# ON THE SPECTRAL RADIUS OF THE ( $L, \kappa$ )-LAZY MARKOV CHAIN 

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#### Abstract

We consider an $(L, \kappa)$-lazy operation on an irreducible Markov transition probability $P$ with state space $S$ where $L \subset S$ and $\kappa \in[0,1)$. For each $x \in L$ and $y \in S$, this $(L, \kappa)$-operation replaces $P(x, y)$, the transition probability from $x$ to $y$, by $\kappa 1_{\{x=y\}}+(1-\kappa) P(x, y)$. We are interested in how $L$ and $\kappa$ influence the spectral radius $\rho_{\kappa}^{L}$ of this new transition probability. We first show that $\rho_{\kappa}^{L}$ is non-decreasing and continuous in $\kappa$. We then show that: (1) If $L$ is nonempty and finite, then $P$ being rho-transient is equivalent to that the growth of $\left(\rho_{\kappa}^{L}\right)_{\kappa \in[0,1)}$ exhibits a phase transition: There exists a critical value $\kappa_{c}(L) \in(0,1)$ such that $\kappa \mapsto \rho_{\kappa}^{L}$ is a constant on $\left[0, \kappa_{c}(L)\right]$ and increases strictly on $\left[\kappa_{c}(L), 1\right)$; (2) For every $\kappa \in(0,1)$, if $S \backslash L$ is nonempty and finite, then $\rho_{\kappa}^{L}=\rho_{\kappa}^{S}$ if and only if $P$ is not strictly rho-recurrent.


## 1. Introduction

We consider irreducible discrete-time Markov chains on a countable infinite state space $S$. In recent years, there are some papers studied the $\kappa$-lazy version of a Markov transition probability $P$ on $S$, which is defined as $\kappa I+(1-\kappa) P$ where $I$ is the identity matrix (see [1, 2, 4, 5]). Intuitively speaking, the corresponding $\kappa$-lazy Markov chain is the original chain being delayed at each step by tossing a coin with heads' probability $\kappa$ : If the chain gets a head, then it won't move, and it gets a tail, it will move according to the transition probability $P$.

In this note, we are going to consider the $(L, \kappa)$-laziness which generalized the notion of the $\kappa$-laziness by only delaying the Markov chain on a subset of $S$. To be more precise, for any $L \subset S$ and $\kappa \in[0,1)$, we denote by $P_{\kappa}^{L}$ the $(L, \kappa)$-lazy version of a transition probability $P$ which is defined so that for any $x, y \in S$,

$$
P_{\kappa}^{L}(x, y):= \begin{cases}\kappa+(1-\kappa) P(x, x), & \text { if } x \in L \text { and } x=y ; \\ (1-\kappa) P(x, y), & \text { if } x \in L \text { and } x \neq y ; \\ P(x, y), & \text { if } x \notin L .\end{cases}
$$

We call $P\left(=P_{0}^{L}\right)$ the underlying transition probability.
We are particularly interested in how the spectral radius of the transition probability $P_{\kappa}^{L}$ is influenced by $L$ and $\kappa$. The Green function $G(x, y \mid z)$ of a transition probability $P$ is defined as

$$
G(x, y \mid z):=\sum_{n=0}^{\infty} \mathbb{P}^{x}\left(X_{n}=y\right) z^{n}, \quad x, y \in S, z \in \mathbb{R}
$$

[^0]where $\left(X_{n}\right)_{n=0}^{\infty}$ is a Markov chain with transition probability $P$ and initial value $X_{0}=x$ under the probability $\mathbb{P}^{x}$. It is proved in [12, Lemma 1.7] that if $P$ is irreducible, then the convergence radius of the Green function $G(x, y \mid \cdot)$ is independent of $x$ and $y$. The reciprocal of this convergence radius is known as the spectral radius of $P$ (see [12] for example). It is noted in [12, (1.8)] that
(1) the spectral radius of an irreducible transition probability takes its value in $(0,1]$. Observe that for $\kappa \in[0,1)$, if $P$ is irreducible, then so is $P_{\kappa}^{L}$.

