Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 163 (2023) 498–534

stochastic processes and their applications

www.elsevier.com/locate/spa

Subcritical superprocesses conditioned on non-extinction[☆]

Rongli Liu[a,1], Yan-Xia Ren[b,2], Renming Song[c,3], Zhenyao Sun[d,*]

[a] *Mathematics and Applied Mathematics, Beijing Jiaotong University, Beijing 100044, PR China*

[b] *LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing 100871, PR China*

[c] *Department of Mathematics, University of Illinois Urbana-Champaign, Urbana, IL 61801, USA*

[d] *School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China*

Received 24 October 2022; received in revised form 25 February 2023; accepted 12 June 2023

Available online 19 June 2023

Abstract

We consider a class of subcritical superprocesses $(X_t)_{t\geq 0}$ with general spatial motions and general branching mechanisms. We study the asymptotic behaviors of $\mathbf{Q}_{t,r}$, the distribution of X_t conditioned on X_{t+r} not being a null measure. We first give the existence of $\lim_{t\to\infty}\mathbf{Q}_{t,r}$ and $\lim_{r\to\infty}\mathbf{Q}_{t,r}$, and then show that an $L\log L$-type condition is equivalent to the existence of the iterated limits: $\lim_{r\to\infty}\lim_{t\to\infty}\mathbf{Q}_{t,r}$ and $\lim_{t\to\infty}\lim_{r\to\infty}\mathbf{Q}_{t,r}$. Finally, when the $L\log L$-type condition holds, we show that those iterated limits, and the double limit $\lim_{r,t\to\infty}\mathbf{Q}_{t,r}$, are the same.

1. Introduction

Motivation

The study of the extinction of stochastic processes related to population dynamics is of great interest in both biology and probability theory. Take a subcritical Galton–Watson process

☆ The research of this project is supported by the National Key R&D Program of China (No. 2020YFA0712900).

* Corresponding author.

E-mail addresses: rlliu@bjtu.edu.cn (R. Liu), yxren@math.pku.edu.cn (Y.-X. Ren), rsong@illinois.edu (R. Song), zhenyao.sun@gmail.com (Z. Sun).

[1] The research of Rongli Liu is supported in part by NSFC (Grant No. 12271374), and the Fundamental Research Funds for the Central Universities, China (Grant No. 2017RC007).

[2] The research of Yan-Xia Ren is supported in part by NSFC (Grant Nos. 12071011, 11731009 and 12231002), and the Fundamental Research Funds for the Central Universities, Peking University LMEQF.

[3] The research of Renming Song is supported in part by a grant from the Simons Foundation, United States (#960480, Renming Song).

https://doi.org/10.1016/j.spa.2023.06.008

0304-4149/

$(Z_n)_{n\in\mathbb{Z}_+}$ as an example. Assume that $Z_0 = 1$ and $m = E[Z_1] \in (0, 1)$. It is well known that the extinction probability $q := \lim_{n\to\infty} P(Z_n = 0)$ is equal to 1. In other words, the probability $P(Z_n > 0)$ decays to 0. A natural question is to find the decay rate of this probability. In 1967, Heathcote, Seneta and Vere-Jones [15] proved that the following three statements are equivalent.

(1.1) $\lim_{n\to\infty} P(Z_n > 0)/m^n > 0$.
(1.2) $\sup E[Z_n|Z_n > 0] < \infty$.
(1.3) $E\left[Z_1 \log^+ Z_1\right] < \infty$.

Condition (1.3) is now known as the $L\log L$ condition and the equivalence of the three statements above is usually called the $L\log L$ criterion. It is also natural to consider $Q_{n,0}$, the distribution of Z_n conditioned on $\{Z_n > 0\}$. In 1967, Heathcote, Seneta and Vere-Jones [15] and Joffe [18] proved that $Q_{n,0}$ has a weak limit $Q_{\infty,0}$ when $n \to \infty$. This result was first obtained by Yaglom [44] in 1947 under some moment condition, and the probability measure $Q_{\infty,0}$ is therefore referred to as the Yaglom limit. One can also consider $Q_{n,m}$, the distribution of Z_n conditioned on $\{Z_{n+m} > 0\}$. As a corollary of the Yaglom limit result, Athreya and Ney [1] showed in 1972 that for every $m \in \mathbb{Z}_+$, $Q_{n,m}$ has a weak limit $Q_{\infty,m}$ when $n \to \infty$. Joffe, in his 1967 paper [18], pointed out that for every $n \in \mathbb{Z}_+$, $Q_{n,m}$ has a weak limit $Q_{n,\infty}$ when $m \to \infty$. Later in 1999, Pakes [31] proved that the $L\log L$ condition (1.3) is equivalent to each of the following two statements.

(1.4) $Q_{\infty,m}$ has a weak limit when $m \to \infty$.
(1.5) $Q_{n,\infty}$ has a weak limit when $n \to \infty$.

Moreover, when (1.3) holds, Pakes [31] showed that $\lim_{m\to\infty} Q_{\infty,m} = \lim_{n\to\infty} Q_{n,\infty}$.

Yaglom limit theorem is now a fundamental topic in the study of Markov processes. A long list of references on Yaglom limit theorems of a variety of models can be found on the website [34] maintained by Pollett. It turns out that Heathcote, Seneta and Vere-Jones' $L\log L$ theorem, as well as Pakes' iterated limit theorem, are also universal among models with the Markovian branching property. Analogs of these results in the context of multitype Galton–Watson processes can be found in [33] and the references therein. Results for continuous-state branching processes can be found in [12,22] and [24].

We are interested in a class of measure-valued branching processes known as superprocesses. The book [25] is a good reference for superprocesses. In recent years, there have been a lot of papers on the large time asymptotic behavior of superprocesses. For laws of large numbers and central limit theorems of some supercritical superprocesses, see [3,6,27,37,38,40] and the references therein. For Yaglom limit results of various critical superprocesses, see [10,35,36,41].

In our recent work [28], we characterized the Yaglom limits of a class of subcritical superprocesses with general spatial motions and general branching mechanisms. The goal of this paper is to establish Heathcote, Seneta and Vere-Jones' $L\log L$ theorem, as well as Pakes' iterated limit theorem, for the same class of subcritical superprocesses.

Model and assumptions

We first recall the definition of superprocesses. For any topological space F, we denote by $C(F)$ the set of continuous real-valued functions on F, and by $\mathscr{B}(F)$ the Borel σ-algebra of F. In general, if $\mathcal{F}$ is a space of real-valued functions, then we use b$\mathcal{F}$, p$\mathcal{F}$ and bp$\mathcal{F}$

to denote the bounded, non-negative, and non-negative bounded elements in $\mathcal{F}$, respectively. Moreover, if $\mathcal{F}$ is a σ-algebra, then we use b$\mathcal{F}$, p$\mathcal{F}$ and bp$\mathcal{F}$ to denote the set of bounded, non-negative, and non-negative bounded $\mathcal{F}$-measurable real functions, respectively. Let E be a Polish space. Let $(\xi_t)_{t\in[0,\zeta)}$ be an E-valued Borel right process with (possibly sub-Markovian) transition semigroup $(P_t)_{t\geq 0}$ and lifetime ζ. Denote $\mathbb{R}_+ := [0,\infty)$. Let ψ be a function on $E \times \mathbb{R}_+$ given by

$$\psi(x,z) = -\beta(x)z + \sigma(x)^2 z^2 + \int_{(0,\infty)} (e^{-zu} - 1 + zu)\pi(x,\mathrm{d}u), \quad x \in E, z \geq 0,$$

where $\beta, \sigma \in \mathrm{b}\mathscr{B}(E)$, and π is a kernel from E to $(0,\infty)$ such that

$$\sup_{x\in E} \int_{(0,\infty)} (u \wedge u^2)\pi(x,\mathrm{d}u) < \infty.$$

For any $f \in \mathrm{bp}\mathscr{B}(E)$, there exists a unique non-negative Borel function $(t,x) \mapsto V_t f(x)$ on $\mathbb{R}_+ \times E$ such that $\sup_{0\leq t\leq t_0, x\in E} V_t f(x) < \infty$ for every $t_0 \geq 0$, and that

$$V_t f(x) + \int_0^t \mathrm{d}s \int_E \psi\big(y, V_{t-s} f(y)\big) P_s(x,\mathrm{d}y) = P_t f(x), \quad t \geq 0, x \in E.$$

The Polish space of all finite Borel measures on E, equipped with the topology of weak convergence, is denoted by $\mathcal{M}$. It is known that there exists an $\mathcal{M}$-valued conservative right process $(X_t)_{t\geq 0}$ with transition semigroup $(Q_t)_{t\geq 0}$ such that for each $\mu \in \mathcal{M}, t \in \mathbb{R}_+$ and $f \in \mathrm{bp}\mathscr{B}(E)$,

$$\int_{\mathcal{M}} \exp\Big\{-\int_E f(x)\eta(\mathrm{d}x)\Big\} Q_t(\mu,\mathrm{d}\eta) = \exp\Big\{-\int_E V_t f(x)\mu(\mathrm{d}x)\Big\}. \tag{1.6}$$

This process $(X_t)_{t\geq 0}$ is known as a (ξ,ψ)-superprocess. We refer our readers to [25] for more details.

For each $x \in E$, denote by Π_x the law of $(\xi_t)_{t\in[0,\zeta)}$ with initial value $\xi_0 = x$. For each $\mu \in \mathcal{M}$, denote by P_μ the law of $(X_t)_{t\geq 0}$ with initial value $X_0 = \mu$. Given any measure γ and function f, we write $\gamma(f)$ for the integral of f with respect to γ whenever it is well-defined. For any $f \in \mathrm{b}\mathscr{B}(E)$, define

$$T_t f(x) = \Pi_x\big[e^{\int_0^t \beta(\xi_s)\mathrm{d}s} f(\xi_t)\mathbf{1}_{\{t<\zeta\}}\big], \quad t \geq 0, x \in E.$$

It is known that $(T_t)_{t\geq 0}$ is a Borel semigroup on E, and that

$$\mu(T_t f) = \mathrm{P}_\mu[X_t(f)], \quad \mu \in \mathcal{M}, t \in \mathbb{R}_+, f \in \mathrm{b}\mathscr{B}(E). \tag{1.7}$$

We call $(T_t)_{t\geq 0}$ the mean semigroup of X. We will always assume the following statement holds.

(1.8) There exist a constant $\lambda \in \mathbb{R}$, a bounded strictly positive Borel function ϕ on E, and a probability measure ν with full support on E, such that $\nu(\phi) = 1$, and that for any $t \geq 0$, $T_t\phi = e^{\lambda t}\phi$ and $\nu T_t = e^{\lambda t}\nu$.

From the expectation formula (1.7) of superprocesses, it is easy to see that, when Assumption (1.8) holds, $(e^{-\lambda t}X_t(\phi))_{t\geq 0}$ is a martingale. Denote by $L_1^+(\nu)$ the collection of $f \in \mathrm{p}\mathscr{B}(E)$ such that $\nu(f) < \infty$. We further assume that the mean semigroup $(T_t)_{t\geq 0}$ satisfies the following condition.

(1.9) There exists a map $(t,x,f) \mapsto H_t f(x)$ from $(0,\infty) \times E \times L_1^+(\nu)$ to $\mathbb{R}$ such that $T_t f(x) = e^{\lambda t}\phi(x)\nu(f)\big(1 + H_t f(x)\big)$ for any $t > 0$, $x \in E$ and $f \in L_1^+(\nu)$; $\sup_{x\in E, f\in L_1^+(\nu)} |H_t f(x)| < \infty$ for any $t > 0$; and $\lim_{t\to\infty} \sup_{x\in E, f\in L_1^+(\nu)} |H_t f(x)| = 0$.

The triplet (λ, ϕ, ν) satisfying (1.8) and (1.9) is unique. In fact, suppose that there is another triplet (λ', ϕ', ν') satisfying (1.8) and (1.9), then $e^{-\lambda t} T_t \mathbf{1}_E(x) \to \phi(x)$ and $e^{-\lambda' t} T_t \mathbf{1}_E(x) \to \phi'(x)$ as $t \to \infty$ for arbitrary $x \in \mathbb{E}$. This can only happen if $(\lambda', \phi') = (\lambda, \phi)$. Also it is clear that for every bounded Borel function f on E,

$$\nu'(f) = \lim_{t\to\infty} \frac{T_t f(x)}{e^{\lambda' t}\phi'(x)} = \lim_{t\to\infty} \frac{T_t f(x)}{e^{\lambda t}\phi(x)} = \nu(f),$$

which says that $\nu = \nu'$.

Assumptions similar to (1.8) and (1.9) are nowadays very common in the study of superprocesses [11,26,28,35,36,41] and other spatial Markovian branching processes [11,13,14,17,42]. In particular, we mention a very recent paper [11] where exactly the same assumptions were used to study the asymptotic behavior of the moments of both the superprocesses and the branching Markov processes. In general, it was explained in our earlier paper [28] that (1.8) and (1.9) hold true if the transition semigroup of the Markov process $(\xi_t)_{t\geq 0}$ is intrinsically ultracontractive. (For the definition and more details on the intrinsically ultracontractivity, see [20,42].) Some interesting examples satisfying (1.8) and (1.9) include multitype irreducible continuous-state branching processes and super-Brownian motions in a bounded Lipschitz domain. Many more examples can be found in [28, Section 1.3] and [41, Section 1.4]. We also mention here that one cannot apply our results to the super-Brownian motion on $\mathbb{R}^d$ because it does not satisfy (1.9).

Under the Assumptions (1.8) and (1.9), we say the superprocess is supercritical, critical, or subcritical, if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Since $(e^{-\lambda t} X_t(\phi))_{t\geq 0}$ is a nonnegative martingale, it has an almost sure finite limit. Thus roughly speaking, the process grows on average if $\lambda > 0$; decays on average if $\lambda < 0$; maintains a stabilizing average if $\lambda = 0$. The above definition is consistent with the similar notion for Galton–Watson processes. See [14] for similar definitions for branching Markov processes. In this paper, we are only concerned with the subcritical case, i.e., we will assume that

$$\lambda < 0. \tag{1.10}$$

Denote by $\mathbf{0}$ the null measure on E. Define $\mathcal{M}^o := \mathcal{M} \setminus \{\mathbf{0}\}$. It is possible that the superprocess is persistent in the sense that $\mathrm{P}_\mu(X_t \neq \mathbf{0}$ for all $t \geq 0) = 1$ for any $\mu \in \mathcal{M}^o$. Note that, if X is persistent, then it is trivial to consider X conditioned on non-extinction. So we use the following assumption to exclude this trivial case:

$$\mathrm{P}_\nu(X_t = \mathbf{0}) > 0, \quad t > 0. \tag{1.11}$$

It can be verified that $\mathrm{P}_\mu(X_t = \mathbf{0}) > 0$ for every $t > 0$ and $\mu \in \mathcal{M}^o$ under the above assumptions. See Remark 2.2 for more details. If the branching mechanism is spatially homogeneous, that is to say the function $\psi(x, z) = \psi(z)$ is independent of $x \in E$, then (1.11) is known to be equivalent to Grey's condition:

$$\text{There exists } z' > 0 \text{ such that } \psi(z) > 0 \text{ for all } z \geq z' \text{ and } \int_{z'}^{\infty} \psi(z)^{-1}\mathrm{d}z < \infty.$$

It is also known that if the branching mechanism $\psi(x, z)$ is bounded below by a spatially homogeneous branching mechanism $\tilde{\psi}$ satisfying Grey's condition, then (1.11) holds. See [41, Lemma 2.3] for more details.

Main results

Given $X_0 = \mu \in \mathcal{M}^o$, we denote by $\mathbf{Q}^{\mu}_{t,r}$ the distribution of X_t conditioned on $\{X_{t+r} \neq \mathbf{0}\}$, i.e.,

$$\mathbf{Q}^{\mu}_{t,r}(A) := \mathrm{P}_{\mu}(X_t \in A | X_{t+r} \neq \mathbf{0}), \quad t, r \in \mathbb{R}_+, A \in \mathscr{B}(\mathcal{M}).$$

Our first result is about the convergence of $\mathbf{Q}^{\mu}_{t,r}$ as $t \to \infty$ with r fixed.

Theorem 1.1. *For any $r \in \mathbb{R}_+$, there exists a probability measure $\mathbf{Q}_{\infty,r}$ on $\mathcal{M}$ such that, for any $\mu \in \mathcal{M}^o$, $\mathbf{Q}^{\mu}_{t,r}$ converges weakly to $\mathbf{Q}_{\infty,r}$ as $t \to \infty$.*

Notice that the case $r = 0$ of Theorem 1.1 was given in [28].

Our next result is about the convergence of $\mathbf{Q}^{\mu}_{t,r}$ as $r \to \infty$ with t fixed. For any given measurable space $(\tilde{\Omega}, \tilde{\mathscr{F}})$, we say a sequence of probability measures $(\mu_n)_{n=1}^{\infty}$ on $(\tilde{\Omega}, \tilde{\mathscr{F}})$ converge strongly (or converge setwise) to a probability measure μ on $(\tilde{\Omega}, \tilde{\mathscr{F}})$ if $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ for any $f \in \mathrm{b}\tilde{\mathscr{F}}$. An equivalent definition can be found in [16, Definition 1.4.1] .

Theorem 1.2. *For any $\mu \in \mathcal{M}^o$ and $t \in \mathbb{R}_+$, there exists a probability measure $\mathbf{Q}^{\mu}_{t,\infty}$ on $\mathcal{M}$ such that $\mathbf{Q}^{\mu}_{t,r}$ converges strongly to $\mathbf{Q}^{\mu}_{t,\infty}$ as $r \to \infty$.*

We then consider the limits of $\mathbf{Q}_{\infty,r}$ and $\mathbf{Q}^{\mu}_{t,\infty}$ as $r \to \infty$ and $t \to \infty$ respectively. Define $\mathcal{E} \in [0, \infty]$ by

$$\mathcal{E} := \int_E \nu(\mathrm{d}x) \int_{(0,\infty)} u\phi(x) \log^+\big(u\phi(x)\big)\pi(x, \mathrm{d}u), \tag{1.12}$$

where $\log^+ z := \max(\log z, 0)$ for every $z > 0$.

Theorem 1.3. *Let $\mu \in \mathcal{M}^o$ be arbitrary. The following five statements are equivalent:*

(1) $\mathcal{E} < \infty$.
(2) $\int_{\mathcal{M}} \eta(\phi)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta) < \infty$.
(3) $\liminf_{t\to\infty} e^{-\lambda t}\mathrm{P}_{\mu}(X_t \neq \mathbf{0}) > 0$.
(4) $\mathbf{Q}_{\infty,r}$ *converges strongly as* $r \to \infty$.
(5) $\mathbf{Q}^{\mu}_{t,\infty}$ *converges weakly as* $t \to \infty$.

Theorem 1.3 can be considered as an analog of Heathcote, Seneta and Vere-Jones' $L \log L$ theorem for superprocesses. In particular, the condition $\mathcal{E} < \infty$ is an analog of the $L \log L$ condition (1.3). The same condition has already appeared in [26] where the first three authors of this paper studied the asymptotic behavior of supercritical superdiffusions. Here, in the subcritical setting, $\mathcal{E} < \infty$ is shown to be equivalent to the exponential decay of the survival probability. (We are using 'liminf' in the third statement because we are not assuming, a priori, existence of the limit. In fact, it is made clear in the next theorem that the limit does exist under the condition $\mathcal{E} < \infty$.) The equivalence of $\mathcal{E} < \infty$ to the existence of the two types of iterated limits in (4) and (5) is in parallel to Pakes' iterated limit theorem [31, Theorems 2.2 and 2.3]. Notice that in (4) and (5), the sense of convergence for these two double limits are different. This difference is not present in the context of Galton–Watson processes because the weak convergence and the strong convergence are equivalent for the probability distributions on the discrete space $\mathbb{N}$. The following theorem says that the two iterated limits coincide, which is in parallel to Pakes' result on the Galton–Watson branching processes. We also give the weak limit for $\mathbf{Q}^{\mu}_{t,r}$ when t and r converge to ∞ together. It seems that this latter result has not been explored before for other Markov branching processes.

Theorem 1.4. *Suppose that $\mathcal{E} < \infty$. Then there exists a probability measure $\mathbf{Q}_{\infty,\infty}$ on $\mathcal{M}$ such that the following statements hold for any $\mu \in \mathcal{M}^o$:*

(1) $\mathbf{Q}_{\infty,\infty}(\mathrm{d}\eta) = \eta(\phi)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta)/\int_{\mathcal{M}} \eta(\phi)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta)$.
(2) $\lim_{t\to\infty} e^{-\lambda t}\mathrm{P}_\mu(X_t \neq \mathbf{0}) = \mu(\phi)/\int_{\mathcal{M}} \eta(\phi)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta)$.
(3) $\mathbf{Q}_{\infty,r}$ converges strongly to $\mathbf{Q}_{\infty,\infty}$ as $r \to \infty$.
(4) $\mathbf{Q}^{\mu}_{t,\infty}$ converges weakly to $\mathbf{Q}_{\infty,\infty}$ as $t \to \infty$.
(5) $\mathbf{Q}^{\mu}_{t,r}$ converges weakly to $\mathbf{Q}_{\infty,\infty}$ as $t, r \to \infty$.

Remark 1.5. If the space E only contains one point, i.e. $E = \{x\}$, the superprocess X degenerates to a continuous-state branching process. In this special case, Assumptions (1.8) and (1.9) hold automatically, and the main results of this paper have already been given by [12,22] and [24].

