M\big[e^{-\rho t^{-1}X_t(\phi)}\big]\, d\rho$$
$$\xrightarrow[t\to\infty]{} \int_0^\theta \frac{1}{(1+c_0\rho)^2}\, d\rho = c_0^{-1}\left(1 - \frac{1}{1+c_0\theta}\right).$$

Hence, by Theorem 1.10 we have

$$\begin{aligned}\mathbf{P}_\mu\big[1-e^{-\theta t^{-1}X_t(\phi)}\big|X_t\neq\mathbf{0}\big] &= \mathbf{P}_\mu(X_t\neq\mathbf{0})^{-1}\mathbf{P}_\mu\big[\big(1-e^{-\theta t^{-1}X_t(\phi)}\big)\mathbf{1}_{X\neq\mathbf{0}}\big]\\ &= \mathbf{P}_\mu(X_t\neq\mathbf{0})^{-1}\mathbf{P}_\mu\bigg[\big(1-e^{-\theta t^{-1}X_t(\phi)}\big)\frac{X_t(\phi)}{X_t(\phi)}\bigg]\\ &= \big(t\mathbf{P}_\mu(X_t\neq\mathbf{0})\big)^{-1}\langle\mu,\phi\rangle\mathbf{P}_\mu^M\bigg[\frac{1-e^{-\theta t^{-1}X_t(\phi)}}{t^{-1}X_t(\phi)}\bigg]\\ &\xrightarrow[t\to\infty]{} 1-\frac{1}{1+c_0\theta},\end{aligned}$$

which completes the proof. □

Now we consider limit of $t^{-1}X_t(f)$ with general weight $f\in bp\mathscr{B}_E^\phi$. The main idea is to use the following decomposition for f: $f(x)=\langle\phi^*,f\rangle_m\phi(x)+\tilde f(x)$, $x\in E$. The following lemma says that $\tilde f$ has no contribution to the limit, and then we can easily get that the conditional limit of $t^{-1}X_t(f)$ as $t\to\infty$ is the contribution of $\langle\phi^*,f\rangle_m t^{-1}X_t(\phi)$, which is known from Lemma 5.6.

Lemma 5.7 *Suppose that Assumptions 1′, 2′ and 3 hold. If $\tilde f\in b\mathscr{B}_E^\phi$ satisfies $\langle\tilde f,\phi^*\rangle=0$, then we have, for any $\mu\in\mathscr{M}_f^\phi$,*

$$\big\{t^{-1}X_t(\tilde f);\mathbf{P}_\mu(\cdot|X_t\neq\mathbf{0})\big\}\xrightarrow[t\to\infty]{}0,\quad \textit{in probability}.$$

Proof If we can show that $\mathbf{P}_\mu[(t^{-1}X_t(\tilde f))^2|X_t\neq\mathbf{0}]\xrightarrow[t\to\infty]{}0$, then the desired result follows by the Chebyshev's inequality

$$\mathbf{P}_\mu\big(|t^{-1}X_t(\tilde f)|\geq\epsilon|X_t\neq\mathbf{0}\big)\leq\epsilon^{-2}\mathbf{P}_\mu\big[\big(t^{-1}X_t(\tilde f)\big)^2|X_t\neq\mathbf{0}\big].$$

By Proposition 4.2 we have that

$$\begin{aligned}&\mathbf{P}_\mu\big[\big(t^{-1}X_t(\tilde f)\big)^2\big|X_t\neq\mathbf{0}\big]\\ &\quad=t^{-2}\mathbf{P}_\mu(X_t\neq\mathbf{0})^{-1}\mathbf{P}_\mu\big[X_t(\tilde f)^2\mathbf{1}_{X_t\neq\mathbf{0}}\big]\\ &\quad=t^{-1}\mathbf{P}_\mu(X_t\neq\mathbf{0})^{-1}\bigg(\frac{\langle\mu,S_t\tilde f\rangle^2}{t}+\langle\mu,\phi\rangle\dot{\mathbb{P}}_\mu\bigg[\big(\phi^{-1}\tilde f\big)(\xi_t)\frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde f)(\xi_s)\,ds\bigg]\bigg).\end{aligned}\tag{5.33}$$