In the rest of this note, we will always assume that the underlying transition probability $P$ is irreducible, and we will denote by $\rho_{\kappa}^{L}$ the spectral radius of $P_{\kappa}^{L}$ with $\rho:=\rho_{0}^{L}$. One can iterate the lazy operations on the underlying transition probability. For example, we denote by $\left(P_{\kappa}^{L}\right)_{\kappa^{\prime}}^{L^{\prime}}$ the $\left(L^{\prime}, \kappa^{\prime}\right)$-lazy version of the $(L, \kappa)$-lazy version of $P$, and by $\left(\rho_{\kappa}^{L}\right)_{\kappa^{\prime}}^{L^{\prime}}$ the spectral radius of $\left(P_{\kappa^{\prime}}^{L}\right)_{\kappa^{\prime}}^{L^{\prime}}$.

The spectral radius $\rho_{\kappa}^{S}$ for the $\kappa$-lazy version of $P$ has been studied in [12, Lemma 9.2]: For $\kappa \in[0,1)$, it holds that

$$
\begin{equation*}
\rho_{\kappa}^{S}=\kappa+(1-\kappa) \rho . \tag{2}
\end{equation*}
$$

Our first result concerns the monotonicity and the continuity of $\kappa \mapsto \rho_{\kappa}^{L}$ for general $L \subset S$.

Theorem 1.1. Let $L \subset S$ and $\kappa \in[0,1)$.
(i) $\rho_{\kappa}^{L}$ is non-decreasing in $\kappa$.
(ii) $\rho_{\kappa}^{L}$ is continuous in $\kappa$.
(iii) $\rho=1 \Leftrightarrow \rho_{\kappa}^{L}=1$, or equivalently speaking, $\rho<1 \Leftrightarrow \rho_{\kappa}^{L}<1$.
(iv) If $L \neq \varnothing$, then $\lim _{\kappa \uparrow 1} \rho_{\kappa}^{L}=1$.

Let us now introduce a classification of the transition probability, which is crucial for the rest of our results. It is proved in [12, Lemma 1.7] that $G(x, y \mid 1 / \rho)$ either $=\infty$ (or $<\infty)$ simultaneously for all $x, y \in S$, and the corresponding $P$ is referred to as rhorecurrent (or rho-transient) transition probability. These concepts appeared in [9] and were studied in [3, 7, 8, 10, 11] and [6, Section 3.2]. One can find specific examples of rho-recurrent/rho-transient Markov chains in [12, Section 7.B].

Theorem 1.2. Suppose $\rho<1$.
(i) If $0<\# L<\infty$ and $P$ is rho-transient, then there exists a unique $\kappa_{c}(L) \in(0,1)$ such that

- For $\kappa \in\left[0, \kappa_{c}(L)\right), \rho_{\kappa}^{L}=\rho$ and $P_{\kappa}^{L}$ is rho-transient;
- For $\kappa \in\left[\kappa_{c}(L), 1\right), \rho_{\kappa}^{L}$ increases strictly in $\kappa$ with $\rho_{\kappa_{c}(L)}^{L}=\rho$, and $P_{\kappa}^{L}$ is rhorecurrent.
(ii) If $0<\# L<\infty$ and $P$ is rho-recurrent, then for every $\kappa \in[0,1), \rho_{\kappa}^{L}$ increases strictly in $\kappa$ and $P_{\kappa}^{L}$ is rho-recurrent.
(iii) If $\#(S \backslash L)<\infty$, then $\rho_{\kappa}^{L}$ increases strictly in $\kappa \in[0,1)$.