Remark 1.6. If $E = \{x_1, \ldots, x_n\}$ is a finite set and the E-valued Markov chain $(\xi_t)_{t\geq 0}$ is irreducible, then the superprocess X degenerates to an irreducible multitype continuous state branching process. In this case, one can verify using the Perron–Frobenius theory that the Assumptions (1.8) and (1.9) hold. If one further assumes that the kernel $\pi(x, \mathrm{d}u) = 0$, then our results (3) and (4) of Theorem 1.4 already appeared in [2, Theorem 3.7].

Remark 1.7. When the branching mechanism ψ is spatially homogeneous, our Theorem 1.2 is an immediate corollary of [25, Theorem 6.8].

Overview of the method

Note that the main results Theorems 1.1–1.4 depend only on the transition semigroup $(Q_t)_{t\geq 0}$ of the superprocess $(X_t)_{t\geq 0}$. Therefore, we can work on any specific realization of $(X_t)_{t\geq 0}$ without loss of generality. According to Lemma A.1, $(Q_t)_{t\geq 0}$ is a Borel semigroup on $\mathcal{M}$. This and [25, Theorem A.33] allow us to realize the superprocess on the space of $\mathcal{M}$-valued right continuous paths. To be more precise, we can, and will, assume the following statements hold throughout the rest of the paper.

(1.13) Ω is the space of $\mathcal{M}$-valued right continuous functions on $\mathbb{R}_+$.
(1.14) $(X_t)_{t\geq 0}$ is the coordinate process of the path space Ω.
(1.15) $(\theta_t)_{t\geq 0}$ are the shift operators on the path space Ω.
(1.16) $\mathscr{F}_t = \sigma(X_s : s \in [0, t])$ and $\mathscr{F} = \sigma(X_s : s \in \mathbb{R}_+)$.
(1.17) For any $\mu \in \mathcal{M}$, P_μ is the probability measure on $(\Omega, \mathscr{F})$ so that under P_μ, $X_0 = \mu$ almost surely and that $(X_t)_{t\geq 0}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$.

Note that for any $H \in b\mathscr{F}$, $\mu \mapsto \mathrm{P}_\mu(H)$ is a measurable function on $\mathcal{M}$. For any probability measure $\mathbf{P}$ on $\mathcal{M}$, we define a probability measure $\mathbf{P}\mathrm{P}$ on $(\Omega, \mathscr{F})$ by

$$(\mathbf{P}\mathrm{P})(A) := \int_{\mathcal{M}} \mathrm{P}_\mu(A)\mathbf{P}(\mathrm{d}\mu), \quad A \in \mathscr{F}.$$

Denote by $(\mathscr{F}^{\mathrm{a}}, (\mathscr{F}^{\mathrm{a}}_t)_{t\geq 0})$ the augmentation of $(\mathscr{F}, (\mathscr{F}_t)_{t\geq 0})$ by the system of probability measures $\{\mathbf{PP} : \mathbf{P}$ is a probability measure on $\mathcal{M}\}$. Then, according to [25, Lemma A.33],

$$X := (\Omega, \mathscr{F}^{\mathrm{a}}, (\mathscr{F}^{\mathrm{a}}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, (\mathrm{P}_\mu)_{\mu\in\mathcal{M}})$$

is a Borel right process with transition semigroup $(Q_t)_{t\geq 0}$, i.e., a (ξ, ψ)-superprocess.

We already proved Theorem 1.1 in the case $r = 0$ in [28]. For the case $r > 0$, we will give a stronger result by considering the shifted two-sided process $(X_{t+u})_{u\in\mathbb{R}}$ with the convention $X_s := \mathbf{0}$ for $s < 0$. We will show in Proposition 2.5 that this two-sided process, conditioned on $\{X_t \neq \mathbf{0}\}$, has a limiting process $(Y_u)_{u\in\mathbb{R}}$ when $t \to \infty$. We will obtain this result by analyzing the Laplace transform of the shifted two-sided process.

We will also establish a stronger version of Theorem 1.2 by considering the (non-shifted) process $(X_u)_{u\geq 0}$ under the condition $\{X_t \neq \mathbf{0}\}$. We will show in Proposition 3.3 that this process has a limiting process $(\widetilde{X}_u)_{u\geq 0}$ when $t \to \infty$. We obtain this stronger result by a martingale change of measure method. The limiting process $(\widetilde{X}_u)_{u\geq 0}$ is interpreted as a superprocess conditioned on living forever, and is referred to as the Q-process. We mention here that the Q-process $(\widetilde{X}_u)_{u\geq 0}$ has a different law compared to the process $(Y_t)_{t\geq 0}$ above. This Q-process also arises in another type of conditioning, see [5]. The study of the Q-process can be traced back to Lamperti and Ney [23] where they considered the Q-process for Galton–Watson processes. For studies on the Q-processes of other models, we refer our readers to [30,32] and the references therein.

For the proofs of Theorems 1.3 and 1.4, we use the spine decomposition theorem for superprocesses. Roughly speaking, the Q-process $(\widetilde{X}_u)_{u\geq 0}$ can be decomposed in terms of an immortal particle which moves according to a Markov process and generates pieces of mass evolving according to the law of the unconditioned superprocess. This representation for the superprocesses was first obtained by [9], and developed and generalized into the spine decomposition theorem by [6–8,26,36,39]. Under Assumption (1.9), this immortal particle will converge in law to its ergodic equilibrium, and the quantitative information about the Q-process can be obtained using the ergodic theorem.

Our proofs of Theorems 1.3 and 1.4 adopt a method which can be traced back to Lyons, Pemantle and Peres [29] where they gave a probabilistic proof of Heathcote, Seneta and Vere-Jones' $L\log L$ theorem for Galton–Watson processes. Let us give some intuition here. Note that for the spine decomposition of the Q-process, each piece of mass being generated will vanish eventually since they are subcritical and non-persistent. When the $L\log L$ condition holds, the rate at which masses are created is smaller than the rate at which masses vanish, and the Q-process will converge to an equilibrium state. When the $L\log L$ condition does not hold, the rate at which masses are created is bigger than the rate at which they vanish, and the Q-process will not converge to any equilibrium because it accumulates more and more mass.

Organization of the paper

In Section 2, we give the proof of Theorem 1.1. In Section 3, we give the proof of Theorem 1.2. Section 4 gives the proofs of Theorems 1.3 and 1.4, and summarizes the spine decomposition theorems in Lemmas 4.11 and 4.17. In Appendix, we gather the proofs of several technical lemmas.

2. A two-sided process: Proof of Theorem 1.1

We first recall some basic results from [28]. Define

$$v_t(x) := -\log \mathrm{P}_{\delta_x}(X_t = \mathbf{0}), \quad t \geq 0, x \in E. \tag{2.1}$$

From (1.6) and the monotone convergence theorem, we get that

$$\mu(v_t) = -\log \mathrm{P}_\mu(X_t = \mathbf{0}), \quad \mu \in \mathcal{M}, t \geq 0. \tag{2.2}$$

In particular, from (1.11), we have that $\nu(v_t) < \infty$ for $t > 0$. The following lemma, which is a corollary of [28, Proposition 2.2], entails that $\{v_t : t > 0\} \subset \mathrm{bp}\mathscr{B}(E)$.

Lemma 2.1. *For any $t > 0$ and $x \in E$, $v_t(x) = \phi(x)\nu(v_t)(1 + C_1(t, x))$, where $C_1(t, x) \in \mathbb{R}$ satisfies that $\lim_{t\to\infty} \sup_{x\in E} |C_1(t, x)| = 0$.*

Remark 2.2. For any $t > 0$ and $\mu \in \mathcal{M}$, from Lemma 2.1 we have $\mu(v_t) < \infty$; and therefore by (2.2), $\mathrm{P}_\mu(X_t = \mathbf{0}) = e^{-\mu(v_t)} > 0$.

We will also use the following fundamental fact for the subcritical superprocess X. It can be verified, for example, using (2.2) and [28, (3.39)] .

Lemma 2.3. *For any $\mu \in \mathcal{M}$, $\lim_{t\to\infty} \mathrm{P}_\mu(X_t = \mathbf{0}) = 1$.*

In [28], we already showed that there exists a probability measure $\mathbf{Q}_{\infty,0}$ on $\mathcal{M}$ such that for every $\mu \in \mathcal{M}^o$, $\mathbf{Q}^\mu_{t,0}$ converges weakly to $\mathbf{Q}_{\infty,0}$ as $t \to \infty$. $\mathbf{Q}_{\infty,0}$ is known as the Yaglom limit of the superprocess X. It was also proved there that $\mathbf{Q}_{\infty,0}$ is the quasi-stationary distribution for $(X_t)_{t\geq 0}$ with extinction rate $-\lambda$, i.e.,

$$(\mathbf{Q}_{\infty,0}\mathrm{P})(X_r \in \mathrm{d}\mu | X_r \neq \mathbf{0}) = \mathbf{Q}_{\infty,0}(\mathrm{d}\mu) \tag{2.3}$$

and

$$(\mathbf{Q}_{\infty,0}\mathrm{P})(X_r \neq \mathbf{0}) = e^{\lambda r} > 0. \tag{2.4}$$

(2.4) allows us to define a probability measure $\mathbf{Q}_{\infty,r}$ on $\mathcal{M}$ for any $r \geq 0$ such that

$$\mathbf{Q}_{\infty,r}[F] = (\mathbf{Q}_{\infty,0}\mathrm{P})[F(X_0)|X_r \neq \mathbf{0}], \quad F \in \mathrm{b}\mathscr{B}(\mathcal{M}). \tag{2.5}$$

We will prove Theorem 1.1 by showing that $\mathbf{Q}_{\infty,r}$ is the weak limit of $\mathbf{Q}^\mu_{t,r}$ when $t \to \infty$ for any $\mu \in \mathcal{M}^o$. In fact, we can prove a proposition which is stronger than Theorem 1.1. To formulate this proposition, we first prove a lemma. We will use the convention $X_t := \mathbf{0}$ for $t < 0$.

Lemma 2.4. *There exists a two-sided $\mathcal{M}$-valued process $(Y_u)_{u\in\mathbb{R}}$ on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbb{P})$ such that for any $t > 0$, the process $(X_{t+u})_{u\geq -t}$ under $(\mathbf{Q}_{\infty,0}\mathrm{P})(\cdot|X_t \neq \mathbf{0})$ has the same finite-dimensional distributions as $(Y_u)_{u\geq -t}$.*

Proof. We say G is a finite-dimensional $[0, \infty]$-valued linear functional of $\mathcal{M}$-valued two-sided paths if the following statement holds.

(2.6) There exist a natural number n, $\{u_i : i = 1, \ldots, n\} \subset \mathbb{R}$, and a list of $[0, \infty]$-valued Borel functions $(f_i)_{i=1}^n$ on E, such that $G(w) = \sum_{i=1}^n w_{u_i}(f_i)$ for every $\mathcal{M}$-valued two-sided path $w = (w_u)_{u\in\mathbb{R}}$.

Fix an arbitrary finite-dimensional $[0, \infty]$-valued linear functional G as above. For any $s \in \mathbb{R}$, define $G_s(w) := G(w_{s+\cdot})$ for any $\mathcal{M}$-valued two-sided path w. Then G_s is also a finite-dimensional $[0, \infty]$-valued linear functional. Fix a time $s \geq 0$ large enough so that $s + u_i \geq 0$

for every $i = 1, \dots, n$. Since X is a time-homogeneous Markov process, using (2.3) and (2.4), we have that for any $t \geq s$,

$$\begin{aligned}
&(\mathbf{Q}_{\infty,0}\mathrm{P})(e^{-G_t(X)}|X_t \neq \mathbf{0}) = e^{-\lambda t} \cdot (\mathbf{Q}_{\infty,0}\mathrm{P})[e^{-G_t(X)}\mathbf{1}_{\{X_t\neq\mathbf{0}\}}] \\
&= e^{-\lambda t} \cdot (\mathbf{Q}_{\infty,0}\mathrm{P})\big[\mathbf{1}_{\{X_{t-s}\neq\mathbf{0}\}}\mathrm{P}_{X_{t-s}}[e^{-G_s(X)}\mathbf{1}_{\{X_s\neq\mathbf{0}\}}]\big] \\
&= e^{-\lambda s} \cdot (\mathbf{Q}_{\infty,0}\mathrm{P})\Big[\mathrm{P}_{X_{t-s}}[e^{-G_s(X)}\mathbf{1}_{\{X_s\neq\mathbf{0}\}}]\Big|X_{t-s} \neq \mathbf{0}\Big] \\
&= e^{-\lambda s} \cdot (\mathbf{Q}_{\infty,0}\mathrm{P})\big[e^{-G_s(X)}\mathbf{1}_{\{X_s\neq\mathbf{0}\}}\big] = (\mathbf{Q}_{\infty,0}\mathrm{P})(e^{-G_s(X)}|X_s \neq \mathbf{0}).
\end{aligned}$$

In other words, given a finite subset $U = \{u_i : i = 1, \dots, n\} \subset \mathbb{R}$ and a large enough $t \geq 0$, the $\mathcal{M}$-valued random vector $(X_{t+u})_{u\in U}$ under the probability $(\mathbf{Q}_{\infty,0}\mathrm{P})(\cdot|X_t \neq \mathbf{0})$ has a distribution, denoted by $\mathcal{D}_U$, which is independent of the choice of t. Using the Markov property, it is easy to verify that this family of finite-dimensional distributions $\mathcal{D} := \{\mathcal{D}_U :$ U is a finite subset of $\mathbb{R}\}$ satisfies the consistency condition for the Kolmogorov extension theorem. Therefore, there exists a two-sided $\mathcal{M}$-valued process $(Y_u)_{u\in\mathbb{R}}$ whose finite-dimensional distributions are given by $\mathcal{D}$.

It is a routine to verify that $(Y_u)_{u\in\mathbb{R}}$ satisfies the desired properties of this lemma. □

Recall that the two-sided indexed process $(X_t)_{t\in\mathbb{R}}$ is defined with the convention that $X_s := \mathbf{0}$ for $s < 0$.

Proposition 2.5. *For any $\mu \in \mathcal{M}^o$, when $t \to \infty$, the $\mathcal{M}$-valued two-sided process $(X_{t+u})_{u\in\mathbb{R}}$ under $\mathrm{P}_\mu(\cdot|X_t \neq \mathbf{0})$ converges to the process $(Y_u)_{u\in\mathbb{R}}$, given in Lemma 2.4, in the sense of finite-dimensional distributions.*

We first explain that the above proposition is indeed stronger than Theorem 1.1.

Proof of Theorem 1.1. Fix arbitrary $r \geq 0$ and $F \in \mathrm{b}C(\mathcal{M})$. Using Proposition 2.5, Lemma 2.4 and the definition (2.5) of the probability $\mathbf{Q}_{\infty,r}$, we have

$$\begin{aligned}
&\mathbf{Q}^\mu_{t,r}[F] = \mathrm{P}_\mu[F(X_{(t+r)-r})|X_{t+r} \neq 0] \\
&\xrightarrow[t\to\infty]{} \mathbb{P}[F(Y_{-r})] = (\mathbf{Q}_{\infty,0}\mathrm{P})[F(X_0)|X_r \neq \mathbf{0}] = \mathbf{Q}_{\infty,r}[F]
\end{aligned}$$

as desired. □

Before we prove Proposition 2.5, we first present the following two lemmas.

Lemma 2.6. *For any $\mu \in \mathcal{M}^o$ and $[0, \infty]$-valued Borel function f on E,*

$$\int_{\mathcal{M}} e^{-\eta(f)}\mathbf{Q}^\mu_{t,0}(\mathrm{d}\eta) \xrightarrow[t\to\infty]{} \int_{\mathcal{M}} e^{-\eta(f)}\mathbf{Q}_{\infty,0}(\mathrm{d}\eta).$$

Lemma 2.6 follows from [28, Proposition 2.3 & (2.9)] .

Lemma 2.7. *For any $\eta, \mu \in \mathcal{M}^o$ and $s \in \mathbb{R}_+$, it holds that*

$$\frac{\mathrm{P}_\eta(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})} \xrightarrow[t\to\infty]{} \frac{e^{-\lambda s}\eta(\phi)}{\mu(\phi)}.$$

Proof. Let $v_t(x)$ be given as in (2.1). It follows from [28, (3.20)] that for any real number s,

$$\lim_{t\to\infty} \frac{\nu(v_{t+s})}{\nu(v_t)} = e^{\lambda s}. \tag{2.7}$$

(Note that $v_t(x) = (V_t\infty)(x)$ in the language of [28].) Therefore we have from (2.7), Lemma 2.1 and the bounded convergence theorem that, for any $\eta, \mu \in \mathcal{M}^o$ and $s \geq 0$,

$$\lim_{t\to\infty} \frac{\eta(v_{t-s})}{\mu(v_t)} = \lim_{t\to\infty} \frac{\nu(v_{t-s}) \int \phi(x)(1 + C_1(t-s,x))\eta(\mathrm{d}x)}{\nu(v_t) \int \phi(x)(1 + C_1(t,x))\mu(\mathrm{d}x)} = \frac{e^{-\lambda s}\eta(\phi)}{\mu(\phi)}.$$

Thus we have by (2.2) and [28, (3.39)] that,

$$\lim_{t\to\infty} \frac{\mathrm{P}_\eta(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})} = \lim_{t\to\infty} \frac{1 - e^{-\eta(v_{t-s})}}{1 - e^{-\mu(v_t)}} = \lim_{t\to\infty} \frac{\eta(v_{t-s})}{\mu(v_t)} = \frac{e^{-\lambda s}\eta(\phi)}{\mu(\phi)}. \qquad \square$$

Proof of Proposition 2.5. To prove the convergence of the processes, we verify the convergence of all the Laplace transforms of the finite-dimensional linear functional. Fix an arbitrary $\mu \in \mathcal{M}^o$ and an arbitrary finite-dimensional $[0,\infty]$-valued linear functional G defined in (2.6). It can be verified using (1.6), the Markov property and induction that there exists a $[0,\infty]$-valued Borel function v_G on E, which depends on the choice of G but not on μ, such that $\mathrm{P}_\mu[\exp\{-G(X)\}] = \exp\{-\mu(v_G)\}$. Fix a time $s \geq 0$ large enough so that $s + u_i \geq 0$ for every $i = 1, \dots, n$. From the Markov property, we can verify that for any $t \geq s$,

$$\begin{aligned}
\mathrm{P}_\mu(e^{-G(X_{t+\cdot})}|X_t \neq \mathbf{0}) &= \frac{\mathrm{P}_\mu[e^{-G(X_{t+\cdot})}\mathbf{1}_{\{X_t\neq\mathbf{0}\}}]}{\mathrm{P}_\mu(X_t \neq 0)} = \frac{\mathrm{P}_\mu\big[\mathrm{P}_{X_{t-s}}[e^{-G(X_{s+\cdot})}\mathbf{1}_{\{X_s\neq\mathbf{0}\}}]\big]}{\mathrm{P}_\mu(X_t \neq 0)} \\
&= \frac{\mathrm{P}_\mu(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})}\mathrm{P}_\mu\Big[\mathrm{P}_{X_{t-s}}[e^{-G(X_{s+\cdot})}\mathbf{1}_{\{X_s\neq\mathbf{0}\}}]\Big|X_{t-s} \neq \mathbf{0}\Big] \\
&= \frac{\mathrm{P}_\mu(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})}\mathrm{P}_\mu\Big[e^{-X_{t-s}(v_{G_s})} - e^{-X_{t-s}(v_{\tilde{G}_s})}\Big|X_{t-s} \neq \mathbf{0}\Big]
\end{aligned} \tag{2.8}$$

where

$$G_s(w) = G(w_{s+\cdot}) = \sum_{i=1}^n w_{s+u_i}(f_i),$$

and

$$\tilde{G}_s(w) := G(w_{s+\cdot}) + w_s(\infty\mathbf{1}_E) = \sum_{i=1}^n w_{s+u_i}(f_i) + w_s(\infty\mathbf{1}_E),$$

are finite-dimensional $[0,\infty]$-valued linear functionals for $\mathcal{M}$-valued two-sided paths w. In fact, (2.8) holds because

$$e^{-G(X_{s+\cdot})}\mathbf{1}_{\{X_s=\mathbf{0}\}} = e^{-[G(X_{s+\cdot})+X_s(\infty\mathbf{1}_E)]} = e^{-\tilde{G}_s(X)},$$

and that for any $\eta \in \mathcal{M}$,

$$\mathrm{P}_\eta[e^{-G(X_{s+\cdot})}\mathbf{1}_{\{X_s\neq\mathbf{0}\}}] = \mathrm{P}_\eta[e^{-G_s(X)} - e^{-\tilde{G}_s(X)}] = e^{-\eta(v_{G_s})} - e^{-\eta(v_{\tilde{G}_s})}.$$

Now we have that

$$\begin{aligned}
&\lim_{t\to\infty} \mathrm{P}_\mu(e^{-G(X_{t+\cdot})}|X_t \neq \mathbf{0}) \\
&\overset{(2.8)}{=} \lim_{t\to\infty} \frac{\mathrm{P}_\mu(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})} \lim_{t\to\infty} \mathrm{P}_\mu\Big[e^{-X_{t-s}(v_{G_s})} - e^{-X_{t-s}(v_{\tilde{G}_s})}\Big|X_{t-s} \neq \mathbf{0}\Big] \\
&\overset{\text{Lemmas 2.6 and 2.7}}{=} e^{-\lambda s}\int_{\mathcal{M}} (e^{-\eta(v_{G_s})} - e^{-\eta(v_{\tilde{G}_s})})\mathbf{Q}_{\infty,0}(\mathrm{d}\eta)
\end{aligned}$$

$$= e^{-\lambda s} \int_{\mathcal{M}} \mathrm{P}_\eta [e^{-G(X_{s+\cdot})} \mathbf{1}_{\{X_s \neq \mathbf{0}\}}] \mathbf{Q}_{\infty,0}(\mathrm{d}\eta)$$
$$\overset{(2.4)}{=} (\mathbf{Q}_{\infty,0}\mathrm{P})[e^{-G(X_{s+\cdot})} | X_s \neq \mathbf{0}] \overset{\text{Lemma } 2.4}{=} \mathbb{E}[e^{-G(Y)}].$$

This and [25, Theorem 1.18] imply the desired result. □

3. The Q-process: Proof of Theorem 1.2

According to (1.7) and (1.8), we have $\mathrm{P}_\mu[X_t(\phi)] = e^{\lambda t}\mu(\phi) \in (0, \infty)$ for any $t \in \mathbb{R}_+$ and $\mu \in \mathcal{M}^o$. This allows us to define, for any $t \in \mathbb{R}_+$ and $\mu \in \mathcal{M}^o$, a probability measure $\mathbf{Q}^\mu_{t,\infty}$ on $(\mathcal{M}, \mathscr{B}(\mathcal{M}))$ such that

$$\mathbf{Q}^\mu_{t,\infty}[F] = \mathrm{P}_\mu \Big[\frac{X_t(\phi)}{e^{\lambda t}\mu(\phi)} \cdot F(X_t) \Big], \quad F \in \mathrm{b}\mathscr{B}(\mathcal{M}). \tag{3.1}$$

We will prove Theorem 1.2 by showing that $\mathbf{Q}^\mu_{t,\infty}$ is the strong limit of $\mathbf{Q}^\mu_{t,r}$ when $r \to \infty$.