Letting $c,\gamma>0$ be the constants in (1.20), we know that

$$\begin{aligned}\big|S_t\tilde f(x)-\langle\phi^*,\tilde f\rangle_m\phi(x)\big| &= \bigg|\int_E\big(q(t,x,y)-\phi(x)\phi^*(y)\big)\tilde f(y)m(dy)\bigg|\\ &\leq\int_E\bigg|\frac{q(t,x,y)}{\phi(x)\phi^*(y)}-1\bigg|\cdot\big|\phi(x)\phi^*(y)\tilde f(y)\big|m(dy)\\ &\leq ce^{-\gamma t}\phi(x)\big\|\phi^{-1}\tilde f\big\|_\infty\int_E\big(\phi\phi^*\big)(y)m(dy)\\ &\xrightarrow[t\to\infty]{}0,\quad x\in E.\end{aligned}\tag{5.34}$$

Therefore, by the dominated convergence theorem,

$$\langle \mu, S_t \tilde{f} \rangle \xrightarrow[t\to\infty]{} \langle \phi^*, \tilde{f} \rangle_m \langle \mu, \phi \rangle = 0.$$

Hence,

$$\frac{\langle \mu, S_t \tilde{f} \rangle}{t} \xrightarrow[t\to\infty]{} 0, \quad x \in E. \tag{5.35}$$

By (5.34) and Lemma 5.1, we know that

$$\begin{aligned} \frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde{f})(\xi_s)\,ds &= \int_0^1 A(\xi_{ut})\cdot(S_{t-ut}\tilde{f})(\xi_{ut})\,du \\ &\xrightarrow[t\to\infty]{L^2(\dot{\mathbb{P}}_x)} \int_0^1 \langle A\phi, \phi\phi^* \rangle_m \langle \phi^*, \tilde{f} \rangle_m\,du = 0. \end{aligned}$$

Hence, by Lemma 4.1 and the bounded convergence theorem we have that

$$\begin{aligned} &\left| \langle \mu, \phi \rangle \dot{\mathbb{P}}_\mu \left[\left(\phi^{-1}\tilde{f}\right)(\xi_t) \frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde{f})(\xi_s)\,ds \right] \right| \\ &\le \int \mu(dx)\phi(x) \left| \dot{\mathbb{P}}_x \left[\left(\phi^{-1}\tilde{f}\right)(\xi_t) \frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde{f})(\xi_s)\,ds \right] \right| \\ &\le \|\phi^{-1}\tilde{f}\|_\infty \cdot \int \mu(dx)\phi(x) \dot{\mathbb{P}}_x \left[\left| \frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde{f})(\xi_s)\,ds \right|^2 \right]^{\frac{1}{2}} \\ &\xrightarrow[t\to\infty]{} 0. \end{aligned} \tag{5.36}$$

Finally, using Theorem 1.10 and combining (5.33), (5.35) and (5.36), we have that

$$\mathbf{P}_\mu\left[\left(t^{-1}X_t(\tilde{f})\right)^2 \middle| X_t \neq \mathbf{0}\right] \xrightarrow[t\to\infty]{} 0$$

as required. □

Proof of Theorem 1.11 Define a function $\tilde{f}$ by

$$\tilde{f}(x) := f(x) - \langle \phi^*, f \rangle_m \phi(x), \quad x \in E. \tag{5.37}$$

It is easy to see that $\tilde{f} \in b\mathscr{B}_E^\phi$ and $\langle \tilde{f}, \phi^* \rangle_m = 0$. It then follows from Lemma 5.6 that

$$\left\{ t^{-1}X_t\left(\langle \phi^*, f \rangle_m \phi\right); \mathbf{P}_\mu(\cdot | X_t \neq \mathbf{0}) \right\} \xrightarrow[t\to\infty]{law} \frac{1}{2}\langle \phi^*, f \rangle_m \langle \phi A, \phi\phi^* \rangle_m \mathbf{e}, \tag{5.38}$$

and from Lemma 5.7 that

$$\left\{ t^{-1}X_t(\tilde{f}); \mathbf{P}_\mu(\cdot | X_t \neq \mathbf{0}) \right\} \xrightarrow[t\to\infty]{\text{in probability}} 0. \tag{5.39}$$

The desired result then follows from (5.37), (5.38), (5.39) and Slutsky's theorem. □

Remark 5.8 In the symmetric case, i.e. when (S_t) are self-adjoint operators, (5.37) is exactly an L^2-orthogonal decomposition.

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