In this note, we will introduce a further classification for the rho-recurrent transition probabilities, which we will use in our next result. Let the $U$-function of a transition
probability $P$ be the power series:

$$
U(x, y \mid z):=\sum_{n=0}^{\infty} \mathbb{P}^{x}\left(\tau^{y}=n\right) z^{n}, \quad x, y \in S, z \in \mathbb{R}
$$

where $\tau^{y}:=\inf \left\{n \geq 1: X_{n}=y\right\}$. Let $r(U \mid x, y)$ be the convergence radius of $U(x, y \mid \cdot)$. (Note that $r(U \mid x, y)$ may depend on $x, y$.) We will prove in Lemma 2.3 that if $P$ is rho-transient, then

$$
\begin{equation*}
r(U \mid x, x)=1 / \rho, \forall x \in S \tag{3}
\end{equation*}
$$

If $P$ is rho-recurrent and (3) holds, then we say $P$ is critically rho-recurrent. If $P$ is rho-recurrent, but (3) does not hold, then we say $P$ is strictly rho-recurrent.

Theorem 1.3. Suppose that $\rho<1$ and $0<\# L<\infty$.
(i) If $P$ is rho-transient, then $P_{\kappa_{c}(L)}^{L}$ is critically rho-recurrent where $\kappa_{c}(L)$ is given as in Theorem 1.2 (i). Moreover, for every $\kappa \in\left(\kappa_{c}(L), 1\right), P_{\kappa}^{L}$ is strictly rho-recurrent.
(ii) If $P$ is rho-recurrent, then for every $\kappa \in(0,1), P_{\kappa}^{L}$ is strictly rho-recurrent.

For $\kappa \in(0,1)$, observe that $\left(P_{\kappa}^{L}\right)_{\kappa}^{S \backslash L}=P_{\kappa}^{S}$. Hence by Theorem 1.1 (i), $\rho_{\kappa}^{L} \leq \rho_{\kappa}^{S}$.
Theorem 1.4. Suppose that $\rho<1, \#(S \backslash L)>0$ and $\kappa \in(0,1)$.
(i) If $\#(S \backslash L)<\infty$, then $\rho_{\kappa}^{L}=\rho_{\kappa}^{S}$ if and only if $P$ is not strictly rho-recurrent.
(ii) If $P$ is strictly rho-recurrent, then $\rho_{\kappa}^{L}<\rho_{\kappa}^{S}$.
(iii) If \#L $<\infty$, then $\rho_{\kappa}^{L}<\rho_{\kappa}^{S}$.

The rest of this note is organized as follows. Section 2 gives some preliminary results that we will use throughout the note. Proof of Theorem 1.1 is provided in Section 3, In Section 4, we show the proofs of Theorem 1.2 (ii), (iii), and Theorem 1.4 (iii). In Section 5 , we give the proofs of Theorem 1.3, Theorem 1.4 (ii), (iii), and Theorem 1.2 (iiii).

Acknowledgment. Part of this research was done while the second author was a Postdoc at the Technion-Israel Institute of Technology, supported by a scholarship from the Israel Council for Higher Education.

The authors want to thank Dayue Chen for helpful conversations.

## 2. Preliminary

This section will introduce some basic results for irreducible Markov chains. Recall that we used notations $\left(\mathbb{P}^{x}\right)_{x \in S}, G(\cdot, \cdot \mid \cdot), \rho, U(\cdot, \cdot \mid \cdot)$, and $r(U \mid \cdot, \cdot)$ to represent the probability of a Markov chain, the Green function, the spectral radius, the U-function, and the convergence radius of the U-function, corresponding to an irreducible transition probability $P$ on $S$, respectively. In the rest of this note, we will use notations $\left(\mathbb{P}_{\kappa}^{L, x}\right)_{x \in S}, G_{\kappa}^{L}(\cdot, \cdot \mid \cdot), \rho_{\kappa}^{L}, U_{\kappa}^{L}(\cdot, \cdot \mid \cdot)$ and $r\left(U_{\kappa}^{L} \mid \cdot, \cdot\right)$ to represent the similar concepts for $P_{\kappa}^{L}$, the $(L, \kappa)$-lazy version of $P$.
2.1. Markov chains. We first observe that for any $x \in S$,

$$
\begin{equation*}
r(U \mid x, x) \geq 1 / \rho \tag{4}
\end{equation*}
$$

by the Cauchy-Hadamard formula. It is then easy to see that $P$ is strictly rho-recurrent if and only if

$$
\begin{equation*}
\exists x \in S, r(U \mid x, x)>1 / \rho \tag{5}
\end{equation*}
$$