In fact, we can prove a result which is stronger than Theorem 1.2. Before presenting this result, we introduce some notation and give a technical lemma. Denote by $\widetilde{\mathrm{P}}^{(t)}_\mu$ the law of the $\mathcal{M}$-valued process $(X_r)_{r\geq 0}$ under $\mathrm{P}_\mu(\cdot | X_t \neq \mathbf{0})$. More precisely, for any $t \geq 0$ and $\mu \in \mathcal{M}^o$, define $\widetilde{\mathrm{P}}^{(t)}_\mu$ as the probability measure on Ω such that

$$\widetilde{\mathrm{P}}^{(t)}_\mu [H] = \mathrm{P}_\mu [H | X_t \neq \mathbf{0}], \quad H \in \mathrm{bp}\mathscr{F}.$$

The following lemma can be verified from [43, Theorem 62.19] .

Lemma 3.1. *For any $\mu \in \mathcal{M}^o$, there exists a unique probability measure $\widetilde{\mathrm{P}}^{(\infty)}_\mu$ on $(\Omega, \mathscr{F})$ such that for any $s \geq 0$ and $H \in \mathrm{p}\mathscr{F}_s$, it holds that*

$$\widetilde{\mathrm{P}}^{(\infty)}_\mu [H] = \mathrm{P}_\mu \Big[\frac{X_s(\phi)}{e^{\lambda s}\mu(\phi)} \cdot H \Big].$$

Remark 3.2. From Lemma 3.1 we have $\widetilde{\mathrm{P}}^{(\infty)}_\mu (X_t \in \cdot) = \mathbf{Q}^\mu_{t,\infty}(\cdot)$ for every $t \geq 0$.

We say a family $(\mathrm{R}^{(t)})_{t\geq 0}$ of probability measures on Ω converges, as $t \to \infty$, locally strongly to a probability measure R on Ω if for any $s \geq 0$ and $H \in \mathrm{bp}\mathscr{F}_s$ it holds that $\lim_{t\to\infty} \mathrm{R}^{(t)}(H) = \mathrm{R}(H)$. The following proposition is the main result of this section, and it is stronger than Theorem 1.2.

Proposition 3.3. *For any $\mu \in \mathcal{M}^o$, $\widetilde{\mathrm{P}}^{(t)}_\mu$ converges to $\widetilde{\mathrm{P}}^{(\infty)}_\mu$ locally strongly as $t \to \infty$.*

Proof of Theorem 1.2. We can verify using Lemma 3.1, Proposition 3.3 that for any $t \in \mathbb{R}_+$, $\mu \in \mathcal{M}^o$ and $F \in \mathrm{b}\mathscr{B}(\mathcal{M})$,

$$\mathbf{Q}^\mu_{t,r}[F] = \widetilde{\mathrm{P}}^{(t+r)}_\mu [F(X_t)] \xrightarrow[r\to\infty]{} \widetilde{\mathrm{P}}^{(\infty)}_\mu [F(X_t)] = \mathbf{Q}^\mu_{t,\infty}[F]. \quad \square$$

Before proving Proposition 3.3, we first prove the following lemma.

Lemma 3.4. *For any $\mu \in \mathcal{M}^o$ and $s \in \mathbb{R}_+$, it holds that*

$$\limsup_{t\to\infty} \sup_{\eta\in\mathcal{M}^o} \frac{1}{\eta(\phi)} \frac{\mathrm{P}_\eta(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})} < \infty.$$

Proof. Let $v_t(x)$ be given as in (2.1). By (2.2) and Lemma 2.1, we have for any $\mu, \eta \in \mathcal{M}^o$, $s \geq 0$ and $t > s$,

$$\begin{aligned}\frac{\mathrm{P}_\eta(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})} &\leq \frac{\eta(v_{t-s})}{\mu(v_t)} \frac{\mu(v_t)}{1 - e^{-\mu(v_t)}} \\ &\leq \frac{\nu(v_{t-s})}{\nu(v_t)} \frac{(1 + \sup_{x \in E} |C_1(t-s,x)|)\eta(\phi)}{\int (1 + C_1(t,x))\phi(x)\mu(\mathrm{d}x)} \frac{\mu(v_t)}{1 - e^{-\mu(v_t)}}.\end{aligned}$$

Using this, [28, (3.20)], Lemma 2.1, the bounded convergence theorem and [28, (3.39)]

$$\sup_{\eta \in \mathcal{M}^o} \frac{1}{\eta(\phi)} \frac{\mathrm{P}_\eta(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})} \leq \frac{\nu(v_{t-s})}{\nu(v_t)} \frac{(1 + \sup_{x \in E} |C_1(t-s,x)|)}{\int (1 + C_1(t,x))\phi(x)\mu(\mathrm{d}x)} \frac{\mu(v_t)}{1 - e^{-\mu(v_t)}} \xrightarrow[t \to \infty]{} \frac{e^{-s\lambda}}{\mu(\phi)}$$

which implies the desired result. □

Proof of Proposition 3.3. Fix arbitrary $\mu \in \mathcal{M}^o$, $s \geq 0$ and $H \in \mathrm{b}\mathscr{F}_s$. It follows from Lemma 3.4 that there exist $C_2(\mu, s) > 0$ and $t_0 > s$ such that for any $t \geq t_0$, P_μ-almost surely,

$$\frac{\mathrm{P}_{X_s}(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})} \leq C_2(\mu, s) X_s(\phi). \tag{3.2}$$

Using the Markov property, Lemma 2.7, (3.2) and the dominated convergence theorem, we have

$$\begin{aligned}\widetilde{\mathrm{P}}_\mu^{(t)}[H] &= \frac{\mathrm{P}_\mu[H \cdot \mathbf{1}_{\{X_t \neq \mathbf{0}\}}]}{\mathrm{P}_\mu(X_t \neq \mathbf{0})} = \mathrm{P}_\mu\Big[H \cdot \frac{\mathrm{P}_{X_s}(X_{t-s} \neq \mathbf{0})}{\mathrm{P}_\mu(X_t \neq \mathbf{0})}\Big] \\ &\xrightarrow[t \to \infty]{} \mathrm{P}_\mu\Big[H \cdot \frac{e^{-\lambda s} X_s(\phi)}{\mu(\phi)}\Big] = \widetilde{\mathrm{P}}_\mu^{(\infty)}[H]\end{aligned}$$

as desired. □

4. $L \log L$ Type results: Proofs of Theorems 1.3 and 1.4

Our proofs of Theorems 1.3 and 1.4 are separated into the following five propositions whose proofs are postponed to Subsections 4.1, 4.2, 4.3, 4.4, and 4.5, respectively.

Proposition 4.1. *There exists a constant $\mathcal{K} \in [0, \infty)$ which is independent of the initial value $\mu \in \mathcal{M}^o$ such that*

$$\lim_{t \to \infty} e^{-\lambda t} \mathrm{P}_\mu(X_t \neq \mathbf{0}) = \mathcal{K} \mu(\phi), \quad \mu \in \mathcal{M}^o. \tag{4.1}$$

In the remainder of this paper, $\mathcal{K}$ always denotes the constant above.

Proposition 4.2. $\mathcal{K} > 0$ *if and only if* $\mathcal{E} < \infty$.

Proposition 4.3. *It holds that*

$$\int_{\mathcal{M}} \eta(\phi) \mathbf{Q}_{\infty,0}(\mathrm{d}\eta) = \mathcal{K}^{-1}.$$

When $\mathcal{K} > 0$, Proposition 4.3 allows us to consider the (unique) probability measure $\mathbf{Q}_{\infty,\infty}$ on $\mathcal{M}$ satisfying

$$\mathbf{Q}_{\infty,\infty}(F) = \int_{\mathcal{M}} F(\eta) \cdot \mathcal{K}\eta(\phi) \mathbf{Q}_{\infty,0}(\mathrm{d}\eta), \quad F \in \mathrm{b}\mathscr{B}(\mathcal{M}). \tag{4.2}$$

Proposition 4.4. *Let $\mu \in \mathcal{M}^o$ be arbitrary. If $\mathcal{K} > 0$, then $\mathbf{Q}^{\mu}_{t,\infty}$ converges weakly to $\mathbf{Q}_{\infty,\infty}$ as $t \to \infty$. If $\mathcal{K} = 0$, then $\mathbf{Q}^{\mu}_{t,\infty}$ does not converge weakly as $t \to \infty$.*

Proposition 4.5. *If $\mathcal{K} > 0$, then $\mathbf{Q}_{\infty,r}$ converges strongly to $\mathbf{Q}_{\infty,\infty}$ as $r \to \infty$. If $\mathcal{K} = 0$, then $\mathbf{Q}_{\infty,r}$ does not converge strongly as $r \to \infty$.*

Proposition 4.6. *If $\mathcal{E} < \infty$, then for any $\mu \in \mathcal{M}^o$ and non-negative continuous function f on E,*

$$\lim_{t,r\to\infty} \mathrm{P}_{\mu}\left(e^{-X_t(f)}|X_{t+r} \neq 0\right) = \int e^{-\eta(f)}\mathbf{Q}_{\infty,\infty}(\mathrm{d}\eta).$$

Proofs of Theorems 1.3 and 1.4. The desired results can be verified directly from Propositions 4.1–4.6. □

4.1. Exponential decay of the survival probability

In order to prove Proposition 4.1, we need the spine decomposition theorem for superprocesses. To formulate this theorem, we first introduce the *Kuznetsov measures* via the following lemma which is proved in [25, Section 8.4] .

Lemma 4.7. *There exists a unique σ-finite kernel $\mathbb{N} = (\mathbb{N}_x(A) : x \in E, A \in \mathscr{F})$ from $(E, \mathscr{B}(E))$ to $(\Omega, \mathscr{F})$ such that*

(1) $\mathbb{N}_x(X_0 \neq \mathbf{0}) = 0$ for any $x \in E$;
(2) $\mathbb{N}_x(X_t = \mathbf{0}$ for all $t \geq 0) = 0$ for every $x \in E$; and
(3) for any $\mu \in \mathcal{M}$, if $\mathbf{N}$ is a Poisson random measure on Ω with intensity $\mu\mathbb{N}$, then $(\mu\mathbf{1}_{\{0\}}(t) + \mathbf{N}(X_t)\mathbf{1}_{(0,\infty)}(t))_{t\geq 0}$ is an $\mathcal{M}$-valued stochastic process of the same finite dimensional distributions as a (ξ, ψ)-superprocess with initial value μ. Here $\mathbf{N}(X_t) = \int_{\Omega} X_t(\omega)\mathbf{N}(\mathrm{d}\omega) = \int_{\Omega} \omega_t\mathbf{N}(\mathrm{d}\omega)$, $t > 0$.

The family of σ-finite measures $(\mathbb{N}_x)_{x\in E}$ is known as the Kuznetsov measures of X. Note that those measures are typically not finite. One way to use them is to transform them into probability measures. Notice that from Lemma 4.7(3) and Campbell's theorem,

$$(\mu\mathbb{N})[X_t(f)] = \mathrm{P}_{\mu}[X_t(f)] = \mu(T_t f), \quad \mu \in \mathcal{M}, t > 0, f \in \mathrm{bp}\mathscr{B}(E). \tag{4.3}$$

Therefore, for any $\mu \in \mathcal{M}^o$ and $t > 0$, there exists a unique probability measure $\widetilde{\mu\mathbb{N}}^{(t)}$ on $(\Omega, \mathscr{F})$ such that for any $H \in \mathrm{b}\mathscr{F}$, $\widetilde{\mu\mathbb{N}}^{(t)}[H] = (\mu\mathbb{N})[H \cdot e^{-\lambda t}\mu(\phi)^{-1}X_t(\phi)]$.

Another ingredient for the spine decomposition theorem is the so-called spine process which is an E-valued Markov process with transition semigroup $(S_t)_{t\geq 0}$ on E defined so that

$$S_t f(x) = e^{-\lambda t}\phi(x)^{-1}T_t(\phi f)(x), \quad t \geq 0, f \in \mathrm{b}\mathscr{B}(E), x \in E. \tag{4.4}$$

The following lemma can be verified using [43, Theorem 62.19] .

Lemma 4.8. *$(S_t)_{t\geq 0}$ is a conservative Borel right semigroup on E.*

In this section, we will add a little twist to the classical spine decomposition theorem by only considering a specific initial value ν, but with a two-sided spine. This is possible thanks to the following lemma whose proof is postponed to the Appendix. For any probability measure μ on E, we define a probability measure $\tilde{\mu}$ on E so that

$$\tilde{\mu}(f) = \mu(\phi)^{-1}\mu(\phi f) \quad \text{for every } f \in \mathrm{bp}\mathscr{B}(E). \tag{4.5}$$

In particular, $\tilde{\nu}(f) = \nu(\phi f)$ for any $f \in \mathrm{bp}\mathscr{B}(E)$. We say an E-valued two-sided process $(g_t)_{t\in\mathbb{R}}$ defined on a probability space $(\Omega_0, \mathscr{G})$ is measurable if $(t,\omega) \mapsto g_t(\omega)$ is a measurable map from $(\mathbb{R}\times\Omega_0, \mathscr{B}(\mathbb{R})\otimes\mathscr{G})$ to $(E, \mathscr{B}(E))$.

Lemma 4.9. *$\tilde{\nu}$ is an invariant probability measure of the semigroup $(S_t)_{t\geq 0}$. In particular, there exists a two-sided E-valued measurable stationary Markov process with transition semigroup $(S_t)_{t\geq 0}$ and one-dimensional distribution $\tilde{\nu}$.*

Proof. It is straight-forward to verify that $\tilde{\nu}$ is an invariant measure for the semigroup $(S_t)_{t\geq 0}$. Using Kolmogorov's extension theorem, we can construct an E-valued two-sided stationary Markov process $(\xi_t^*)_{t\in\mathbb{R}}$, canonically on the product space $E^{\mathbb{R}}$ with transition semigroup $(S_t)_{t\geq 0}$ and one-dimensional distribution $\tilde{\nu}$.

To finish the proof, we only have to construct a measurable process $(\tilde{\xi}_t)_{t\in\mathbb{R}}$ which is a modification of $(\xi_t^*)_{t\in\mathbb{R}}$. To do this, we consider the compact metric space $\tilde{E}$ which is the Ray-Knight completion of E with respect to the right semigroup $(S_t)_{t\geq 0}$. (We refer our readers to [25, p. 318] for the precise construction.) Denote by ρ the corresponding metric of $\tilde{E}$. Thanks to [25, Theorem A.30] and Lemma 4.8, we have $E \in \mathscr{B}(\tilde{E},\rho)$ and $\mathscr{B}(E) = \mathscr{B}(E,\rho)$; and therefore, $(\xi_t^*)_{t\in\mathbb{R}}$ is also an $\tilde{E}$-valued process. According to [25, Theorem A.32 & Proposition A.7] for any natural number n, the $\tilde{E}$-valued process $(\xi_t^*)_{t\in[-n,\infty)}$ admits an $\tilde{E}$-càdlàg modification. Thus $(\xi_t^*)_{t\in\mathbb{R}}$ admits an $\tilde{E}$-càdlàg modification, denoted by $(\xi_t^{**})_{t\in\mathbb{R}}$. Finally, fixing an element $x_0 \in E$, taking the measurable map $\psi : x \mapsto x\mathbf{1}_{x\in E} + x_0\mathbf{1}_{x\in\tilde{E}\setminus E}$ from $(\tilde{E}, \mathscr{B}(\tilde{E}))$ to $(E, \mathscr{B}(E))$, we can verify that $\tilde{\xi}_t := \psi(\xi_t^{**}), t \in \mathbb{R}$ is an E-valued measurable modification of the process $(\xi_t^*)_{t\in\mathbb{R}}$ as desired. $\square$

Roughly speaking, the spine decomposition theorem says that the $\mathcal{M}$-valued process $(X_t)_{t\geq 0}$ under the probability $\widetilde{\mathrm{P}}_\mu^{(\infty)}$ can be decomposed in law as the sum of a copy of the original (ξ,ψ)-superprocess and an $\mathcal{M}$-valued immigration process along the trajectory of an immortal moving particle. Note that we will only consider the case when μ is taken as ν in this section. To formulate this theorem, we construct random elements $\big(W^{(0)}, \tilde{\xi}, \mathcal{N}, (s_k, y_k, W^{(k)})_{k=1}^\infty\big)$, on a probability space with probability measure Q, so that the following statements (4.6)–(4.10) hold.

(4.6) $\tilde{\xi} = (\tilde{\xi}_t)_{t\in\mathbb{R}}$ is a two-sided E-valued measurable stationary Markov process with transition semigroup $(S_t)_{t\geq 0}$ and one-dimensional distribution $\tilde{\nu}$.

(4.7) Conditioned on $\tilde{\xi}$, $(s_k, y_k)_{k=1}^\infty$ is a sequence of $\mathbb{R}\times\mathbb{R}_+$-valued random elements such that $\mathcal{D} := \sum_{k=1}^\infty \delta_{(s_k,y_k)}$ is a Poisson random measure on $\mathbb{R}\times\mathbb{R}_+$ with intensity $\mathrm{d}s \otimes y\pi(\tilde{\xi}_s, \mathrm{d}y)$.

(4.8) Conditioned on $\tilde{\xi}$ and $(s_k, y_k)_{k=1}^\infty$, $(W^{(k)})_{k=1}^\infty$ is a sequence of independent $\mathcal{M}$-valued right-continuous stochastic processes such that, for each natural number k, $W^{(k)} = (W_t^{(k)})_{t\geq 0}$ has distribution $\mathrm{P}_{y\delta_x}$ where $y = y_k$ and $x = \tilde{\xi}_{s_k}$.

(4.9) Conditioned on $\tilde{\xi}$, $\mathcal{N}$ is a Poisson random measure on $\mathbb{R}\times\Omega$, independent of $(s_k, y_k, W^{(k)})_{k=1}^\infty$, with intensity $2\sigma(\tilde{\xi}_s)^2\mathrm{d}s \otimes \mathbb{N}_{\tilde{\xi}_s}(\mathrm{d}w)$.

(4.10) $W^{(0)} = (W^{(0)}_t)_{t\geq 0}$ is a $\mathcal{M}$-valued right-continuous process with law P_ν, independent of $\big(\tilde{\xi}, \mathcal{N}, (s_k, y_k, W^{(k)})_{k=1}^{\infty}\big)$.

Remark 4.10. The existence of the above random elements $\big(W^{(0)}, \tilde{\xi}, \mathcal{N}, (s_k, y_k, W^{(k)})_{k=1}^{\infty}\big)$ follows from the existence of the spine process (Lemma 4.9), the superprocesses [25], and the Poisson random measures [21, Theorem 2.4] . The precise construction of a probability space that carries those structures will be omitted since it is tedious but straightforward.

Notice that there are two types of immigration along the spine $\tilde{\xi}$. The discrete immigration is given by $(W^{(k)})_{k=1}^{\infty}$, while the continuous immigration is governed by the Poisson random measure $\mathcal{N}$. We are interested in the total contributions, at a given time t, of all the immigration whose earliest immigrant ancestor is born in a time interval $(a, b]$. More precisely, we define, for each $-\infty \leq a < b \leq t < \infty$ and $f \in \mathrm{bp}\mathscr{B}(E)$,

$$Z^{(a,b]}_t(f) := \sum_{k=1}^{\infty} W^{(k)}_{t-s_k}(f)\mathbf{1}_{(a,b]}(s_k) + \int_{(a,b]\times\Omega} w_{t-s}(f)\mathcal{N}(\mathrm{d}s, \mathrm{d}w). \tag{4.11}$$

It can be verified using Lemma 4.11 that when $a > -\infty$, $Z^{(a,b]}_t$ is an $\mathcal{M}$-valued random element. However, this does not hold in general if $a = -\infty$. In particular, $Z^{(-\infty,0]}_0(\phi)$ might take ∞ as a value. With the convention that $\infty^{-1} = 0$ and $0^{-1} = \infty$, we define a constant

$$\mathcal{K} := \mathrm{Q}[Z^{(-\infty,0]}_0(\phi)^{-1}]. \tag{4.12}$$

We will prove Proposition 4.1 by showing that $\mathcal{K}$ is finite and fulfills (4.1).

The spine decomposition theorem will be summarized in the following lemma. For its proof, we refer our readers to [36, Theorem 1.5 & Corollary 1.6]. We define $Z^{(0,0]}_t = \mathbf{0}$ for any $t \geq 0$.

Lemma 4.11. *The $\mathcal{M}$-valued process $(W^{(0)}_t + Z^{(0,t]}_t)_{t\geq 0}$ under Q has the same finite-dimensional distributions as the coordinate process $(X_t)_{t\geq 0}$ under $\widetilde{\mathrm{P}}^{(\infty)}_\nu$. Moreover, for any $t_0 > 0$, the $\mathcal{M}$-valued process $(Z^{(0,t]}_t)_{t\in[0,t_0]}$ under Q has the same finite-dimensional distributions as the coordinate process $(X_t)_{t\in[0,t_0]}$ under $\widetilde{\nu\mathbb{N}}^{(t_0)}$.*

We are now ready to give the proof of Proposition 4.1.

Proof of Proposition 4.1. *Step 1.* One can verify that for any $-\infty < a < b \leq t < \infty$ and $s \in \mathbb{R}$, the $\mathcal{M}$-valued random elements $Z^{(a,b]}_t$ and $Z^{(a+s,b+s]}_{t+s}$ have the same distribution. This is due to the fact that both the discrete immigration (4.7)–(4.8) and the continuous immigration (4.9) are defined in a time-homogeneous way along the spine $(\tilde{\xi}_t)_{t\in\mathbb{R}}$ which is a stationary process (4.6).

Step 2. Let $\mathcal{K}$ be given as in (4.12). We will show that $\mathcal{K} < \infty$ and (4.1) holds when $\mu = \nu$. The main idea is to work with the reciprocal of the additive martingale $e^{-\lambda t}X_t(\phi)$ under the measure $\widetilde{\mathrm{P}}^{(\infty)}_\nu$ to analyze the survival probability. In fact, for any $t \geq 0$, from Lemmas 3.1, 4.11, and Step 1, we have

$$\begin{aligned} &e^{-\lambda t}\mathrm{P}_\nu(X_t \neq \mathbf{0}) = \widetilde{\mathrm{P}}^{(\infty)}_\nu[X_t(\phi)^{-1}] \\ &= \mathrm{Q}\big[\big(W^{(0)}_t(\phi) + Z^{(0,t]}_t(\phi)\big)^{-1}\big] = \mathrm{Q}\big[\big(W^{(0)}_t(\phi) + Z^{(-t,0]}_0(\phi)\big)^{-1}\big]. \end{aligned} \tag{4.13}$$

From (1.8), (4.10) and the Markov property of superprocesses, we can verify that the process $(e^{-\lambda t}W_t^{(0)}(\phi))_{t\geq 0}$ is a non-negative Q-martingale. So by the martingale convergence theorem and (1.10), we have Q-almost surely $W_t^{(0)}(\phi)\to 0$ as $t\to\infty$. From the fact that $t\mapsto Z_0^{(-t,0]}(\phi)$ is a non-decreasing process with almost sure limit $Z_0^{(-\infty,0]}(\phi)$ in $[0,\infty]$, we have almost surely

$$(W_t^{(0)}(\phi)+Z_0^{(-t,0]}(\phi))^{-1}\xrightarrow[t\to\infty]{} Z_0^{(-\infty,0]}(\phi)^{-1}\in[0,\infty].$$

Now, we can apply the dominated convergence theorem in (4.13) and get the desired result in this step. In fact, the family of non-negative random variables $\{(W_t^{(0)}(\phi)+Z_0^{(-t,0]}(\phi))^{-1}:t\geq 1\}$ is dominated by $Z_0^{(-1,0]}(\phi)^{-1}$, which is Q-integrable since, according to Step 1, Lemmas 4.11, 4.7(3), Campbell's theorem and (1.8), we have

$$\begin{aligned}&\mathrm{Q}[Z_0^{(-1,0]}(\phi)^{-1}]=\mathrm{Q}[Z_1^{(0,1]}(\phi)^{-1}]=\nu\widetilde{\mathbb{N}}^{(1)}[X_1(\phi)^{-1}]\\&=e^{-\lambda}\cdot(\nu\mathbb{N})(X_1\neq\mathbf{0})=-e^{-\lambda}\log\mathrm{P}_\nu(X_1=\mathbf{0})<\infty.\end{aligned}\tag{4.14}$$

Final step. To see (4.1) holds for all $\mu\in\mathcal{M}^o$, we use Step 2 and Lemma 2.7. □

4.2. The $L\log L$ criterion

Let $\big(W^{(0)},\tilde{\xi},\mathcal{N},(s_k,y_k,W^{(k)})_{k=1}^\infty\big)$ be the random elements constructed in Section 4.1. Our proof of Proposition 4.2 will rely on the following two lemmas.

Lemma 4.12. *There exist $s_0,\epsilon,\theta>0$ and $\delta>0$ such that for any $x\in E, s>s_0$ and $y\geq e^{\epsilon s}/\phi(x)$, it holds that $\mathrm{P}_{y\delta_x}(X_s(\phi)>\theta)>\delta$.*

Proof. From [28, (3.20)] we know that there exist $t_0,a,\epsilon>0$ such that for all $s\geq t_0$, we have $\nu(V_s\phi)\geq a\exp(-\epsilon s)$. According to [28, Proposition 2.2] we know that there exists $s_0'>0$ such that for all $s\geq s_0'$ and $x\in E$ we have $V_s\phi(x)\geq\frac{1}{2}\phi(x)\nu(V_s\phi)$. Now take $s_0:=t_0\vee s_0'$, we have for all $s\geq s_0$ and $x\in E$, $V_s\phi(x)\geq\dfrac{a}{2}\phi(x)e^{-\epsilon s}$. Let $\theta\in(0,a/2)$. We have for any $s>s_0$, $x\in E$ and $y\geq\frac{e^{\epsilon s}}{\phi(x)}$ that

$$\begin{aligned}&\mathrm{P}_{y\delta_x}\big(w_s(\phi)>\theta\big)=\mathrm{P}_{y\delta_x}\big(e^{-w_s(\phi)}<e^{-\theta}\big)\\&=1-\mathrm{P}_{y\delta_x}(e^{-w_s(\phi)}\geq e^{-\theta})\overset{\text{Chebyshev}}{\geq}1-e^{\theta}\mathrm{P}_{y\delta_x}[e^{-w_s(\phi)}]\\&=1-e^{\theta}e^{-yV_s\phi(x)}\geq 1-e^{\theta}e^{-y\frac{a}{2}\phi(x)e^{-\epsilon s}}\geq 1-e^{\theta-a/2}=:\delta>0\end{aligned}$$

as desired. □

Lemma 4.13. *(1) If $\mathcal{E}<\infty$, then for any $\epsilon>0$,*

$$\sum_{k=1}^\infty\mathbf{1}_{(-\infty,0]}(s_k)\cdot y_ke^{\epsilon s_k}\cdot\phi(\tilde{\xi}_{s_k})<\infty,\quad\text{Q-}a.s.$$

(2) *If $\mathcal{E}=\infty$, then for any $\epsilon>0$ and $s_0\geq 0$,*

$$\int_{-\infty}^{-s_0}\mathrm{d}s\int_{e^{-\epsilon s}\phi(\tilde{\xi}_s)^{-1}}^\infty y\pi(\tilde{\xi}_s,\mathrm{d}y)=\infty,\quad\text{Q-}a.s.$$

Lemma 4.13 is similar to [26, Lemma 3.2] and its proof is pretty long. We postpone its proof to the Appendix.

Proof of Proposition 4.2.

Step 1. Assuming $\mathcal{E} < \infty$, we will show that $\mathcal{K} > 0$. To do this, we verify using Campbell's theorem, (1.8), (4.3), (4.6) and (4.9) that

$$\begin{aligned} &\mathrm{Q}\Big[\int_{(-\infty,0]\times\Omega} w_{-s}(\phi)\mathcal{N}(\mathrm{d}s,\mathrm{d}w)\Big] = \mathrm{Q}\Big[\int_{-\infty}^{0} 2\sigma(\tilde{\xi}_s)^2\mathrm{d}s \int_{\Omega} w_{-s}(\phi)\mathbb{N}_{\tilde{\xi}_s}(\mathrm{d}w)\Big] \\ &= \mathrm{Q}\Big[\int_{-\infty}^{0} 2\sigma(\tilde{\xi}_s)^2 e^{-\lambda s}\phi(\tilde{\xi}_s)\mathrm{d}s\Big] = 2\int_{-\infty}^{0} e^{-\lambda s}\tilde{\nu}(\sigma^2\phi)\mathrm{d}s < \infty, \end{aligned} \tag{4.15}$$

where in the last inequality we used the fact that $\sigma, \phi \in \mathrm{b}\mathscr{B}(E)$ and $\lambda < 0$. Then we define the σ-algebra

$$\mathscr{G} := \sigma(\tilde{\xi}, (s_k, y_k)_{k=1}^{\infty}), \tag{4.16}$$

and verify from (1.7), (1.8), (4.8), and Lemma 4.13 that Q-almost surely,

$$\begin{aligned} &\mathrm{Q}\Big(\sum_{k=1}^{\infty} W_{-s_k}^{(k)}(\phi)\mathbf{1}_{(-\infty,0]}(s_k)\Big|\mathscr{G}\Big) = \sum_{k=1}^{\infty} \mathrm{P}_{y_k\delta_{\tilde{\xi}_{s_k}}}[X_{-s_k}(\phi)]\mathbf{1}_{(-\infty,0]}(s_k) \\ &= \sum_{k=1}^{\infty} y_k\cdot(T_{-s_k}\phi)(\tilde{\xi}_{s_k})\cdot\mathbf{1}_{(-\infty,0]}(s_k) = \sum_{k=1}^{\infty} y_k e^{-\lambda s_k}\phi(\tilde{\xi}_{s_k})\cdot\mathbf{1}_{(-\infty,0]}(s_k) < \infty. \end{aligned} \tag{4.17}$$

From (4.15) and (4.17), we can verify that **Q**-almost surely,

$$Z_0^{(-\infty,0]}(\phi) = \int_{(-\infty,0]\times\Omega} w_{-s}(\phi)\mathcal{N}(\mathrm{d}s,\mathrm{d}w) + \sum_{k=1}^{\infty} W_{-s_k}^{(k)}(\phi)\mathbf{1}_{(-\infty,0]}(s_k) < \infty.$$

It follows from (4.12) that $\mathcal{K} = \mathrm{Q}[Z_0^{(-\infty,0]}(\phi)^{-1}] > 0$ as desired.

Step 2. Assuming $\mathcal{E} = \infty$, we will show that $\mathcal{K} = 0$. Let s_0, ϵ, θ and $\delta > 0$ be given as in Lemma 4.12. We claim that in this case

$$n_\theta := \#\{k : k \geq 1, k \in \mathbb{Z}, s_k \leq 0, W_{-s_k}^{(k)}(\phi) > \theta\} = \infty, \quad \text{Q-a.s.} \tag{4.18}$$

Using this claim, we immediately have that $Z_0^{(-\infty,0]}(\phi) \geq \theta n_\theta = \infty$ almost surely, which implies the desired result since $\mathcal{K} = \mathrm{Q}[Z_0^{(-\infty,0]}(\phi)^{-1}]$. Now we prove the claim (4.18). From (4.8), we have Q-almost surely,

$$\begin{aligned} &\mathrm{Q}[e^{-n_\theta}|\mathscr{G}] = \prod_{k=1}^{\infty} \mathrm{Q}[\exp\{-\mathbf{1}_{(-\infty,0]}(s_k)\mathbf{1}_{\{W_{-s_k}^{(k)}(\phi)>\theta\}}\}|\mathscr{G}] \\ &= \prod_{k=1}^{\infty} \mathrm{P}_{y_k\delta_{\tilde{\xi}_{s_k}}}[\exp\{-\mathbf{1}_{(-\infty,0]}(s_k)\mathbf{1}_{\{X_{-s_k}(\phi)>\theta\}}\}] = \exp\Big\{-\int_{\mathbb{R}\times\mathbb{R}_+} F(s,y)\mathcal{D}(\mathrm{d}s,\mathrm{d}y)\Big\} \end{aligned}$$

where for any $(s, y) \in \mathbb{R}\times\mathbb{R}_+$, the random variable $F(s, y)$ is given by

$$F(s,y) := -\log \mathrm{P}_{y\delta_{\tilde{\xi}_s}}[\exp\{-\mathbf{1}_{(-\infty,0]}(s)\mathbf{1}_{\{X_{-s}(\phi)>\theta\}}\}],$$

and where $\mathcal{D}$ is the Poisson random measure defined as in (4.7). Now by (4.7) and Campbell's theorem,

$$\begin{aligned}
\mathrm{Q}[e^{-n_\theta}|\tilde{\xi}] &= \exp\Big(-\int_{\mathbb{R}} \mathrm{d}s \int_{(0,\infty)} (1-e^{-F(s,y)})y\pi(\tilde{\xi}_s,\mathrm{d}y)\Big)\\
&= \exp\Big(-\int_{\mathbb{R}} \mathrm{d}s \int_{(0,\infty)} \mathrm{P}_{y\delta_{\tilde{\xi}_s}}[1-\exp\{-\mathbf{1}_{(-\infty,0]}(s)\mathbf{1}_{\{X_{-s}(\phi)>\theta\}}\}]y\pi(\tilde{\xi}_s,\mathrm{d}y)\Big)\\
&= \exp\Big(-(1-e^{-1})\int_{-\infty}^0 \mathrm{d}s \int_{(0,\infty)} \mathrm{P}_{y\delta_{\tilde{\xi}_s}}(X_{-s}(\phi)>\theta)y\pi(\tilde{\xi}_s,\mathrm{d}y)\Big). \qquad (4.19)
\end{aligned}$$

Note that from Lemmas 4.12 and 4.13(2), we have Q-almost surely,

$$\int_{-\infty}^0 \mathrm{d}s \int_0^\infty \mathrm{P}_{y\delta_{\tilde{\xi}_s}}(X_{-s}(\phi)>\theta)y\pi(\tilde{\xi}_s,\mathrm{d}y) \ge \delta \int_{-\infty}^{-s_0} \mathrm{d}s \int_{\phi(\tilde{\xi}_s)^{-1}e^{-\epsilon s}}^\infty y\pi(\tilde{\xi}_s,\mathrm{d}y) = \infty.$$

Now from this and (4.19), we have $\mathrm{Q}[e^{-n_\theta}] = 0$ which implies the desired claim (4.18). □

4.3. First moment of the Yaglom law

Let

$$\big(W^{(0)},\tilde{\xi},\mathcal{N},(s_k,y_k,W^{(k)})_{k=1}^\infty\big)$$

be the random elements constructed in Section 4.1. Our proof of Proposition 4.3 in the case $\mathcal{K}>0$ relies on the following lemma.

Lemma 4.14. *If $\mathcal{K}>0$, then $Z_0^{(-\infty,0]}$ is an $\mathcal{M}$-valued random element.*

Proof. From (1.9), we know that there exists a $t_0>0$ such that

$$C_3 := \sup\{|H_t f(x)| : t\ge t_0,\ f\in L_1^+(\nu),\ x\in E\} < \infty.$$

Step 1. We will show that Q-almost surely $Z_0^{(-t_0,0]}(\mathbf{1}_E)<\infty$. In fact, from Step 1 of the proof of Proposition 4.1, we know that, under Q, $Z_0^{(-t_0,0]}(\mathbf{1}_E)$ and $Z_{t_0}^{(0,t_0]}(\mathbf{1}_E)$ have the same distribution. From Lemma 4.11, we know that they are both stochastically dominated by the random variable $X_{t_0}(\mathbf{1}_E)$ under $\widetilde{\mathrm{P}}_\nu^{(\infty)}$. Thus the desired result in this step is valid.

Step 2. We will show that Q-almost surely

$$\int_{(-\infty,-t_0]\times\Omega} w_{-s}(\mathbf{1}_E)\mathcal{N}(\mathrm{d}s,\mathrm{d}w)<\infty.$$

By (4.9) and Campbell's theorem, we get that

$$\begin{aligned}
&\mathrm{Q}\Big[\int_{(-\infty,-t_0]\times\Omega} w_{-s}(\mathbf{1}_E)\mathcal{N}(\mathrm{d}s,\mathrm{d}w)\Big] = \mathrm{Q}\Big[\int_{-\infty}^{-t_0} 2\sigma(\tilde{\xi}_s)^2\mathrm{d}s\int_\Omega w_{-s}(\mathbf{1}_E)\mathbb{N}_{\tilde{\xi}_s}(\mathrm{d}w)\Big]\\
&\overset{(4.3)}{=} \mathrm{Q}\Big[\int_{-\infty}^{-t_0} 2\sigma(\tilde{\xi}_s)^2(T_{-s}\mathbf{1}_E)(\tilde{\xi}_s)\mathrm{d}s\Big] \overset{(4.6)}{\le} 2\|\sigma\|_\infty^2\int_{-\infty}^{-t_0}\mathrm{d}s\int_E (T_{-s}\mathbf{1}_E)(x)\nu(\mathrm{d}x)\\
&\overset{(1.8),(1.9)}{\le} 2\|\sigma\|_\infty^2\int_{-\infty}^{-t_0}\mathrm{d}s\int_E e^{-\lambda s}\phi(x)\nu(\mathbf{1}_E)\big(1+(H_{-s}\mathbf{1}_E)(x)\big)\nu(\mathrm{d}x)\\
&\overset{(1.9)}{\le} 2(1+C_3)\|\sigma\|_\infty^2\int_{-\infty}^{-t_0} e^{-\lambda s}\mathrm{d}s \overset{(1.10)}{<} \infty.
\end{aligned}$$

Step 3. We will show that if $\mathcal{K} > 0$ then Q-almost surely

$$\sum_{k=1}^{\infty} W^{(k)}_{-s_k}(\mathbf{1}_E)\mathbf{1}_{(-\infty,-t_0]}(s_k) < \infty.$$

Recall the definition (4.16) of the σ-algebra $\mathscr{G}$. Then by (1.7), (1.9), (4.8) and Lemma 4.13, we have Q-almost surely that

$$\begin{aligned}
\mathrm{Q}\Big(\sum_{k=1}^{\infty} W^{(k)}_{-s_k}(\mathbf{1}_E)\mathbf{1}_{(-\infty,-t_0]}(s_k)\Big|\mathscr{G}\Big) &= \sum_{k=1}^{\infty} y_k \cdot (T_{-s_k}\mathbf{1}_E)(\tilde{\xi}_{s_k}) \cdot \mathbf{1}_{(-\infty,-t_0]}(s_k) \\
&= \sum_{k=1}^{\infty} y_k e^{-\lambda s_k}\phi(\tilde{\xi}_{s_k})\big(1 + (H_{-s_k}\mathbf{1}_E)(x)\big)\mathbf{1}_{(-\infty,-t_0]}(s_k) \\
&\le (1 + C_3)\sum_{k=1}^{\infty} y_k e^{-\lambda s_k}\phi(\tilde{\xi}_{s_k})\mathbf{1}_{(-\infty,-t_0]}(s_k) < \infty.
\end{aligned}$$

Final Step. From Steps 1, 2 and 3, we know that $Z_0^{(-\infty,0]}(\mathbf{1}_E) < \infty$ almost surely provided $\mathcal{K} > 0$. Then one can use a routine measure theoretic argument to get the desired result of this lemma. □

When $\mathcal{K} > 0$, the above lemma allows us to define a probability measure $\widehat{\mathbf{Q}}$ on $\mathcal{M}$ as the distribution of $Z_0^{(-\infty,0]}$ under the probability Q. It was mentioned in Section 4 that, after establishing Proposition 4.3, one can also construct a probability measure $\mathbf{Q}_{\infty,\infty}$ using (4.2) provided $\mathcal{K} > 0$. It will be explained later in Remark 4.15 that $\widehat{\mathbf{Q}}$ and $\mathbf{Q}_{\infty,\infty}$ are exactly the same.

We are now ready to give the proof of Proposition 4.3.

Proof of Proposition 4.3.