The following lemma connects the Green function, the U-function, and the spectral radius. Some of these results were known in the literature (see [12, Lemma 1.13] for examples). Here, we include their proofs for the sake of completeness.
Lemma 2.1. For $x \in S$ and $z>0$,
(i) $U(x, x \mid z)<1 \Leftrightarrow G(x, x \mid z)<\infty$.
(ii) If $G(x, x \mid z)<\infty$, then

$$
G(x, x \mid z)=\frac{1}{1-U(x, x \mid z)} .
$$

(iii) $U(x, x \mid z)>1 \Leftrightarrow z>1 / \rho$.
(iv) $1 / \rho=\max \{z>0: U(x, x \mid z) \leq 1\}$.

Proof. For $N \in \mathbb{Z}^{+}$,

$$
\begin{aligned}
\sum_{n=0}^{N} \mathbb{P}^{x}\left(X_{n}=x\right) z^{n} & =1+\sum_{n=1}^{N} \sum_{m=0}^{n} \mathbb{P}^{x}\left(\tau^{x}=m\right) \mathbb{P}^{x}\left(X_{n-m}=x\right) z^{n} \\
& =1+\sum_{m=0}^{N} \mathbb{P}^{x}\left(\tau^{x}=m\right) z^{m} \sum_{n=m}^{N} \mathbb{P}^{x}\left(X_{n-m}=x\right) z^{n-m} \\
& =1+\sum_{m=0}^{N} \mathbb{P}^{x}\left(\tau^{x}=m\right) z^{m} \sum_{n=0}^{N-m} \mathbb{P}^{x}\left(X_{n}=x\right) z^{n}
\end{aligned}
$$

Note that

$$
\begin{equation*}
\sum_{n=0}^{N} \mathbb{P}^{x}\left(X_{n}=x\right) z^{n} \geq 1+\sum_{m=0}^{\lfloor N / 2\rfloor} \mathbb{P}^{x}\left(\tau^{x}=m\right) z^{m} \sum_{n=0}^{\lfloor N / 2\rfloor} \mathbb{P}^{x}\left(X_{n}=x\right) z^{n} \tag{6}
\end{equation*}
$$

If $G(x, x \mid z)<\infty$, then by taking $N$ to infinity, we have $U(x, x \mid z)<1$. By taking $N$ to infinity in both

$$
\begin{equation*}
\sum_{n=0}^{N} \mathbb{P}^{x}\left(X_{n}=x\right) z^{n} \leq 1+\sum_{m=0}^{N} \mathbb{P}^{x}\left(\tau^{x}=m\right) z^{m} \sum_{n=0}^{N} \mathbb{P}^{x}\left(X_{n}=x\right) z^{n} \tag{7}
\end{equation*}
$$

and (6), we have (iii).
Now we assume that $G(x, x \mid z)=\infty$. By (7), we have

$$
\sum_{m=0}^{N} \mathbb{P}^{x}\left(\tau^{x}=m\right) z^{m} \geq \frac{\sum_{n=0}^{N} \mathbb{P}^{x}\left(X_{n}=x\right) z^{n}-1}{\sum_{n=0}^{N} \mathbb{P}^{x}\left(X_{n}=x\right) z^{n}}
$$