Step 1. In this step, we will prove Proposition 4.3 in the case $\mathcal{K} > 0$. Let $F \in \mathrm{b}C(\mathcal{M})$ and $t \ge 0$ be arbitrary. Using Lemmas 3.1, 4.11 and Step 1 of the proof of Proposition 4.1, we have

$$\begin{aligned}
\mathrm{P}_\nu[\mathbf{1}_{\{X_t \neq \mathbf{0}\}}F(X_t)] &= \widetilde{\mathrm{P}}^{(\infty)}_\nu[e^{\lambda t}X_t(\phi)^{-1}F(X_t)] \\
&= e^{\lambda t}\mathrm{Q}\Big[\frac{F(W^{(0)}_t + Z^{(0,t]}_t)}{W^{(0)}_t(\phi) + Z^{(0,t]}_t(\phi)}\Big] = e^{\lambda t}\mathrm{Q}\Big[\frac{F(W^{(0)}_t + Z^{(-t,0]}_0)}{W^{(0)}_t(\phi) + Z^{(-t,0]}_0(\phi)}\Big].
\end{aligned} \tag{4.20}$$

It follows from (4.10) and Lemma 2.3 that the $\mathcal{M}$-valued process $(W^{(0)}_t)_{t\ge 0}$ converges to $\mathbf{0}$ in probability when $t \uparrow \infty$ (with respect to any separable metric compatible with the topology of the Polish space $\mathcal{M}$.) It is also clear from (4.11), Lemma 4.14 and monotonicity that the following statement holds.

$$\text{The } \mathcal{M}\text{-valued process } (Z^{(-t,0]}_0)_{t>0} \text{ converges to } Z^{(-\infty,0]}_0 \text{ almost surely as } t \uparrow \infty. \tag{4.21}$$

So by the continuous mapping theorem, we have

$$\frac{F(W^{(0)}_t + Z^{(-t,0]}_0)}{W^{(0)}_t(\phi) + Z^{(-t,0]}_0(\phi)} \xrightarrow[t\to\infty]{} Z^{(-\infty,0]}_0(\phi)^{-1}F(Z^{(-\infty,0]}_0)$$

in probability. Notice also that the family of random variables

$$\left\{\left|\frac{F(W_t^{(0)}+Z_0^{(-t,0]})}{W_t^{(0)}(\phi)+Z_0^{(-t,0]}(\phi)}\right| : t \geq 1\right\}$$

is dominated by the random variable $Z_0^{(-1,0]}(\phi)^{-1} \cdot \sup_{\eta\in\mathcal{M}} |F(\eta)|$ which is Q-integrable by (4.14). Now we can apply the dominated convergence theorem in (4.20) and get that

$$\lim_{t\to\infty} e^{-\lambda t} \mathrm{P}_\nu[\mathbf{1}_{\{X_t\neq \mathbf{0}\}} F(X_t)] = \mathrm{Q}[Z_0^{(-\infty,0]}(\phi)^{-1} F(Z_0^{(-\infty,0]})].$$

Thus from Theorem 1.1 and Proposition 4.1,

$$\begin{aligned}
&\int_{\mathcal{M}} F(\eta)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta) = \lim_{t\to\infty} \mathrm{P}_\nu[F(X_t)|X_t \neq \mathbf{0}] \\
&= \frac{\lim_{t\to\infty} e^{-\lambda t} \mathrm{P}_\nu[\mathbf{1}_{\{X_t\neq\mathbf{0}\}} F(X_t)]}{\lim_{t\to\infty} e^{-\lambda t}\mathrm{P}_\nu(X_t \neq \mathbf{0})} = \mathcal{K}^{-1}\mathrm{Q}[Z_0^{(-\infty,0]}(\phi)^{-1} F(Z_0^{(-\infty,0]})] \\
&= \mathcal{K}^{-1} \int_{\mathcal{M}} \eta(\phi)^{-1} F(\eta)\widehat{\mathbf{Q}}(\mathrm{d}\eta). \qquad (4.22)
\end{aligned}$$

Since F is arbitrary, we can replace $F(\eta)$ in (4.22) by $F(\eta)\cdot(\eta(\phi)\wedge n)$ where n is an arbitrary natural number. Taking $n \uparrow \infty$ and using the monotone convergence theorem, we then arrive at

$$\int_{\mathcal{M}} F(\eta)\cdot\eta(\phi)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta) = \mathcal{K}^{-1}\int_{\mathcal{M}} F(\eta)\widehat{\mathbf{Q}}(\mathrm{d}\eta), \quad F \in \mathrm{bp}C(\mathcal{M}) \qquad (4.23)$$

which implies the desired result in this step.

Step 2. In this step, we will prove Proposition 4.3 in the case $\mathcal{K} = 0$. According to Lemma 2.1, there exists $t_0 > 0$ such that for any $t \geq t_0$ and $x \in E$, $2\nu(v_t)\phi(x) \geq v_t(x)$. Now for any $t \geq t_0$, we have from (2.2) and (2.4) that

$$\begin{aligned}
&\int_{\mathcal{M}} \eta(2\nu(v_t)\phi)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta) \geq \int_{\mathcal{M}} \eta(v_t)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta) \geq \int_{\mathcal{M}} (1-e^{-\eta(v_t)})\mathbf{Q}_{\infty,0}(\mathrm{d}\eta) \\
&= \int_{\mathcal{M}} \mathrm{P}_\eta(X_t \neq \mathbf{0})\mathbf{Q}_{\infty,0}(\mathrm{d}\eta) = e^{\lambda t}.
\end{aligned}$$

From (2.2), Lemma 2.3 and Proposition 4.1 we have that

$$e^{-\lambda t}\nu(v_t) = e^{-\lambda t}\mathrm{P}_\nu(X_t \neq \mathbf{0}) \cdot \frac{-\log(1-\mathrm{P}_\nu(X_t\neq\mathbf{0}))}{\mathrm{P}_\nu(X_t\neq\mathbf{0})} \xrightarrow[t\to\infty]{} \mathcal{K}\nu(\phi) = 0.$$

Now we have that

$$\int_{\mathcal{M}} \eta(\phi)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta) \geq e^{\lambda t}/(2\nu(v_t)) \xrightarrow[t\to\infty]{} \infty$$

which implies the desired result for this step. $\square$

Remark 4.15. From the definition (4.2) of $\mathbf{Q}_{\infty,\infty}$ and (4.23), it is clear that $\mathbf{Q}_{\infty,\infty} = \widehat{\mathbf{Q}}$ when $\mathcal{K} > 0$.

4.4. Limit of the Q-process

Using the spine decomposition theorem (Lemma 4.11), we can get the following lemma.

Lemma 4.16. *If $\mathcal{E} < \infty$, then $\mathbf{Q}^{\nu}_{t,\infty}$ converges weakly to $\mathbf{Q}_{\infty,\infty}$ as $t \to \infty$.*

Proof. We note that for any $f \in \mathrm{b}p C(E)$,

$$\int_{\mathcal{M}} e^{-\eta(f)}\mathbf{Q}^{\nu}_{t,\infty}(\mathrm{d}\eta) = \widetilde{\mathrm{P}}^{(\infty)}_{\nu}[e^{-X_t(f)}] = \mathrm{Q}[e^{-W^{(0)}_t(f)-Z^{(0,t]}_t(f)}] = \mathrm{P}_{\nu}[e^{-X_t(f)}]\mathrm{Q}[e^{-Z^{(-t,0]}_0(f)}]$$

where the first equality is due to (3.1) and Lemma 3.1, the second equality is due to Lemma 4.11, and the third equality is due to (4.10) and Step 1 of the proof of Proposition 4.1. Thus,

$$\begin{aligned}\lim_{t\to\infty}\int_{\mathcal{M}} e^{-\eta(f)}\mathbf{Q}^{\nu}_{t,\infty}(\mathrm{d}\eta) &= \lim_{t\to\infty}\mathrm{P}_{\nu}[e^{-X_t(f)}]\lim_{t\to\infty}\mathrm{Q}[e^{-Z^{(-t,0]}_0(f)}]\\ &= \mathrm{Q}[e^{-Z^{(-\infty,0]}_0(f)}] = \int_{\mathcal{M}} e^{-\eta(f)}\mathbf{Q}_{\infty,\infty}(\mathrm{d}\eta),\end{aligned}$$

where the second equality is due to Lemma 2.3 and (4.21), and the last equality is due to Remark 4.15. Now the desired result follows from [25, Theorem 1.18] . □

The above lemma is a special case of Proposition 4.4. For the general case, we will use the spine decomposition theorem for superprocesses with arbitrary initial value $\mu \in \mathcal{M}^o$. Now the corresponding spine process will not be stationary, and cannot be extended into a two-sided process in general. Therefore, in this subsection, we construct the random elements $(W^{(0)}, \tilde{\xi}, \mathcal{N}, (s_k, y_k, W^{(k)})_{k=1}^{\infty})$ in a different probability space with respect to a new probability measure Q_{μ}, under which statements (4.24), (4.7) (with $\mathbb{R}$ replaced by $\mathbb{R}_+$), (4.8), (4.9) (with $\mathbb{R}$ replaced by $\mathbb{R}_+$) and (4.10) (with P_{ν} replaced by P_{μ}) hold.

(4.24) $\tilde{\xi} = (\tilde{\xi}_t)_{t\geq 0}$ is an E-valued right continuous Markov process with transition semigroup $(S_t)_{t\geq 0}$ and initial distribution $\tilde{\mu}$ given by (4.5).

We present the spine decomposition for superprocesses with arbitrary initial value in the following lemma. We refer our readers to [36] for its proof. For any $0 \leq a < b \leq t < \infty$, and $f \in \mathrm{b}p\mathscr{B}(E)$, let the random variable $Z^{(a,b]}_t(f)$ be defined as in (4.11).

Lemma 4.17. *Let $\mu \in \mathcal{M}^o$ be arbitrary. The $\mathcal{M}$-valued process $(W^{(0)}_t + Z^{(0,t]}_t)_{t\geq 0}$ under Q_{μ} has the same finite-dimensional distributions as the coordinate process $(X_t)_{t\geq 0}$ under $\widetilde{\mathrm{P}}^{(\infty)}_{\mu}$. Moreover, for any $t > 0$, the $\mathcal{M}$-valued process $(Z^{(0,s]}_s)_{s\in[0,t]}$ under Q_{μ} has the same finite-dimensional distributions as the coordinate process $(X_s)_{s\in[0,t]}$ under $\widetilde{\mu\mathbb{N}}^{(t)}$.*

For any $t \geq 0$, $f \in \mathrm{b}p\mathscr{B}(E)$ and $x \in E$, define $\mathcal{L}_t f(x) := \mathrm{Q}_{\delta_x}[e^{-Z^{(0,t]}_t(f)}]$ and $\mathcal{L}_{\infty} f(x) := \limsup_{t\to\infty}\mathcal{L}_t f(x)$.

Lemma 4.18. *For any $x \in E$ and $f \in \mathrm{b}p\mathscr{B}(E)$, $\lim_{t\to\infty}\mathcal{L}_t f(x) = \tilde{\nu}(\mathcal{L}_{\infty} f)$.*

Proof. Step 1. We will show that for any $f \in \mathrm{b}p\mathscr{B}(E)$ and $x \in E$, $\mathcal{L}_{\infty} f(x) \leq \tilde{\nu}(\mathcal{L}_{\infty} f)$. To this end, let us take arbitrary $f \in \mathrm{b}p\mathscr{B}(E)$, $s, t \in \mathbb{R}_+$ and $x \in E$, and verify that

$$\mathcal{L}_{t+s} f(x) = \mathrm{Q}_{\delta_x}[e^{-Z^{(0,t+s]}_{t+s}(f)}] \leq \mathrm{Q}_{\delta_x}[e^{-Z^{(t,t+s]}_{t+s}(f)}] = \mathrm{Q}_{\delta_x}\left[\mathrm{Q}_{\delta_{\tilde{\xi}_t}}[e^{-Z^{(0,s]}_s(f)}]\right] = S_t\mathcal{L}_s f(x).$$

In fact, only the second equality above needs more explanation. From (4.7) (with $\mathbb{R}$ replaced by $\mathbb{R}_+$), (4.8), (4.9) (with $\mathbb{R}$ replaced by $\mathbb{R}_+$) and Campbell's formula, we can verify that

$$\begin{aligned}
\mathrm{Q}_{\delta_x}[e^{-Z_{t+s}^{(t,t+s]}(f)}] &= \mathrm{Q}_{\delta_x}\Big[\exp\Big\{-\int_{(t,t+s]\times(0,\infty)}(-\log \mathrm{P}_{y\delta_{\tilde{\xi}_r}}[e^{-X_{t+s-r}(f)}])\mathcal{D}(\mathrm{d}r,\mathrm{d}y)\Big\}\times \\
&\qquad \exp\Big\{-\int_{(t,t+s]\times\Omega} w_{t+s-r}(f)\mathcal{N}(\mathrm{d}r,\mathrm{d}w)\Big\}\Big] \\
&= \mathrm{Q}_{\delta_x}\Big[\exp\Big\{-\int_{(t,t+s]\times(0,\infty)}\mathrm{P}_{y\delta_{\tilde{\xi}_r}}[1-e^{-X_{t+s-r}(f)}]\mathrm{d}r\otimes y\pi(\tilde{\xi}_r,\mathrm{d}y)\Big\}\times \\
&\qquad \exp\Big\{-\int_{(t,t+s]\times\Omega}(1-e^{-w_{t+s-r}(f)})2\sigma(\tilde{\xi}_r)^2\mathrm{d}r\otimes\mathbb{N}_{\tilde{\xi}_r}(\mathrm{d}w)\Big\}\Big] \\
&= \mathrm{Q}_{\delta_x}[e^{-F((\tilde{\xi}_{t+r})_{r\in(0,s]})}]
\end{aligned}$$

where for any E-valued process $(x_r)_{r\in(0,s]}$, the functional F is defined by

$$\begin{aligned}
F((x_r)_{r\in(0,s]}) := &\int_{(0,s]}\mathrm{d}r\int_{(0,\infty)}\mathrm{P}_{y\delta_x}[1-e^{-X_{s-r}(f)}]y\pi(x_r,\mathrm{d}y)+ \\
&\int_{(0,s]}2\sigma(x_r)^2\mathrm{d}r\int_{\Omega}(1-e^{-w_{s-r}(f)})\mathbb{N}_{x_r}(\mathrm{d}w).
\end{aligned}$$

Therefore, by the Markov property of the spine process $\tilde{\xi}$, we have

$$\begin{aligned}
&\mathrm{Q}_{\delta_x}[e^{-Z_{t+s}^{(t,t+s]}(f)}] = \mathrm{Q}_{\delta_x}[e^{-F((\tilde{\xi}_{t+r})_{r\in(0,s]})}] \\
&= \mathrm{Q}_{\delta_x}[\mathrm{Q}_{\delta_{\tilde{\xi}_t}}[e^{-F((\tilde{\xi}_r)_{r\in(0,s]})}]] = \mathrm{Q}_{\delta_x}\big[\mathrm{Q}_{\delta_{\tilde{\xi}_t}}[e^{-Z_s^{(0,s]}(f)}]\big].
\end{aligned}$$

Now by (1.9),

$$\mathcal{L}_{t+s}f(x)\le S_t\mathcal{L}_s f(x)\overset{(4.4)}{=}\phi(x)^{-1}e^{-\lambda t}T_t(\phi\mathcal{L}_s f)(x)=\big(1+H_t(\phi\mathcal{L}_s f)(x)\big)\tilde{\nu}(\mathcal{L}_s f).$$

Noticing that, from (1.9), $\lim_{t\to\infty}\sup_{x\in E,g\in \mathrm{b}\mathscr{B}(E)}|H_t g(x)|=0$. Therefore, letting $t\to\infty$, we have

$$\mathcal{L}_\infty f(x)\le\tilde{\nu}(\mathcal{L}_s f). \tag{4.25}$$

Now, taking $\limsup_{s\to\infty}$ in (4.25), using the reverse Fatou's lemma, we get the desired result for this step.

Step 2. We will show that for any $f\in \mathrm{bp}\mathscr{B}(E)$, the Borel function $\mathcal{L}_t f$ on E converges to the constant $\tilde{\nu}(\mathcal{L}_\infty f)$ in probability as $t\to\infty$ under $\tilde{\nu}$. First note that, if $\tilde{\nu}(\mathcal{L}_\infty f)=0$ then this is trivial from Step 1. So let us assume that $\tilde{\nu}(\mathcal{L}_\infty f)>0$ for the rest of this step. Take arbitrary $\varepsilon_1,\varepsilon_2\in(0,1)$, $t\ge 0$, and define

$$\begin{aligned}
U_t &:= U_t^{\varepsilon_1} := \{x\in E:\mathcal{L}_t f(x)>(1+\varepsilon_1)\tilde{\nu}(\mathcal{L}_\infty f)\}; \\
L_t &:= L_t^{\varepsilon_2} := \{x\in E:\mathcal{L}_t f(x)<(1-\varepsilon_2)\tilde{\nu}(\mathcal{L}_\infty f)\}.
\end{aligned}$$

From the reverse Fatou's lemma and Step 1, we have

$$\limsup_{t\to\infty}\tilde{\nu}(U_t)\le\tilde{\nu}(\limsup_{t\to\infty}\mathbf{1}_{U_t})=\tilde{\nu}(\cap_{t_0\ge 0}\cup_{t\ge t_0}U_t)=\tilde{\nu}(\emptyset)=0.$$

Now we only need to show that $\lim_{t\to\infty}\tilde{\nu}(L_t)=0$. From (4.25), and the fact that the function $\mathcal{L}_t f$ is bounded by 1, we have

$$\tilde{\nu}(\mathcal{L}_\infty f)\le\tilde{\nu}(\mathcal{L}_t f)=\tilde{\nu}(\mathcal{L}_t f\cdot\mathbf{1}_{L_t})+\tilde{\nu}(\mathcal{L}_t f\cdot\mathbf{1}_{U_t})+\tilde{\nu}(\mathcal{L}_t f\cdot\mathbf{1}_{(L_t\cup U_t)^c})$$
$$\le(1-\varepsilon_2)\tilde{\nu}(\mathcal{L}_\infty f)\tilde{\nu}(L_t)+\tilde{\nu}(U_t)+(1+\varepsilon_1)\tilde{\nu}(\mathcal{L}_\infty f)\big(1-\tilde{\nu}(L_t)-\tilde{\nu}(U_t)\big)$$
$$\le(1+\varepsilon_1)\tilde{\nu}(\mathcal{L}_\infty f)-(\varepsilon_1+\varepsilon_2)\tilde{\nu}(\mathcal{L}_\infty f)\tilde{\nu}(L_t)+\tilde{\nu}(U_t).$$

Taking $\liminf_{t\to\infty}$, we have

$$\tilde{\nu}(\mathcal{L}_\infty f)\le(1+\varepsilon_1)\tilde{\nu}(\mathcal{L}_\infty f)-(\varepsilon_1+\varepsilon_2)\tilde{\nu}(\mathcal{L}_\infty f)\limsup_{t\to\infty}\tilde{\nu}(L_t^{\varepsilon_2}).$$

Letting $\varepsilon_1\to 0$, we have

$$\tilde{\nu}(\mathcal{L}_\infty f)\le\tilde{\nu}(\mathcal{L}_\infty f)-\varepsilon_2\tilde{\nu}(\mathcal{L}_\infty f)\limsup_{t\to\infty}\tilde{\nu}(L_t^{\varepsilon_2}).$$

This is impossible unless $\limsup_{t\to\infty}\tilde{\nu}(L_t^{\varepsilon_2})=0$ holds as desired.

Step 3. We will show that for any $f\in \mathrm{bp}\mathscr{B}(E)$ and $x\in E$, $\liminf_{t\to\infty}\mathcal{L}_t f(x)\ge\tilde{\nu}(\mathcal{L}_\infty f)$. To this end, we fix arbitrary $f\in \mathrm{bp}\mathscr{B}(E)$ and $x\in E$, and note that for any $t,s>0$,

$$\mathcal{L}_{t+s}f(x)\ge \mathrm{Q}_{\delta_x}\big[\exp\{-Z_{t+s}^{(t,t+s]}(f)\};\,Z_{t+s}^{(0,t]}=\mathbf{0}\big]$$
$$=\mathrm{Q}_{\delta_x}\Big[\mathrm{Q}_{\delta_x}\big[\exp\{-Z_{t+s}^{(t,t+s]}(f)\}\big|\tilde{\xi}_t\big]\cdot\mathrm{Q}_{\delta_x}\big(Z_{t+s}^{(0,t]}=\mathbf{0}\big|\tilde{\xi}_t\big)\Big]$$
$$=\mathrm{Q}_{\delta_x}\big[\mathcal{L}_s f(\tilde{\xi}_t)\cdot\mathrm{Q}_{\delta_x}\big(Z_{t+s}^{(0,t]}=\mathbf{0}\big|\tilde{\xi}_t\big)\big].\tag{4.26}$$

We claim that for any $t>0$ and $s\ge 0$, the random measure $Z_{t+s}^{(0,t]}$ under $\mathrm{Q}_{\delta_x}(\cdot|\tilde{\xi}_t)$ has the same distribution as X_s under $\mathrm{Q}_{\delta_x}[\mathrm{P}_{Z_t^{(0,t]}}(\cdot)|\tilde{\xi}_t]$. In fact, from the Markov property of $(X_t)_{t\ge0}$, we have

$$\mathrm{P}_\mu[e^{-X_{t+s}(f)}]=\mathrm{P}_\mu[\mathrm{P}_{X_t}[e^{-X_s(f)}]]=\mathrm{P}_\mu[e^{-X_t(V_s f)}]$$

for arbitrary $\mu\in\mathcal{M}$; and similarly from [25, (8.44)] , we have

$$\int_\Omega(1-e^{-w_{t+s}(f)})\mathbb{N}_x(\mathrm{d}w)=\int_{\mathcal{M}^2}(1-e^{-\mu_2(f)})\mathbb{N}_x(X_t\in\mathrm{d}\mu_1)\mathrm{P}_{\mu_1}(X_s\in\mathrm{d}\mu_2)$$
$$=\int_{\mathcal{M}}(1-e^{-w_t(V_s f)})\mathbb{N}_x(\mathrm{d}w).$$

Therefore, we can verify from (4.7) (with $\mathbb{R}$ replaced by $\mathbb{R}_+$), (4.8), (4.9) (with $\mathbb{R}$ replaced by $\mathbb{R}_+$) and Campbell's formula that

$$\mathrm{Q}_{\delta_x}[\mathrm{P}_{Z_t^{(0,t]}}(e^{-X_s(f)})|\tilde{\xi}_t]=\mathrm{Q}_{\delta_x}[e^{-Z_t^{(0,t]}(V_s f)}|\tilde{\xi}_t]$$
$$=\mathrm{Q}_{\delta_x}\Bigg[\exp\Bigg\{-\int_0^t\mathrm{d}r\int_{(0,\infty)}\mathrm{P}_{y\delta_{\tilde{\xi}_r}}[1-e^{-X_{t-r}(V_s f)}]y\pi(\tilde{\xi}_r,\mathrm{d}r)-$$
$$\int_0^t 2\sigma(\tilde{\xi}_r)^2\mathrm{d}r\int_\Omega(1-e^{-w_{t-r}(V_s f)})\mathbb{N}_{\tilde{\xi}_r}(\mathrm{d}w)\Bigg\}\Bigg|\tilde{\xi}_t\Bigg]$$

$$= \mathrm{Q}_{\delta_x}\Big[\exp\Big\{-\int_0^t \mathrm{d}r \int_{(0,\infty)} \mathrm{P}_{y\delta_{\tilde{\xi}_r}}[1-e^{-X_{t+s-r}(f)}]y\pi(\tilde{\xi}_r,\mathrm{d}r) -$$

$$\int_0^t 2\sigma(\tilde{\xi}_r)^2\mathrm{d}r \int_\Omega (1-e^{-w_{t+s-r}(f)})\mathbb{N}_{\tilde{\xi}_r}(\mathrm{d}w)\Big\}\Big|\tilde{\xi}_t\Big]$$

$$= \mathrm{Q}_{\delta_x}[e^{-Z^{(0,t]}_{t+s}(f)}|\tilde{\xi}_t]$$

as claimed.

In particular, due to Lemma 2.3 and the bounded convergence theorem, for each $t > 0$, it holds that

$$\mathrm{Q}_{\delta_x}\big(Z^{(0,t]}_{t+s} = \mathbf{0}\big|\tilde{\xi}_t\big) = \mathrm{Q}_{\delta_x}[\mathrm{P}_{Z^{(0,t]}_t}(X_s = \mathbf{0})|\tilde{\xi}_t] \to 1 \quad \text{as } s \to \infty.$$

On the other hand, for any $\varepsilon > 0$, defining a function $F_{s,\varepsilon} \in \mathrm{bp}\mathscr{B}(E)$ by

$$F_{s,\varepsilon}(y) := \mathbf{1}_{\{|\mathcal{L}_s f(y) - \tilde{\nu}(\mathcal{L}_\infty f)| > \varepsilon\}}, \quad y \in E,$$

we can verify from (4.24), (1.9) and Step 2 that, for any $t > 0$,

$$\mathrm{Q}_{\delta_x}\big(|\mathcal{L}_s f(\tilde{\xi}_t) - \tilde{\nu}(\mathcal{L}_\infty f)| > \varepsilon\big) = S_t F_{s,\varepsilon}(x) = \phi(x)^{-1}e^{-\lambda t}T_t(\phi \cdot F_{s,\varepsilon})(x)$$
$$= \tilde{\nu}(F_{s,\varepsilon})\big(1 + (H_t F_{s,\varepsilon})(x)\big) \le \tilde{\nu}(F_{s,\varepsilon})\big(1 + \sup_{g \in L_1^+(\nu), x \in E} |H_t g(x)|\big) \xrightarrow[s\to\infty]{} 0.$$

Therefore, for any $t > 0$, $\mathcal{L}_s f(\tilde{\xi}_t) \cdot \mathrm{Q}_{\delta_x}\big(Z^{(0,t]}_{t+s} = \mathbf{0}\big|\tilde{\xi}_t\big)$ converges to $\tilde{\nu}(\mathcal{L}_\infty f)$ in probability with respect to Q_{δ_x} as $s \to \infty$. By taking limit inferior in (4.26) as $s \to \infty$ and using the bounded convergence theorem, we get the desired result for this step.

Final Step. Combine the results in Steps 1 and 3. □

We are now ready to give the proof of Proposition 4.4.

Proof of Proposition 4.4 in the case $\mathcal{K} > 0$. *Step 1.* Let $f \in \mathrm{bp}\mathscr{B}(E)$ and $t \ge 0$ be arbitrary. We will show that $\mathrm{Q}_\mu[e^{-Z^{(0,t]}_t(f)}] = \tilde{\mu}(\mathcal{L}_t f)$. To do this, we note that by Lemma 4.17,

$$\mathcal{L}_t f(x) = \widetilde{\delta_x \mathbb{N}}^{(t)}[e^{-X_t(f)}] = \mathbb{N}_x\Big[e^{-X_t(f)}\frac{X_t(\phi)}{e^{\lambda t}\phi(x)}\Big], \quad x \in E.$$

Therefore, again by Lemma 4.17 we have

$$\mathrm{Q}_\mu[e^{-Z^{(0,t]}_t(f)}] = \widetilde{\mu\mathbb{N}}^{(t)}[e^{-X_t(f)}] = \int_E \mathbb{N}_x\Big[e^{-X_t(f)}\frac{X_t(\phi)}{e^{\lambda t}\mu(\phi)}\Big]\mu(\mathrm{d}x)$$
$$= \mu(\phi)^{-1}\int_E (\mathcal{L}_t f)(x) \cdot \phi(x)\mu(\mathrm{d}x) = \tilde{\mu}(\mathcal{L}_t f).$$

Step 2. We will show that for any $\mu \in \mathcal{M}^o$ and $f \in \mathrm{bp}\mathscr{B}(E)$,

$$\lim_{t\to\infty}\int_{\mathcal{M}} e^{-\eta(f)}\mathbf{Q}^\mu_{t,\infty}(\mathrm{d}\eta) = \tilde{\nu}(\mathcal{L}_\infty f).$$

In fact, from Lemmas 3.1, 4.17 and Step 1, we have that

$$\int_{\mathcal{M}} e^{-\eta(f)}\mathbf{Q}^\mu_{t,\infty}(\mathrm{d}\eta) = \widetilde{\mathrm{P}}^{(\infty)}_\mu[e^{-X_t(f)}]$$
$$= \mathrm{Q}_\mu[e^{-W^{(0)}_t(f) - Z^{(0,t]}_t(f)}] = \mathrm{P}_\mu[e^{-X_t(f)}]\tilde{\mu}(\mathcal{L}_t f), \quad t \ge 0.$$

Now the desired result in this step follows from Lemmas 2.3, 4.18 and the bounded convergence theorem.

Final Step. From Lemma 4.16 for any $f \in \mathrm{bp}C(E)$,

$$\lim_{t\to\infty}\int_{\mathcal{M}} e^{-\eta(f)}\mathbf{Q}^{\nu}_{t,\infty}(\mathrm{d}\eta) = \int_{\mathcal{M}} e^{-\eta(f)}\mathbf{Q}_{\infty,\infty}(\mathrm{d}\eta).$$

Combining this with Step 2 and [25, Theorem 1.18], we get the desired result. □

Proof of Proposition 4.4 in the case $\mathcal{K}=0$. We give a proof by contradiction. Assume that there exists $\mu \in \mathcal{M}^o$ such that $\mathbf{Q}^{\mu}_{t,\infty}$ converges weakly to a probability measure, say $\mathbf{Q}^*$ on $\mathcal{M}$. Then we have that

$$\int_{\mathcal{M}} e^{-\eta(\phi)}\mathbf{Q}^{\mu}_{t,\infty}(\mathrm{d}\eta) \geq \int_{\mathcal{M}} e^{-\eta(\|\phi\|_\infty \mathbf{1}_E)}\mathbf{Q}^{\mu}_{t,\infty}(\mathrm{d}\eta) \xrightarrow[t\to\infty]{} \int_{\mathcal{M}} e^{-\eta(\|\phi\|_\infty \mathbf{1}_E)}\mathbf{Q}^{*}(\mathrm{d}\eta) > 0.$$

On the other hand, from $xe^{-x} \leq 1$ for every $x \geq 0$, and Proposition 4.1, we have

$$\int_{\mathcal{M}} e^{-\eta(\phi)}\mathbf{Q}^{\mu}_{t,\infty}(\mathrm{d}\eta) \leq \int_{\mathcal{M}} \eta(\phi)^{-1}\mathbf{Q}^{\mu}_{t,\infty}(\mathrm{d}\eta) = e^{-\lambda t}\mu(\phi)^{-1}\mathrm{P}_{\mu}(X_t \neq \mathbf{0}) \xrightarrow[t\to\infty]{} 0$$

which is a contradiction.

4.5. Limit of the two-sided process

Proof of Proposition 4.5 in the case $\mathcal{K}>0$. From (2.4) and (2.5) we have

$$\begin{aligned}\mathbf{Q}_{\infty,r}[F] &= \int_{\mathcal{M}^o} e^{-\lambda r}\mathrm{P}_\eta[F(X_0)\mathbf{1}_{\{X_r\neq\mathbf{0}\}}]\mathbf{Q}_{\infty,0}(\mathrm{d}\eta)\\ &= \int_{\mathcal{M}^o} F(\eta)e^{-\lambda r}\mathrm{P}_\eta(X_r\neq\mathbf{0})\mathbf{Q}_{\infty,0}(\mathrm{d}\eta), \quad F\in \mathrm{bp}\mathscr{B}(\mathcal{M}), r\geq 0. \end{aligned} \tag{4.27}$$

Note that by Proposition 4.3, $\eta(\phi)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta)$ is a finite measure concentrated on $\mathcal{M}^o$, and that by Lemma 3.4 and Proposition 4.1, for any $F \in \mathrm{bp}\mathscr{B}(\mathcal{M})$, and r large enough,

$$\sup_{\eta\in\mathcal{M}^o} F(\eta)e^{-\lambda r}\mathrm{P}_\eta(X_r\neq\mathbf{0})\frac{1}{\eta(\phi)} \leq \|F\|_\infty e^{-\lambda r}\mathrm{P}_\nu(X_r\neq\mathbf{0}) \sup_{\eta\in\mathcal{M}^o}\frac{1}{\eta(\phi)}\frac{\mathrm{P}_\eta(X_r\neq\mathbf{0})}{\mathrm{P}_\nu(X_r\neq\mathbf{0})} < \infty.$$

Now by Proposition 4.1 and the bounded convergence theorem we have

$$\lim_{r\to\infty}\mathbf{Q}_{\infty,r}[F] = \int_{\mathcal{M}^o} F(\eta)\cdot\mathcal{K}\eta(\phi)\mathbf{Q}_{\infty,0}(\mathrm{d}\eta) = \mathbf{Q}_{\infty,\infty}[F], \quad F\in\mathrm{bp}\mathscr{B}(\mathcal{M})$$

as desired. □

Proof of Proposition 4.5 in the case $\mathcal{K}=0$. We give a proof by contradiction. Assume that $\mathbf{Q}_{\infty,r}$ converges strongly to a probability measure, say $\mathbf{Q}^*$, as $r\to\infty$. Taking $F(\eta) := e^{-\eta(\phi)}$, $\eta\in\mathcal{M}$ we have $\lim_{r\to\infty}\mathbf{Q}_{\infty,r}[F] = \mathbf{Q}^*[F] > 0$. On the other hand, we first observe from $\sup_{x\geq 0} xe^{-x} \leq 1$ that $F(\eta)\leq \eta(\phi)^{-1}$ for every $\eta\in\mathcal{M}$. Then, noticing that (4.27) still holds in this case, and also noticing from Proposition 4.1 and Lemma 3.4 that

$$\sup_{\eta\in\mathcal{M}^o} F(\eta)e^{-\lambda r}\mathrm{P}_\eta(X_r\neq\mathbf{0}) \leq e^{-\lambda r}\mathrm{P}_\nu(X_r\neq\mathbf{0}) \sup_{\eta\in\mathcal{M}^o}\frac{1}{\eta(\phi)}\frac{\mathrm{P}_\eta(X_r\neq\mathbf{0})}{\mathrm{P}_\nu(X_r\neq\mathbf{0})} \xrightarrow[r\to\infty]{} 0.$$

Using the bounded convergence theorem we have $\lim_{r\to\infty}\mathbf{Q}_{\infty,r}[F] = 0$, which is a contradiction. □

4.6. Double limit

Lemma 4.19. *If $\mathcal{E} < \infty$, then*

$$\lim_{t\to\infty} \sup_{r\geq 0} \frac{e^{\lambda r}\nu(v_t)}{\nu(v_{t+r})} = 1.$$

Proof. According to [28, (3.10)] ,

$$\frac{e^{\lambda r}\nu(v_t)}{\nu(v_{t+r})} = \exp\left\{\int_t^{t+r} \frac{\nu(\Psi_0 v_s)}{\nu(v_s)}\mathrm{d}s\right\}, \quad t > 0, r \geq 0,$$

where $\Psi_0 v_s(x) := \psi_0(x, v_s(x))$, $x \in E$, $s > 0$, and

$$\psi_0(x,\lambda) := \sigma(x)^2\lambda^2 + \int_0^\infty \left(e^{-\lambda u} - 1 + \lambda u\right)\pi(x, \mathrm{d}u), \quad x \in E, \lambda \geq 0.$$

To prove the desired result, it suffices to show

$$\int_t^\infty \frac{\nu(\Psi_0 v_s)}{\nu(v_s)}\mathrm{d}s < \infty, \qquad \text{for some } t \geq 0. \tag{4.28}$$

It is easy to see that for any $x \in E$,

$$\frac{\partial\psi_0(x,\lambda)}{\partial\lambda} = 2\sigma(x)^2\lambda + \int_0^\infty \left(1 - e^{-\lambda u}\right) u\pi(x, \mathrm{d}u)$$

is a nonnegative increasing function with respect to λ and $\psi_0(x, 0) = 0$. Thanks to the mean value theorem,

$$\psi_0(x,\lambda) = \psi_0(x,\lambda) - \psi_0(x, 0) \leq \lambda\frac{\partial\psi_0(x,\lambda)}{\partial\lambda}, \quad x \in E, \lambda \geq 0.$$

Therefore,

$$\Psi_0 v_s(x) \leq v_s(x) \left.\frac{\partial\psi_0(x,\lambda)}{\partial\lambda}\right|_{\lambda=v_s(x)}, \quad x \in E, s > 0.$$

By Lemma 2.1, there exists $T_0 > 0$ such that $v_s(x) \leq 2\phi(x)\nu(v_s)$ for any $x \in E$ and $s > T_0$. For $s > T_0$,

$$\nu(\Psi_0 v_s) \leq 2\nu(v_s)\nu\left(\phi \left.\frac{\partial\psi_0(\cdot,\lambda)}{\partial\lambda}\right|_{\lambda=v_s(\cdot)}\right).$$

Note that for any $x \in E$ and $s > T_0$,

$$\begin{aligned}\phi(x) \left.\frac{\partial\psi_0(\cdot,\lambda)}{\partial\lambda}\right|_{\lambda=v_s(x)} &= 2\sigma(x)^2\phi(x)v_s(x) + \int_0^\infty \left(1 - e^{-v_s(x)u}\right)\phi(x)u\pi(x, \mathrm{d}u)\\ &\leq 2\|\sigma^2\phi\|_\infty v_s(x) + \int_0^\infty \left(1 - e^{-2\nu(v_s)\phi(x)u}\right)\phi(x)u\pi(x, \mathrm{d}u).\end{aligned}$$

Define a measure ρ on $(0, \infty)$ so that for any non-negative Borel function f on $(0, \infty)$,

$$\int_{(0,\infty)} f(u)\rho(\mathrm{d}u) = \int_E \nu(\mathrm{d}x)\int_{(0,\infty)} f(\phi(x)u)\pi(x, \mathrm{d}u).$$

Then for any $s > T_0$,

$$\nu(\Psi_0 v_s) \leq 2\nu(v_s)\left[2\|\sigma^2\phi\|_\infty\nu(v_s) + \int_0^\infty \left(1 - e^{-2\nu(v_s)u}\right)u\rho(\mathrm{d}u)\right].$$

Now the integral on the left hand side of (4.28) can be bounded by

$$\int_t^\infty \frac{\nu(\Psi_0 v_s)}{\nu(v_s)}\mathrm{d}s \le 4\|\sigma^2\phi\|_\infty \mathrm{I}_t + 2\mathrm{II}_t, \quad t > T_0, r > 0,$$

where

$$\mathrm{I}_t := \int_t^\infty \nu(v_s)\mathrm{d}s, \text{ and } \mathrm{II}_t := \int_t^\infty \mathrm{d}s \int_0^\infty \left(1 - e^{-2\nu(v_s)u}\right) u\rho(\mathrm{d}u).$$

From (2.2) and Propositions 4.1 and 4.2, we know that $e^{-\lambda t}\nu(v_t) \to \mathcal{K} > 0$ as $t \to \infty$. In particular, there exist $T_1 \ge T_0$ and $C_4 > 0$ such that $\nu(v_s) \le C_4 e^{\lambda s}$ for every $s \ge T_1$. This yields $\mathrm{I}_t < \infty$ for $t \ge T_1$. Note that $\int_0^\infty \left(1 - e^{-2\theta u}\right) u\rho(\mathrm{d}u)$ is an increasing function with respect to θ. Thus for sufficiently large t so that $t \ge T_1$ and $2C_4 e^{\lambda t} \le 1$,

$$\begin{aligned}\mathrm{II}_t &\le \int_t^\infty \mathrm{d}s \int_0^\infty \left(1 - e^{-2C_4 e^{\lambda s} u}\right) u\rho(\mathrm{d}u) = \frac{1}{|\lambda|}\int_0^\infty u\rho(\mathrm{d}u) \int_0^{2C_4 u e^{\lambda t}} \frac{1-e^{-z}}{z}\mathrm{d}z \\ &\le \frac{1}{|\lambda|}\int_0^\infty u\rho(\mathrm{d}u)\int_0^u \frac{1-e^{-z}}{z}\mathrm{d}s \le \frac{1}{|\lambda|}\int_0^1 u^2\rho(\mathrm{d}u) + \frac{1}{|\lambda|}\int_1^\infty u(1+\log u)\rho(\mathrm{d}u).\end{aligned}$$

When $\mathcal{E} < \infty$, it is easy to see that $\mathrm{II}_t < \infty$. The proof is complete. □

Lemma 4.20. *Suppose $\mathcal{E} < \infty$. Then for any $\mu \in \mathcal{M}^o$,*

$$\limsup_{t\to\infty} \sup_{r\ge 0} \sup_{\eta\in\mathcal{M}^o} \frac{\mu(\phi)e^{\lambda t}\mathrm{P}_\eta(X_r \ne 0)}{\eta(\phi)\mathrm{P}_\mu(X_{t+r}\ne 0)} \le 16.$$

Proof. It is well known that $1-e^{-u} \le u$ and that there is a $\delta > 0$ such that $1-e^{-u} \ge u/2$ for $u \in [0,\delta]$. By [28, (3.39)] we have $\lim_{t\to\infty}\nu(v_t) = 0$. By Lemma 2.1, given $\mu \in \mathcal{M}^o$, there exists $T_2(\mu) > 0$ such that $\mu(v_t) \le \delta$ for $t \ge T_2(\mu)$. Therefore, for any $\eta, \mu \in \mathcal{M}^o$, $t \ge T_2(\mu)$ and $r \ge 0$, we have

$$\frac{\mathrm{P}_\eta(X_r \ne 0)}{\mathrm{P}_\mu(X_{t+r}\ne 0)} = \frac{1-e^{-\eta(v_r)}}{1-e^{-\mu(v_{t+r})}} \le \frac{2\eta(v_r)}{\mu(v_{t+r})}.$$

The uniform lower and upper bounds of $v_r(x)$ are given in Lemma 2.1 as well: There is a $T_3 \ge 0$, such that $\phi(x)\nu(v_t)/2 \le v_t(x) \le 2\phi(x)\nu(v_t)$ for $t \ge T_3$ and $x \in E$. From Lemma 4.19, there exists a $T_4 > 0$, such that $e^{\lambda t}\nu(v_r)/\nu(v_{t+r}) \le 2$ for every $t \ge T_4$ and $r > 0$. Now for any $\eta, \mu \in \mathcal{M}^o$, $t \ge T_2(\mu) \vee T_3 \vee T_4$ and $r > 0$,

$$\frac{\mu(\phi)e^{\lambda t}\mathrm{P}_\eta(X_r \ne 0)}{\eta(\phi)\mathrm{P}_\mu(X_{t+r}\ne 0)} \le \frac{\mu(\phi)e^{\lambda t}}{\eta(\phi)}\frac{4\eta(\phi)\nu(v_r)}{\mu(\phi)\nu(v_{t+r})/2} \le \frac{8e^{\lambda t}\nu(v_r)}{\nu(v_{t+r})} \le 16.$$

The proof is complete. □

Proof of Proposition 4.6. Fix arbitrary $\mu \in \mathcal{M}^o$ and non-negative continuous function f on E. For any $t, r > 0$,

$$\begin{aligned}\mathrm{P}_\mu\left(e^{-X_t(f)}|X_{t+r}\ne 0\right) &= \frac{\mathrm{P}_\mu\left(e^{-X_t(f)}\mathbf{1}_{\{X_{t+r}\ne 0\}}\right)}{\mathrm{P}_\mu(X_{t+r}\ne 0)} \\ &= \frac{\mathrm{P}_\mu\left(e^{-X_t(f)}\mathrm{P}_{X_t}(X_r\ne 0)\right)}{\mathrm{P}_\mu(X_{t+r}\ne 0)} = \widetilde{\mathrm{P}}_\mu^{(\infty)}\left(e^{-X_t(f)}\frac{\mu(\phi)e^{\lambda t}\mathrm{P}_{X_t}(X_r\ne 0)}{X_t(\phi)\mathrm{P}_\mu(X_{t+r}\ne 0)}\right)\end{aligned}$$

where the probability $\widetilde{\mathrm{P}}_{\mu}^{(\infty)}$ is given as in Lemma 3.1. According to the Skorohod representation theorem (see [19, Theorem 5.31] , for example) there exists an $\mathcal{M}$-valued process $(\hat{X}_t)_{t\geq 0}$ converging almost surely to an $\mathcal{M}$-valued random element $\hat{X}_\infty$ on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathrm{P}})$ so that the law of $\hat{X}_\infty$ is $\mathbf{Q}_{\infty,\infty}$ and the law of $\hat{X}_t$ is $\mathbf{Q}^{\mu}_{t,\infty}$ for every $t \geq 0$. Now by Remark 3.2 we have

$$\mathrm{P}_\mu\left(e^{-X_t(f)}|X_{t+r}\neq 0\right) = \hat{\mathrm{P}}\left(e^{-\hat{X}_t(f)}\frac{\mu(\phi)e^{\lambda t}\mathrm{P}_{\hat{X}_t}(X_r\neq 0)}{\hat{X}_t(\phi)\mathrm{P}_\mu(X_{t+r}\neq 0)}\right), \quad t,r\geq 0.$$

Note that, by Lemma 4.20, there exists $T_5(\mu) > 0$ such that for any $t \geq T_5(\mu)$ and $r \geq 0$,

$$e^{-\hat{X}_t(f)}\frac{\mu(\phi)e^{\lambda t}\mathrm{P}_{\hat{X}_t}(X_r\neq 0)}{\hat{X}_t(\phi)\mathrm{P}_\mu(X_{t+r}\neq 0)}\leq 17.$$

Also notice that $e^{-\hat{X}_t(f)}$ converges almost surely to $e^{-\hat{X}_\infty(f)}$ as $t\to\infty$. So now if one can show that almost surely

$$\lim_{t,r\to\infty}\frac{\mu(\phi)e^{\lambda t}\mathrm{P}_{\hat{X}_t}(X_r\neq 0)}{\hat{X}_t(\phi)\mathrm{P}_\mu(X_{t+r}\neq 0)} = 1, \tag{4.29}$$

then by the bounded convergence theorem we get the desire result for this proposition.