Let $N \rightarrow \infty$, we have $U(x, x \mid z) \geq 1$. Therefore (ii) is proved.
For $\Rightarrow$ of (iiii), by the strictly increasing property of $U(x, x \mid \cdot)$, it suffices to show that $U(x, x \mid 1 / \rho) \leq 1$. Noticing that the coefficients of the power series $U(x, x \mid \cdot)$ is nonnegative, by the monotone convergence theorem, we have

$$
\lim _{z \uparrow 1 / \rho} U(x, x \mid z)=U(x, x \mid 1 / \rho) .
$$

By (i) we have that $U(x, x \mid z)<1$ for $z<1 / \rho$. Therefore $U(x, x \mid 1 / \rho) \leq 1$ as desired.
For $\Leftarrow$ of (iiii), $z>1 / \rho$ implies that $G(x, x \mid z)=\infty$, and, by (ii), further implies that $U(x, x \mid z) \geq 1$. We only have to exclude $U(x, x \mid z)=1$ by contradiction: If it holds, then for $w \in(1 / \rho, z), U(x, x \mid w)<1$, which by (ii), contradicts the fact that $G(x, x \mid w)<\infty$. (iv) can be directly concluded from (iiii).

As a corollary, we have another equivalent condition for the rho-recurrence and the rho-transience.

Corollary 2.2. For any $x \in S, P$ is rho-recurrent $\Leftrightarrow U(x, x \mid 1 / \rho)=1$; and $P$ is rhotransient $\Leftrightarrow U(x, x \mid 1 / \rho)<1$.

Proof. From Lemma 2.1 (ii), we only have to prove $\Rightarrow$ of the first statement. When $P$ is rho-recurrent, by Lemma 2.1 (ii) we know that $U(x, x \mid 1 / \rho) \geq 1$. By Lemma 2.1 (iiii) we have $U(x, x \mid 1 / \rho) \leq 1$. We are done.

Lemma 2.3. If $P$ is rho-transient, then $r(U \mid x, x)=1 / \rho$ for every $x \in S$.
Proof. For the sake of contradiction and (4), we assume that there exists $x \in S$ such that $r(U \mid x, x)>1 / \rho$. Then by the continuity of the power series inside of its convergence radius and Corollary [2.2, there exists $z>1 / \rho$ such that $U(x, x \mid z)<1$. Now by Lemma 2.1 (ii), we have $G(x, x \mid z)<\infty$ which contradicts the fact that $1 / \rho$ is the convergence radius of $G(x, x \mid \cdot)$.
2.2. ( $L, \kappa$ )-laziness. By Lemma 2.1, the U-function is a good tool for studying the spectral radius.

Lemma 2.4. Let $x \in S, \kappa \in[0,1)$ and $z \geq 0$.
(i) If $L=\{x\}$, then

$$
U_{\kappa}^{L}(x, x \mid z)=\kappa z+(1-\kappa) U(x, x \mid z) .
$$

(ii) If $x \notin L, L \neq \varnothing$ and $\kappa z<1$, then

$$
U_{\kappa}^{L}(x, x \mid z)=\mathbb{P}^{x}\left(\tau^{x}=1\right) z+\sum_{k \geq 2, \vec{l} \in \mathcal{T}(k, x)} P(\vec{l}) z^{k}\left(\frac{1-\kappa}{1-\kappa z}\right)^{\vec{l}(L)},
$$

where for $k \geq 2$,

$$
\begin{aligned}
\mathcal{T}(k, x):= & \left\{\left(x_{0}, x_{1}, x_{2}, \cdots, x_{k-1}, x_{k}\right) \in S^{k+1}:\right. \\
& \left.x_{0}=x ; x_{i} \neq x, \forall i=1, \cdots, k-1 ; x_{k}=x\right\},
\end{aligned}
$$

and for $\vec{l}=\left(x_{0}, x_{1}, x_{2}, \cdots, x_{k-1}, x_{k}\right) \in \mathcal{T}(k, x)$,

$$
P(\vec{l}):=\prod_{i=0}^{k-1} P\left(x_{i}, x_{i+1}\right), \nRightarrow \vec{l}(L):=\#\left\{i: i=1, \cdots, k-1 ; x_{i} \in L\right\} .
$$

(iii) If $x \notin L, L \neq \varnothing$ and $\kappa z \geq 1$, then $U_{\kappa}^{L}(x, x \mid z)=\infty$.

Proof. For (i) , it is done by the following:

$$
\mathbb{P}_{\kappa}^{\{x\}, x}\left(\tau^{x}=n\right)= \begin{cases}\kappa+(1-\kappa) \mathbb{P}^{x}\left(\tau^{x}=1\right), & \text { if } n=1 \\ (1-\kappa) \mathbb{P}^{x}\left(\tau^{x}=n\right), & \text { if } n \geq 2\end{cases}
$$

For (iii) and (iii), noticing that $x \notin L$ and $L \neq \varnothing$, we assert that

$$
\mathbb{P}_{\kappa}^{L, x}\left(\tau^{x}=n\right)= \begin{cases}\mathbb{P}^{x}\left(\tau^{x}=1\right), & \text { if } n=1 ;  \tag{8}\\ \sum_{k=2}^{n} \sum_{\vec{l} \in \mathcal{T}(k, x)} P(\vec{l})(1-\kappa)^{\sharp \vec{l}(L)}(-\kappa)^{n-k}\binom{-\# \vec{l}(L)}{n-k}, & \text { if } n \geq 2,\end{cases}
$$

where we used the generalized binomial series: For $k \in \mathbb{Z}^{+}$and arbitrary $\alpha \in \mathbb{R}$,

$$
\binom{\alpha}{k}:=\frac{\alpha(\alpha-1)(\alpha-2) \cdots(\alpha-k+1)}{k!}, \text { in particular }\binom{\alpha}{0}=1 .
$$

Let us explain (8) when $n \geq 2$. A Markov chain $X_{t}$ with transition probability $P_{\kappa}^{L}$ can be constructed in the following way: At each time $t \in \mathbb{Z}^{+}$, taking an independent uniform r.v. $\theta_{t}$ in $[0,1]$, if $X_{t-1} \in L$ and $\theta_{t} \leq \kappa$, then we set $X_{t}:=X_{t-1}$ and say that the Markov chain takes a lazy step at time $t$; else if $X_{t-1} \notin L$ or $\theta_{t}>\kappa$, we sample $X_{t}$ according to the probability $\left\{P\left(X_{t-1}, y\right): y \in S\right\}$ and say that the chain takes a non-lazy step.

We refer to the excursion the trajectory of the Markov chain $\left(X_{t}\right)$ up to the time $\tau^{x}$. We refer to the non-lazy excursion the trajectory of the Markov chain $X_{t}$ formed only by the non-lazy steps up to the time $\tau^{x}$. It is observed that on the event $\left\{\tau^{x}=n\right\}$, the excursion takes its value in $\mathcal{T}(n, x)$ while the non-lazy excursion takes its value in $\cup_{2 \leq k \leq n} \mathcal{T}(k, x)$.

For a given $2 \leq k \leq n$ and $l \in \mathcal{T}(k, x)$, by the elementary combinatorics, the number of possible excursions in the event $\left\{\tau^{x}=n, l\right.$ is the non-lazy excursion $\}$ is $\binom{-\pi l(L)}{n-k}(-1)^{n-k}$, and each of those excursions happens with the same probability $P(\vec{l})(1-\kappa)^{\mu l(L)} \kappa^{n-k}$. Therefore $\mathbb{P}^{x}\left(\tau^{x}=n, l\right.$ is the non-lazy excursion $)=P(\vec{l})(1-\kappa)^{\# \vec{l}(L)}(-\kappa)^{n-k}\binom{-\vec{l}(L)}{n-k}$. Now (8) holds.