Let us now verify (4.29). From (2.2), we have

$$\frac{\mu(\phi)e^{\lambda t}\mathrm{P}_{\hat{X}_t}(X_r\neq 0)}{\hat{X}_t(\phi)\mathrm{P}_\mu(X_{t+r}\neq 0)} = \frac{1-e^{-\hat{X}_t(v_r)}}{\hat{X}_t(v_r)}\frac{\mu(v_{t+r})}{1-e^{-\mu(v_{t+r})}}\frac{\mu(\phi)e^{\lambda t}\hat{X}_t(v_r)}{\hat{X}_t(\phi)\mu(v_{t+r})}, \quad t,r\geq 0. \tag{4.30}$$

By Lemma 2.1, we have for $t, r \geq 0$,

$$\nu(v_r)\hat{X}_t(\phi)\Big(1-\sup_{x\in E}|C_1(r,x)|\Big)\leq \hat{X}_t(v_r)\leq \nu(v_r)\hat{X}_t(\phi)\Big(1+\sup_{x\in E}|C_1(r,x)|\Big),$$

with $\lim_{r\to\infty}\sup_{x\in E}|C_1(r,x)| = 0$. By [28, (3.39)] we have $\lim_{r\to\infty}\nu(v_r) = 0$ and that

$$\limsup_{t\to\infty}\hat{X}_t(\phi)\leq\lim_{t\to\infty}\hat{X}_t(\|\phi\|_\infty\mathbf{1}_E) = \hat{X}_\infty(\|\phi\|_\infty\mathbf{1}_E)<\infty, \quad \text{a.s.}$$

Thus $\lim_{t,r\to\infty}\hat{X}_t(v_r) = 0$ almost surely. Therefore,

$$\lim_{t,r\to\infty}\frac{1-e^{-\hat{X}_t(v_r)}}{\hat{X}_t(v_r)} = 1, \quad \text{a.s.}$$

Similarly we have for every $t, r \geq 0$,

$$\nu(v_{t+r})\mu(\phi)\Big(1-\sup_{x\in E}|C_1(t+r,x)|\Big)\leq\mu(v_{t+r})\leq\nu(v_{t+r})\mu(\phi)\Big(1+\sup_{x\in E}|C_1(t+r,x)|\Big),$$

and

$$\lim_{t,r\to\infty}\frac{\mu(v_{t+r})}{1-e^{-\mu(v_{t+r})}} = 1.$$

Using Lemma 4.19 for the third fraction on the right hand side of (4.30), we get

$$\begin{aligned}\limsup_{t,r\to\infty}\frac{\mu(\phi)e^{\lambda t}\hat{X}_t(v_r)}{\hat{X}_t(\phi)\mu(v_{t+r})}&\leq\limsup_{t,r\to\infty}\frac{\mu(\phi)e^{\lambda t}\nu(v_r)\hat{X}_t(\phi)(1+\sup_{x\in E}|C_1(r,x)|)}{\hat{X}_t(\phi)\nu(v_{t+r})\mu(\phi)(1-\sup_{x\in E}|C_1(t+r,x)|)}\\&=\limsup_{t,r\to\infty}\frac{e^{\lambda t}\nu(v_r)}{\nu(v_{t+r})} = 1.\end{aligned}$$

On the other hand,

$$\liminf_{t,r\to\infty}\frac{\mu(\phi)e^{\lambda t}\hat{X}_t(v_r)}{\hat{X}_t(\phi)\mu(v_{t+r})}\geq\liminf_{t,r\to\infty}\frac{\mu(\phi)e^{\lambda t}\nu(v_r)\hat{X}_t(\phi)(1-\sup_{x\in E}|C_1(r,x)|)}{\hat{X}_t(\phi)\nu(v_{t+r})\mu(\phi)(1+\sup_{x\in E}|C_1(t+r,x)|)}$$
$$=\liminf_{t,r\to\infty}\frac{e^{\lambda t}\nu(v_r)}{\nu(v_{t+r})}=1.$$

The proof is complete. □

Acknowledgments

We thank the referees for the helpful comments and suggestions. Part of the research for this paper was done while the third-named author was visiting Jiangsu Normal University, where he was partially supported by a grant from the National Natural Science Foundation of China (11931004, Yingchao Xie) and by the Priority Academic Program Development of Jiangsu Higher Education Institutions, China. Part of the research for this paper was down while the fourth-named author was a Postdoc at the Technion—Israel Institute of Technology where he was supported by a fellowship of the Israel Council for Higher Education.

Appendix

Lemma A.1. *The transition semigroup $(Q_t)_{t\geq 0}$ given in (1.6) preserves* $b\mathscr{B}(\mathcal{M})$.

Proof. Denote $\overline{E} := E\cup\{\partial\}$, where ∂ is an isolated point not contained in E. The Polish space of all finite Borel measures on $\overline{E}$, equipped with the topology of weak convergence, is denoted by $\mathcal{M}(\overline{E})$. Define a conservative Borel right transition semigroup $(\overline{P}_t)_{t\geq 0}$ on $(\overline{E},\mathscr{B}(\overline{E}))$ using [25, (A.20)] . Let $\overline{\xi}$ be a Borel right process with transition semigroup $(\overline{P}_t)_{t\geq 0}$. Define $\overline{\psi}$ as the extension of ψ on $\overline{E}\times\mathbb{R}_+$ such that $\overline{\psi}(\partial,\cdot)\equiv 0$. Let $\overline{X}$ be a $(\overline{\xi},\overline{\psi})$-superprocess whose transition semigroup is denoted by $(\overline{Q}_t)_{t\geq 0}$. According to [25, Theorem 5.11] , $(\overline{Q}_t)_{t\geq 0}$ is a Borel right transition semigroup on $\mathcal{M}(\overline{E})$. Define a map $\Gamma:\mathcal{M}(\overline{E})\to\mathcal{M}$ so that for any $\mu\in\mathcal{M}(\overline{E})$, the measure $\Gamma\mu$ is the restriction of the set function μ on $\mathscr{B}(E)$. Define a map $\Lambda:\mathcal{M}\to\mathcal{M}(\bar{E})$ so that for any $\mu\in\mathcal{M}$, the measure $\Lambda\mu$ on $\overline{E}$ is the unique extension of μ on $\mathscr{B}(\overline{E})$ so that $\Lambda\mu(\{\partial\})=0$. Obviously we have $\Gamma\circ\Lambda$ is the identity map on $\mathcal{M}$; and from the fact that ∂ is an isolated point in $\overline{E}$ we know Γ and Λ are continuous maps. Fix an arbitrary $t\geq 0$ and $F\in b\mathscr{B}(\mathcal{M})$. It can be verified (see the proof of [25, Theorem 5.12]) that $\overline{Q}_t(F\circ\Gamma)(\overline{\mu})=(Q_tF)\circ\Gamma(\overline{\mu})$ for each $\overline{\mu}\in\mathcal{M}(\overline{E})$. From this we can verify that $Q_tF=(\overline{Q}_t(F\circ\Gamma))\circ\Lambda$ is a real valued bounded Borel function on $\mathcal{M}$. Therefore, the semigroup $(Q_t)_{t\geq 0}$ preserves $b\mathscr{B}(\mathcal{M})$ as desired. □

Proof of Lemma 4.13 (1). Fix arbitrary $0<\delta<\epsilon$. From (4.7), we have

$$\sum_{k=1}^{\infty}\mathbf{1}_{(-\infty,0]}(s_k)y_ke^{\epsilon s_k}\phi(\tilde{\xi}_{s_k})=\int_{\mathbb{R}\times\mathbb{R}_+}\mathbf{1}_{\{s\leq 0\}}ye^{\epsilon s}\phi(\tilde{\xi}_s)\mathcal{D}(\mathrm{d}s,\mathrm{d}y)=\mathrm{I}+\mathrm{II},$$

where

$$\mathrm{I}:=\int_{\mathbb{R}\times\mathbb{R}_+}\mathbf{1}_{\{y\leq\frac{e^{-\delta s}}{\phi(\tilde{\xi}_s)},s\leq 0\}}ye^{\epsilon s}\phi(\tilde{\xi}_s)\mathcal{D}(\mathrm{d}s,\mathrm{d}y),$$

$$\mathrm{II} := \int_{\mathbb{R}\times\mathbb{R}_+} \mathbf{1}_{\{y > \frac{e^{-\delta s}}{\phi(\tilde{\xi}_s)}, s\le 0\}} y e^{\epsilon s}\phi(\tilde{\xi}_s)\mathcal{D}(\mathrm{d}s, \mathrm{d}y).$$

We first show that $\mathrm{II} < \infty$, Q-a.s. In fact, from (4.6) and (1.12),

$$\begin{aligned}
\mathrm{Q}\Big[\int_{-\infty}^0 \mathrm{d}s \int_{\frac{e^{-\delta s}}{\phi(\tilde{\xi}_s)}}^\infty y\pi(\tilde{\xi}_s, \mathrm{d}y)\Big] &= \int_{-\infty}^0 \mathrm{d}s \int_E \tilde{\nu}(\mathrm{d}x) \int_{\frac{e^{-\delta s}}{\phi(x)}}^\infty y\pi(x, \mathrm{d}y) \\
&= \int_{-\infty}^0 \mathrm{d}s \int_E \phi(x)\nu(\mathrm{d}x) \int_{\frac{e^{-\delta s}}{\phi(x)}}^\infty y\pi(x, \mathrm{d}y) \\
&= \int_E \nu(\mathrm{d}x) \int_{\frac{1}{\phi(x)}}^\infty y\phi(x)\pi(x, \mathrm{d}y) \int_{-\frac{\log(y\phi(x))}{\delta}}^0 \mathrm{d}s = \frac{\mathcal{E}}{\delta} < \infty.
\end{aligned}$$

Therefore, we have Q-almost surely

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}_+} \Big[1 \wedge \Big(\mathbf{1}_{\{y > \frac{e^{-\delta s}}{\phi(\tilde{\xi}_s)}, s\le 0\}} y e^{\epsilon s}\phi(\tilde{\xi}_s)\Big)\Big]\mathrm{d}s \cdot y\pi(\tilde{\xi}_s, \mathrm{d}y) \\
&\le \int_{\mathbb{R}\times\mathbb{R}_+} \mathbf{1}_{\{y > \frac{e^{-\delta s}}{\phi(\tilde{\xi}_s)}, s\le 0\}} \mathrm{d}s \cdot y\pi(\tilde{\xi}_s, \mathrm{d}y) < \infty.
\end{aligned}$$

Now from (4.7) and [21, Theorem 2.7(i)] we have $\mathrm{Q}(\mathrm{II} < \infty | \tilde{\xi}) = 1$. We then show that $\mathrm{I} < \infty$, Q-a.s. In fact, from (4.6) and (4.7),

$$\begin{aligned}
\mathrm{Q}[\mathrm{I}] &= \mathrm{Q}\big[\mathrm{Q}[\mathrm{I}|\xi]\big] = \mathrm{Q}\Big[\int_{-\infty}^0 \mathrm{d}s \int_0^{\frac{e^{-\delta s}}{\phi(\tilde{\xi}_s)}} y e^{\epsilon s}\phi(\tilde{\xi}_s) y\pi(\tilde{\xi}_s, \mathrm{d}y)\Big] \\
&= \mathrm{Q}\Big[\int_{-\infty}^0 \mathrm{d}s \int_0^{1\wedge\frac{e^{-\delta s}}{\phi(\tilde{\xi}_s)}} e^{\epsilon s} y^2\phi(\tilde{\xi}_s)\pi(\tilde{\xi}_s, \mathrm{d}y) + \int_{-\infty}^0 \mathrm{d}s \int_{1\wedge\frac{e^{-\delta s}}{\phi(\tilde{\xi}_s)}}^{\frac{e^{-\delta s}}{\phi(\tilde{\xi}_s)}} e^{\epsilon s} y\phi(\tilde{\xi}_s) y\pi(\tilde{\xi}_s, \mathrm{d}y)\Big] \\
&\le \mathrm{Q}\Big[\int_{-\infty}^0 \mathrm{d}s \int_0^1 e^{\epsilon s} y^2\phi(\tilde{\xi}_s)\pi(\tilde{\xi}_s, \mathrm{d}y) + \int_{-\infty}^0 \mathrm{d}s \int_1^\infty e^{(\epsilon-\delta)s} y\pi(\tilde{\xi}_s, \mathrm{d}y)\Big] \\
&\le \|\phi\|_\infty \int_{-\infty}^0 e^{\epsilon s}\mathrm{d}s \int_E \tilde{\nu}(\mathrm{d}x) \int_0^1 y^2\pi(x, \mathrm{d}y) + \int_{-\infty}^0 e^{(\epsilon-\delta)s}\mathrm{d}s \int_E \tilde{\nu}(\mathrm{d}x) \int_1^\infty y\pi(x, \mathrm{d}y) \\
&< \infty.
\end{aligned}$$

Now the desired result of this lemma follows. □

Proof of Lemma 4.13 (2). Fix the arbitrary $\epsilon > 0$ and $s_0 \ge 0$. Define a constant

$$K := \max\{\|\phi\|_\infty, e^{\epsilon s_0}\}, \tag{A.1}$$

and random variables

$$\eta_T := \int_{-T}^0 \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^\infty y\pi(\tilde{\xi}_s, \mathrm{d}y), \quad T \in (0, \infty].$$

Step 1. We will show that $\mathrm{Q}[\eta_\infty] = \infty$. In particular, this implies that there exists a $t_1 > 0$ such that $\mathrm{Q}[\eta_T] > 0$ for all $T \geq t_1$. To show that $\mathrm{Q}[\eta_\infty] = \infty$, we note from (4.6) and Fubini's theorem that

$$\begin{aligned}
\mathrm{Q}[\eta_\infty] &= \int_{-\infty}^{0} \mathrm{d}s \int_E \tilde{\nu}(\mathrm{d}x) \int_{\frac{Ke^{-\epsilon s}}{\phi(x)}}^{\infty} y\pi(x,\mathrm{d}y) \\
&= \int_E \phi(x)\nu(\mathrm{d}x) \int_{\frac{K}{\phi(x)}}^{\infty} y\pi(x,\mathrm{d}y) \int_{-\frac{1}{\epsilon}\log(\frac{y\phi(x)}{K})}^{0} \mathrm{d}s \\
&= \frac{1}{\epsilon}\int_E \phi(x)\nu(\mathrm{d}x) \int_{\frac{K}{\phi(x)}}^{\infty} \Big(\log\big(y\phi(x)\big) - \log K\Big) y\pi(x,\mathrm{d}y) \\
&\geq \frac{1}{\epsilon}\int_E \nu(\mathrm{d}x) \int_{\frac{K}{\phi(x)}}^{\infty} y\phi(x)\log\big(y\phi(x)\big)\pi(x,\mathrm{d}y) - \frac{A}{\epsilon},
\end{aligned} \tag{A.2}$$

where

$$A := \ln K \cdot \sup_{x\in E} \int_1^\infty y\pi(x,\mathrm{d}y) < \infty. \tag{A.3}$$

Since

$$\int_E \nu(\mathrm{d}x) \int_{\frac{1}{\phi(x)}}^{\infty} y\phi(x)\log\big(y\phi(x)\big)\pi(x,\mathrm{d}y) = \mathcal{E} = \infty$$

and

$$\int_E \nu(\mathrm{d}x) \int_{\frac{1}{\phi(x)}}^{\frac{K}{\phi(x)}} y\phi(x)\log\big(y\phi(x)\big)\pi(x,\mathrm{d}y) \leq K\log K \int_E \nu(\mathrm{d}x) \int_{\frac{1}{\|\phi\|_\infty}}^{\infty} \pi(x,\mathrm{d}y) < \infty,$$

we get that

$$\int_E \nu(\mathrm{d}x) \int_{\frac{K}{\phi(x)}}^{\infty} y\phi(x)\log\big(y\phi(x)\big)\pi(x,\mathrm{d}y) = \infty.$$

Now the desired result in this step follows from (A.2), (A.3) and above.

Step 2. We will show that $\mathrm{Q}[\eta_T] < \infty$ for all $T \in (0,\infty)$. From (4.6), (A.1) and Fubini's theorem, we have

$$\begin{aligned}
\mathrm{Q}[\eta_T] &= \int_{-T}^{0} \mathrm{d}s \int_E \tilde{\nu}(\mathrm{d}x) \int_{\frac{Ke^{-\epsilon s}}{\phi(x)}}^{\infty} y\pi(x,\mathrm{d}y) \leq \int_{-T}^{0} \mathrm{d}s \int_E \tilde{\nu}(\mathrm{d}x) \int_1^\infty y\pi(x,\mathrm{d}y) \\
&\leq T\cdot \sup_{x\in E}\int_1^\infty y\pi(x,\mathrm{d}y) < \infty.
\end{aligned} \tag{A.4}$$

Step 3. We will show that there exists a $t_2 > 0$ such that for any $t > t_2$, $x \in E$, and $f \in \mathrm{bp}\mathscr{B}(E)$, it holds that $S_t f(x) \leq 2\tilde{\nu}(f)$. In fact, let H be as in (1.9), then there exists a $t_2 > 0$ such that for any $t > t_2$, $x \in E$ and $f \in \mathrm{bp}\mathscr{B}(E)$, $|H_t(\phi f)(x)| \leq 1$. Now for any $t > t_2$, $x \in E$ and $f \in \mathrm{bp}\mathscr{B}(E)$, we can verify from (4.4) and (1.9) that

$$S_t f(x) = \frac{1}{\phi(x)e^{\lambda t}} T_t(\phi f)(x) = \nu(\phi f)(1 + H_t(\phi f)(x)) \leq 2\tilde{\nu}(f).$$

Step 4. Let $t_0 := \max\{t_1, t_2\}$. We will show that there exists a constant $C_5 > 0$ such that for all $T > t_0$, it holds that $\mathrm{Q}[\eta_T^2] \le C_5 \mathrm{Q}[\eta_T]^2$. To do this we note that for any $T > t_0$,

$$\begin{aligned}\mathrm{Q}[\eta_T^2] &= \mathrm{Q}\Big[\Big(\int_{-T}^{0} \mathrm{d}t \int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r)\Big) \cdot \Big(\int_{-T}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} u\pi(\tilde{\xi}_s, \mathrm{d}u)\Big)\Big] \\ &= 2\mathrm{Q}\Big[\int_{-T}^{0} \mathrm{d}t \int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r) \int_{t}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} u\pi(\tilde{\xi}_s, \mathrm{d}u)\Big] = \mathrm{III} + \mathrm{IV}\end{aligned}$$

where

$$\mathrm{III} := 2\mathrm{Q}\Big[\int_{-T}^{0} \mathrm{d}t \int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r) \int_{t}^{(t+t_0)\wedge 0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} u\pi(\tilde{\xi}_s, \mathrm{d}u)\Big]$$

and

$$\mathrm{IV} := 2\mathrm{Q}\Big[\int_{-T}^{0} \mathrm{d}t \int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r) \int_{(t+t_0)\wedge 0}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} u\pi(\tilde{\xi}_s, \mathrm{d}u)\Big].$$

We first show that there exists a constant $C_6 > 0$ such that for all $T > t_0$, it holds that $\mathrm{III} \le C_6 \mathrm{Q}[\eta_T]^2$. In fact, note $t \mapsto \mathrm{Q}[\eta_t]$ is non-decreasing, we have for any $T > t_0$,

$$\begin{aligned}\mathrm{III} &\le 2\mathrm{Q}\Big[\int_{-T}^{0} \mathrm{d}t \int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r) \int_{t}^{t+t_0} \mathrm{d}s \int_{1}^{\infty} u\pi(\tilde{\xi}_s, \mathrm{d}u)\Big] \\ &\le 2t_0\Big(\sup_{x\in E} \int_{1}^{\infty} u\pi(x, \mathrm{d}u)\Big)\mathrm{Q}[\eta_T] \le \frac{2t_0}{\mathrm{Q}[\eta_{t_0}]}\Big(\sup_{x\in E} \int_{1}^{\infty} u\pi(x, \mathrm{d}u)\Big)\mathrm{Q}[\eta_T]^2.\end{aligned}$$