By Fubini's theorem for nonnegative series $\binom{(-\| \vec{l}(L)}{n-k}(-1)^{n-k}$ is nonnegative), we have that

$$
\begin{aligned}
U_{\kappa}^{L}(x, x \mid z) & =\mathbb{P}^{x}\left(\tau^{x}=1\right) z+\sum_{n \geq 2} \sum_{k=2}^{n} \sum_{\vec{l} \in \mathcal{T}(k, x)} P(\vec{l})(1-\kappa)^{\# \vec{l}(L)}(-\kappa)^{n-k}\binom{-\# \vec{l}(L)}{n-k} z^{n} \\
& =\mathbb{P}^{x}\left(\tau^{x}=1\right) z+\sum_{k \geq 2, \vec{l} \in \mathcal{T}(k, x)} P(\vec{l})(1-\kappa)^{\# \vec{l}(L)} z^{k} \sum_{n \geq k}\binom{-\# \vec{l}(L)}{n-k}(-\kappa z)^{n-k} .
\end{aligned}
$$

For $\kappa z<1$, by the generalized binomial theorem, we have that

$$
\sum_{n \geq k}\binom{-\# \vec{l}(L)}{n-k}(-\kappa z)^{n-k}=(1-\kappa z)^{-\# \vec{l}(L)} .
$$

Now we have (iii).
If $\kappa z \geq 1$, then by $L \neq \varnothing$ and the irreducibility, there exist $k_{0} \geq 2$ and $\vec{l}_{0} \in \mathcal{T}\left(k_{0}, x\right)$ such that $P\left(\vec{l}_{0}\right)>0$ and $\# \vec{l}_{0}(L)>0$. As $\kappa<1$ and $z>0$, we have that

$$
P\left(\vec{l}_{0}\right)(1-\kappa)^{\# \vec{l}_{0}(L)} z^{k_{0}} \sum_{n \geq k_{0}}\binom{-\# \vec{l}_{0}(L)}{n-k_{0}}(-\kappa z)^{n-k_{0}}=\infty .
$$

Thus we have (iiii).
Corollary 2.5. If $L=\{x\}$, then $r\left(U_{\kappa}^{L} \mid x, x\right)=r(U \mid x, x)$ for every $\kappa \in[0,1)$.
Let us state two more results when $L=S$.
Lemma 2.6 ([12], Lemma 9.2). For $\kappa \in[0,1), x \in S$ and $z \in\left[0,1 / \rho_{\kappa}^{S}\right)$,

$$
G_{\kappa}^{S}(x, x \mid z)=\frac{1}{1-\kappa z} G\left(x, x \left\lvert\, \frac{(1-\kappa) z}{1-\kappa z}\right.\right) .
$$

Lemma 2.7. For $\kappa \in[0,1), x \in S$ and $z \in\left[0, r\left(U_{\kappa}^{S} \mid x, x\right)\right)$,

$$
U_{\kappa}^{S}(x, x \mid z)=(1-\kappa z) U\left(x, x \left\lvert\, \frac{(1-\kappa) z}{1-\kappa z}\right.\right)+\kappa z,
$$

and

$$
1 / r\left(U_{\kappa}^{S} \mid x, x\right)=\kappa+(1-\kappa) / r(U \mid x, x) .
$$

Proof. It is similar to the proof of Lemma 9.2 in [12], noticing by (8) that

$$
\mathbb{P}_{\kappa}^{S, x}\left(\tau_{x}=n\right)= \begin{cases}\kappa+(1-\kappa) \mathbb{P}^{x}\left(\tau_{x}=1\right), & n=1 \\ \sum_{k=2}^{n} \mathbb{P}^{x}\left(\tau_{x}=k\right)(-\kappa)^{n-k}(1-\kappa)^{k}\binom{-k+1}{n-k}, & n \geq 2\end{cases}
$$