Note that from Step 1,

$$\frac{2t_0}{\mathrm{Q}[\eta_{t_0}]}\Big(\sup_{x\in E} \int_{1}^{\infty} u\pi(x, \mathrm{d}u)\Big) < \infty.$$

We now show that for any $T > t_0$, $\mathrm{IV} \le 4\mathrm{Q}[\eta_T]^2$. For fixed $T > t_0$, it follows from Fubini's theorem and (4.6) that

$$\begin{aligned}\mathrm{IV} &= \mathrm{Q}\Big[2\int_{-T}^{-t_0} \mathrm{d}t \int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r) \int_{t+t_0}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} u\pi(\tilde{\xi}_s, \mathrm{d}u)\Big] \\ &= 2\int_{-T}^{-t_0} \mathrm{Q}\Big[\int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r) \int_{t+t_0}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} u\pi(\tilde{\xi}_s, \mathrm{d}u)\Big]\mathrm{d}t \\ &= 2\int_{-T}^{-t_0} \mathrm{Q}\Big[\mathrm{Q}\Big[\int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r) \int_{t+t_0}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} u\pi(\tilde{\xi}_s, \mathrm{d}u)\Big|\tilde{\mathscr{F}}_t\Big]\Big]\mathrm{d}t \\ &= 2\int_{-T}^{-t_0} \mathrm{Q}\Big[\int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r) \int_{t+t_0}^{0} \mathrm{Q}\Big[\int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} u\pi(\tilde{\xi}_s, \mathrm{d}u)\Big|\tilde{\mathscr{F}}_t\Big]\mathrm{d}s\Big]\mathrm{d}t \\ &= 2\int_{-T}^{-t_0} \mathrm{Q}\Big[\int_{\frac{Ke^{-\epsilon t}}{\phi(\tilde{\xi}_t)}}^{\infty} r\pi(\tilde{\xi}_t, \mathrm{d}r) \int_{t+t_0}^{0} S_{s-t}h_s(\tilde{\xi}_t)\mathrm{d}s\Big]\mathrm{d}t \\ &= 2\int_{-T}^{-t_0} \mathrm{d}t \int_{E} \tilde{\nu}(\mathrm{d}y) \int_{\frac{Ke^{-\epsilon t}}{\phi(y)}}^{\infty} r\pi(y, \mathrm{d}r) \int_{t+t_0}^{0} S_{s-t}h_s(y)\mathrm{d}s, \end{aligned} \tag{A.5}$$

where, for any $t \in \mathbb{R}$, $\tilde{\mathscr{F}}_t := \sigma(\tilde{\xi}_s : s \in (-\infty, t])$, and for any $s \le 0$ and $y \in E$,

$$h_s(y) := \int_{\frac{Ke^{-\epsilon s}}{\phi(y)}}^{\infty} u\pi(y, \mathrm{d}u) \le \sup_{x\in E} \int_1^{\infty} u\pi(x, \mathrm{d}u) < \infty. \tag{A.6}$$

Note for any $t \in (-T, -t_0)$ and $s \in (t + t_0, 0)$, we have $s - t \ge t_0 \ge t_2$, which, together with Step 3, implies that for any $y \in E$, $S_{s-t}h_s(y) \le 2\tilde{\nu}(h_s)$. Therefore, from (A.4), (A.5) and (A.6), we have

$$\begin{aligned} \mathrm{IV} &\le 4\int_{-T}^{-t_0} \mathrm{d}t \int_E \tilde{\nu}(\mathrm{d}y) \int_{\frac{Ke^{-\epsilon t}}{\phi(y)}}^{\infty} r\pi(y, \mathrm{d}r) \int_{t+t_0}^{0} \tilde{\nu}(h_s)\mathrm{d}s \\ &\le 4\int_{-T}^{0} \tilde{\nu}(h_t)\mathrm{d}t \cdot \int_{-T}^{0} \tilde{\nu}(h_s)\mathrm{d}s = 4\mathrm{Q}[\eta_T]^2. \end{aligned}$$

Now the desired result in this step follows.

Step 5. We show that $\mathrm{Q}(\eta_\infty = \infty) > 0$. Note that for any $T \ge 0$, from the Cauchy–Schwartz inequality,

$$\begin{aligned} &\sqrt{\mathrm{Q}[\eta_T^2]\mathrm{Q}\Big(\eta_T \ge \frac{1}{2}\mathrm{Q}[\eta_T]\Big)} \ge \mathrm{Q}[\eta_T \mathbf{1}_{\{\eta_T \ge \frac{1}{2}\mathrm{Q}[\eta_T]\}}] \\ &= \mathrm{Q}[\eta_T] - \mathrm{Q}[\eta_T \mathbf{1}_{\{\eta_T < \frac{1}{2}\mathrm{Q}[\eta_T]\}}] \ge \mathrm{Q}[\eta_T] - \mathrm{Q}\Big[\frac{1}{2}\mathrm{Q}[\eta_T]\mathbf{1}_{\{\eta_T < \frac{1}{2}\mathrm{Q}[\eta_T]\}}\Big] \ge \frac{1}{2}\mathrm{Q}[\eta_T]. \end{aligned}$$

Let t_0 be as in Step 4. Then, there exists a $C_7 > 0$ such that for any $T \ge t_0$,

$$\mathrm{Q}\Big(\eta_\infty \ge \frac{1}{2}\mathrm{Q}[\eta_T]\Big) \ge \mathrm{Q}\Big(\eta_T \ge \frac{1}{2}\mathrm{Q}[\eta_T]\Big) \ge \frac{\mathrm{Q}[\eta_T]^2}{4\mathrm{Q}[\eta_T^2]} \ge C_7. \tag{A.7}$$

By the monotone convergence theorem and Step 1, we have

$$\mathrm{Q}[\eta_T] \xrightarrow[T\to\infty]{} \mathrm{Q}[\eta_\infty] = \infty.$$

Now by (A.7) and the monotone convergence theorem again,

$$\mathrm{Q}(\eta_\infty = \infty) = \lim_{T\to\infty} \mathrm{Q}\Big(\eta_\infty \ge \frac{1}{2}\mathrm{Q}[\eta_T]\Big) \ge C_7.$$

Step 6. We will show that $\tilde{\nu}$ is an ergodic measure of the semigroup $(S_t)_{t\ge 0}$ in the sense of [4, Section 3.2]. To do this, we claim that for any $\varphi \in L^2(\tilde{\nu})$ satisfying $S_t\varphi = \varphi$ in $L^2(\tilde{\nu})$ for all $t \ge 0$, it holds that φ is a constant $\tilde{\nu}$-a.e. In fact for $\tilde{\nu}$- almost every $x \in E$, from (1.9) and (4.4), we have

$$\begin{aligned} \varphi(x) = S_t\varphi(x) &= \frac{1}{e^{\lambda t}\phi(x)}T_t(\phi\varphi^+)(x) - \frac{1}{e^{\lambda t}\phi(x)}T_t(\phi\varphi^-)(x) \\ &= \nu(\phi\varphi^+)\big(1 + H_t(\phi\varphi^+)(x)\big) - \nu(\phi\varphi^-)\big(1 + H_t(\phi\varphi^-)(x)\big) \xrightarrow[t\to\infty]{} \nu(\phi\varphi), \end{aligned}$$

which implies the desired claim. Now the desired result in this step follows from [4, Theorem 3.2.4.].

Step 7. We will show that $\{\eta_\infty = \infty\}$ is an invariant event for this ergodic process $(\tilde{\xi}_t)_{t\in\mathbb{R}}$ under Q in the sense that, for any $t \in \mathbb{R}$, $\mathrm{Q}(A_0 \Delta A_t) = 0$ where

$$A_t := \Big\{\int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+t})}}^{\infty} y\pi(\tilde{\xi}_{s+t}, \mathrm{d}y) = \infty\Big\}, \quad t \in \mathbb{R}.$$

We first claim that

$$A_r \subset A_{r-t}, \quad r \in \mathbb{R}, t > 0. \tag{A.8}$$

In fact, on the event A_r we have

$$\begin{aligned}
&\int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r-t})}}^{\infty} y\pi(\tilde{\xi}_{s+r-t}, \mathrm{d}y) \geq \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon(s-t)}}{\phi(\tilde{\xi}_{s+r-t})}}^{\infty} y\pi(\tilde{\xi}_{s+r-t}, \mathrm{d}y) \\
&= \int_{-\infty}^{-t} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r})}}^{\infty} y\pi(\tilde{\xi}_{s+r}, \mathrm{d}y) \\
&= \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r})}}^{\infty} y\pi(\tilde{\xi}_{s+r}, \mathrm{d}y) - \int_{-t}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r})}}^{\infty} y\pi(\tilde{\xi}_{s+r}, \mathrm{d}y) \\
&\geq \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r})}}^{\infty} y\pi(\tilde{\xi}_{s+r}, \mathrm{d}y) - |t| \cdot \sup_{x\in E} \int_{1}^{\infty} y\pi(x, \mathrm{d}y) = \infty,
\end{aligned}$$

as claimed. We then claim that for any $r \in \mathbb{R}$ and $t \geq 0$, Q-almost surely,

$$\int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon(s+t)}}{\phi(\tilde{\xi}_{s+r+t})}}^{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r+t})}} y\pi(\tilde{\xi}_{s+r+t}, \mathrm{d}y) < \infty. \tag{A.9}$$

In fact, by (4.6),

$$\begin{aligned}
&\mathrm{Q}\Big[\int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon(s+t)}}{\phi(\tilde{\xi}_{s+r+t})}}^{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r+t})}} y\pi(\tilde{\xi}_{s+r+t}, \mathrm{d}y)\Big] = \int_{-\infty}^{0} \mathrm{d}s \int_{E} \tilde{\nu}(\mathrm{d}x) \int_{\frac{Ke^{-\epsilon(s+t)}}{\phi(x)}}^{\frac{Ke^{-\epsilon s}}{\phi(x)}} y\pi(x, \mathrm{d}y) \\
&= \int_{E} \tilde{\nu}(\mathrm{d}x) \int_{\frac{Ke^{-\epsilon t}}{\phi(x)}}^{\infty} y\pi(x, \mathrm{d}y) \int_{-\frac{1}{\epsilon}\log\frac{y\phi(x)e^{\epsilon t}}{K}}^{-\frac{1}{\epsilon}\log\frac{y\phi(x)}{K}} \mathrm{d}s \\
&= t \int_{E} \tilde{\nu}(\mathrm{d}x) \int_{\frac{Ke^{-\epsilon t}}{\phi(x)}}^{\infty} y\pi(x, \mathrm{d}y) \leq t \cdot \sup_{x\in E} \int_{e^{-\epsilon t}}^{\infty} y\pi(x, \mathrm{d}y) < \infty
\end{aligned}$$

which implies the claim. Finally, we claim that

$$A_r \cap \Omega_{r,t} \subset A_{r+t}, \quad r \in \mathbb{R}, t > 0, \tag{A.10}$$

where $\Omega_{r,t}$ is the event that (A.9) holds. In fact, on the event $A_r \cap \Omega_{r,t}$ we have

$$\begin{aligned}
&\int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r+t})}}^{\infty} y\pi(\tilde{\xi}_{s+r+t}, \mathrm{d}y) \\
&= \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon(s+t)}}{\phi(\tilde{\xi}_{s+r+t})}}^{\infty} y\pi(\tilde{\xi}_{s+r+t}, \mathrm{d}y) - \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon(s+t)}}{\phi(\tilde{\xi}_{s+r+t})}}^{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r+t})}} y\pi(\tilde{\xi}_{s+r+t}, \mathrm{d}y) \\
&= \int_{-\infty}^{t} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r})}}^{\infty} y\pi(\tilde{\xi}_{s+r}, \mathrm{d}y) - \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon(s+t)}}{\phi(\tilde{\xi}_{s+r+t})}}^{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r+t})}} y\pi(\tilde{\xi}_{s+r+t}, \mathrm{d}y) \\
&\geq \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r})}}^{\infty} y\pi(\tilde{\xi}_{s+r}, \mathrm{d}y) - \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon(s+t)}}{\phi(\tilde{\xi}_{s+r+t})}}^{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s+r+t})}} y\pi(\tilde{\xi}_{s+r+t}, \mathrm{d}y) = \infty
\end{aligned}$$

as claimed. Now, for any $r < t$ in $\mathbb{R}$, from (A.8), we know that $\mathrm{Q}(A_t \setminus A_r) = 0$; from (A.9) and (A.10), we know that $\mathrm{Q}(A_r \setminus A_t) = 0$. Therefore, the desired result in this step follows.

Final Step. From steps 5, 7 and [4, Theorem 1.2.4.(i)] , we get that

$$\int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} y\pi(\tilde{\xi}_s, \mathrm{d}y) = \infty, \quad \text{Q-a.s.} \tag{A.11}$$

From (4.6), we know that $(\tilde{\xi}_s)_{s\in\mathbb{R}}$ has the same distribution as $(\tilde{\xi}_{s-s_0})_{s\in\mathbb{R}}$. Therefore we have from (A.11) that

$$\int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s-s_0})}}^{\infty} y\pi(\tilde{\xi}_{s-s_0}, \mathrm{d}y) = \infty, \quad \text{Q-a.s.}$$

Now we have Q-a.s.,

$$\begin{aligned}
&\int_{-\infty}^{-s_0} \mathrm{d}s \int_{\frac{e^{-\epsilon s}}{\phi(\tilde{\xi}_s)}}^{\infty} y\pi(\tilde{\xi}_s, \mathrm{d}y) = \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{e^{-\epsilon s}e^{\epsilon s_0}}{\phi(\tilde{\xi}_{s-s_0})}}^{\infty} y\pi(\tilde{\xi}_{s-s_0}, \mathrm{d}y) \\
&\geq \int_{-\infty}^{0} \mathrm{d}s \int_{\frac{Ke^{-\epsilon s}}{\phi(\tilde{\xi}_{s-s_0})}}^{\infty} y\pi(\tilde{\xi}_{s-s_0}, \mathrm{d}y) = \infty
\end{aligned}$$

as desired. □

References

[1] K.B. Athreya, P.E. Ney, Branching processes, Springer-Verlag, New York-Heidelberg, 1972, p. xi+287, Die Grundlehren der mathematischen Wissenschaften, Band 196.

[2] N. Champagnat, S. Rœlly, Limit theorems for conditioned multitype Dawson-Watanabe processes and Feller diffusions, Electron. J. Probab. 13 (25) (2008) 777–810.

[3] Z.-Q. Chen, Y.-X. Ren, H. Wang, An almost sure scaling limit theorem for Dawson-Watanabe superprocesses, J. Funct. Anal. 254 (7) (2008) 1988–2019.

[4] G. Da Prato, J. Zabczyk, Ergodicity for Infinite-Dimensional Systems, in: London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, 1996, p. xii+339.

[5] J. Delmas, O. Hénard, A williams decomposition for spatially dependent super-processes, Electron. J. Probab. 18 (37) (2013) 43.

[6] M. Eckhoff, A.E. Kyprianou, M. Winkel, Spines, skeletons and the strong law of large numbers for superdiffusions, Ann. Probab. 43 (5) (2015) 2545–2610.

[7] J. Engländer, A.E. Kyprianou, Local extinction versus local exponential growth for spatial branching processes, Ann. Probab. 32 (1A) (2004) 78–99.

[8] A.M. Etheridge, D.R.E. Williams, A decomposition of the $(1+\beta)$-superprocess conditioned on survival, Proc. Roy. Soc. Edinburgh Sect. A 133 (4) (2003) 829–847.

[9] S.N. Evans, Two representations of a conditioned superprocess, Proc. Roy. Soc. Edinburgh Sect. A 123 (5) (1993) 959–971.

[10] S.N. Evans, E. Perkins, Measure-valued Markov branching processes conditioned on nonextinction, Israel J. Math. 71 (3) (1990) 329–337.

[11] I. Gonzalez, E. Horton, A.E. Kyprianou, Asymptotic moments of spatial branching processes, Probab. Theory Related Fields 184 (3–4) (2022) 805–858.

[12] D.R. Grey, Asymptotic behaviour of continuous time, continuous state-space branching processes, J. Appl. Probab. 11 (1974) 669–677.

[13] S.C. Harris, E. Horton, A.E. Kyprianou, Stochastic methods for the neutron transport equation II: almost sure growth, Ann. Appl. Probab. 30 (2020) 2815–2845.

[14] S.C. Harris, E. Horton, A.E. Kyprianou, M. Wang, Yaglom limit for critical neutron transport, arXiv:2103.02237.

[15] C.R. Heathcote, E. Seneta, D. Vere-Jones, A refinement of two theorems in the theory of branching processes, Teor. Verojatnost. I Primenen. 12 (1967) 341–346.

[16] O. Hernández-Lerma, J.B. Lasserre, Markov Chains and Invariant Probabilities, in: Progress in Mathematics, vol. 211, Birkhäuser Verlag, Basel, 2003, p. xvi+205.

[17] E. Horton, A.E. Kyprianou, D. Villemonais, Stochastic methods for the neutron transport equation I: linear semigroup asymptotics, Ann. Appl. Probab. 30 (2020) 2573–2612.

[18] A. Joffe, On the Galton-Watson branching process with mean less than one, Ann. Math. Stat. 38 (1967) 264–266.

[19] O. Kallenberg, Foundations of modern probability, Probability Theory and Stochastic Modelling, vol. 99, Springer, Cham, 2021, p. xii+946, Third edition.

[20] P. Kim, R. Song, Intrinsic ultracontractivity of non-symmetric diffusion semigroups in bounded domains, Tohoku Math. J. (2) 60 (4) (2008) 527–547.

[21] A.E. Kyprianou, Fluctuations of Lévy Processes with Applications, second ed., in: Universitext, Springer, Heidelberg, 2014, p. xviii+455, Introductory lectures.

[22] A. Lambert, Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct, Electron. J. Probab. 12 (14) (2007) 420–446.

[23] J. Lamperti, P. Ney, Conditioned branching processes and their limiting diffusions, Teor. Verojatnost. I Primenen. 13 (1968) 126–137.

[24] Z. Li, Asymptotic behaviour of continuous time and state branching processes, J. Aust. Math. Soc. Ser. A 68 (1) (2000) 68–84.

[25] Z. Li, Measure-Valued Branching Markov Processes, in: Probability and its Applications (New York), Springer, Heidelberg, 2011, p. xii+350.

[26] R.-L. Liu, Y.-X. Ren, R. Song, $L\log L$ Criterion for a class of superdiffusions, J. Appl. Probab. 46 (2) (2009) 479–496.

[27] R.-L. Liu, Y.-X. Ren, R. Song, Strong law of large numbers for a class of superdiffusions, Acta Appl. Math. 123 (2013) 73–97.

[28] R. Liu, Y.-X. Ren, R. Song, Z. Sun, Quasi-stationary distributions for subcritical superprocesses, Stochastic Process. Appl. 132 (2021) 108–134.

[29] R. Lyons, R. Pemantle, Y. Peres, Conceptual proofs of $L\log L$ criteria for mean behavior of branching processes, Ann. Probab. 23 (3) (1995) 1125–1138.

[30] S. Méléard, D. Villemonais, Quasi-stationary distributions and population processes, Probab. Surv. 9 (2012) 340–410.

[31] A.G. Pakes, Revisiting conditional limit theorems for the mortal simple branching process, Bernoulli 5 (6) (1999) 969–998.

[32] S. Pénisson, Conditional limit theorems for multitype branching processes and illustration in epidemiological risk analysis, 2010, https://tel.archives-ouvertes.fr/tel-00570458/document.

[33] S. Pénisson, Beyond the Q-process: various ways of conditioning the multitype Galton-Watson process, ALEA Lat. Am. J. Probab. Math. Stat. 13 (1) (2016) 223–237.

[34] P.K. Pollett, Quasi-stationary distributions: a bibliography, 2015, http://www.maths.uq.edu.au/~pkp/papers/qsds/qsds.html.

[35] Y.-X. Ren, R. Song, Z. Sun, Limit theorems for a class of critical superprocesses with stable branching, Stochastic Process. Appl. 130 (7) (2020) 4358–4391.

[36] Y.-X. Ren, R. Song, Z. Sun, Spine decompositions and limit theorems for a class of critical superprocesses, Acta Appl. Math. 165 (2020) 91–131.

[37] Y.-X. Ren, R. Song, Z. Sun, J. Zhao, Stable central limit theorems for super Ornstein-Uhlenbeck processes, Electron. J. Probab. 24 (141) (2019) 42.

[38] Y.-X. Ren, R. Song, Z. Sun, J. Zhao, Stable central limit theorems for super Ornstein-Uhlenbeck processes, II, Acta Math. Sin. (Engl. Ser.) 38 (3) (2022) 487–498.

[39] Y.-X. Ren, R. Song, T. Yang, Spine decomposition and $L\log L$ criterion for superprocesses with non-local branching mechanisms, ALEA Lat. Am. J. Probab. Math. Stat. 19 (1) (2022) 163–208.

[40] Y.-X. Ren, R. Song, R. Zhang, Central limit theorems for supercritical superprocesses, Stochastic Process. Appl. 125 (2) (2015) 428–457.

[41] Y.-X. Ren, R. Song, R. Zhang, Limit theorems for some critical superprocesses, Illinois J. Math. 59 (1) (2015) 235–276.

[42] Y.-X. Ren, R. Song, R. Zhang, Central limit theorems for supercritical branching nonsymmetric Markov processes, Ann. Probab. 45 (1) (2017) 564–623.

[43] M. Sharpe, General Theory of Markov Processes, in: Pure and Applied Mathematics, vol. 133, Academic Press Inc., Boston, MA, 1988, p. xii+419.

[44] A.M. Yaglom, Certain limit theorems of the theory of branching random processes, Dokl. Akad. Nauk SSSR 56 (1947) 795–798.