Lemma 2.8. For $\kappa \in[0,1), P$ is rho-transient (critically rho-recurrent, or strictly rhorecurrent, respectively), if and only if so is $P_{\kappa}^{S}$.
Proof. By (21) and (11), $\rho_{\kappa}^{S}>\kappa$, and thus $1-\kappa / \rho_{\kappa}^{S}>0$. By the monotone convergence theorem and Lemma 2.6,

$$
\begin{aligned}
G_{\kappa}^{S}\left(x, x \mid 1 / \rho_{\kappa}^{S}\right) & =\lim _{z \uparrow 1 / \rho_{\kappa}^{S}} G_{\kappa}^{S}(x, x \mid z)=\lim _{z \uparrow 1 / \rho_{\kappa}^{S}} \frac{1}{1-\kappa z} G\left(x, x \left\lvert\, \frac{(1-\kappa) z}{1-\kappa z}\right.\right) \\
& =\frac{1}{1-\kappa / \rho_{\kappa}^{S}} G(x, x \mid 1 / \rho) .
\end{aligned}
$$

Therefore $P$ being rho-transient (or rho-recurrent) is equivalent to that $P_{\kappa}^{S}$ being rhotransient (or rho-recurrent).

If $P$ is critically (or strictly) rho-recurrent, then for any $x \in S, r(U \mid x, x)=1 / \rho$ (or by (5), there exists $x$ such that $r(U \mid x, x)>1 / \rho)$. By Lemma 2.7 and (2), $r\left(U_{\kappa}^{S} \mid x, x\right)=1 / \rho_{\kappa}^{S}$ (or $r\left(U_{\kappa}^{S} \mid x, x\right)>1 / \rho_{\kappa}^{S}$ ). The other direction is the same. Now the proof is done.

## 3. Proof of Theorem 1.1

Proof of Theorem 1.1. (ii). Given $0 \leq \kappa<\kappa^{\prime}<1$, we want to prove $\rho_{\kappa}^{L} \leq \rho_{\kappa^{\prime}}^{L}$. If $L=\emptyset$, then it is trivial. If $L=S$, then by (11) and (2), it is done. We assume that $L \neq S$ and $L \neq \varnothing$, and choose $x \in S \backslash L$.

We claim that: $\forall z \in\left[1,1 / \kappa^{\prime}\right), U_{\kappa}^{L}(x, x \mid z) \leq U_{\kappa^{\prime}}^{L}(x, x \mid z)$. In fact, as $x \notin L$ and $1>\kappa^{\prime} z>$ $\kappa z$, by Lemma 2.4 (iii), both $U_{\kappa}^{L}(x, x \mid z)$ and $U_{\kappa^{\prime}}^{L}(x, x \mid z)$ can be expanded. As $1>\kappa^{\prime} z>\kappa z$, we have $\frac{1-\kappa}{1-\kappa z} \leq \frac{1-\kappa^{\prime}}{1-\kappa^{\prime} z}$. Therefore the claim holds by the above mentioned expansions.

By Lemma 2.1 (iiii), we have $U_{\kappa^{\prime}}^{L}\left(x, x \mid 1 / \rho_{\kappa^{\prime}}^{L}\right) \leq 1$. Then as $x \notin L$ and $L \neq \varnothing$, by Lemma 2.4 (iiii), we have $1 / \rho_{\kappa^{\prime}}^{L}<1 / \kappa^{\prime}$. Therefore by the above claim, $U_{\kappa}^{L}\left(x, x \mid 1 / \rho_{\kappa^{\prime}}^{L}\right) \leq$ $U_{\kappa^{\prime}}^{L}\left(x, x \mid 1 / \rho_{\kappa^{\prime}}^{L}\right) \leq 1$. By Lemma 2.1 (iiii), $1 / \rho_{\kappa^{\prime}}^{L} \leq 1 / \rho_{\kappa^{\prime}}^{L}$.
(iii). Firstly, we are going to prove the right continuity. Set $0 \leq \kappa<\kappa^{\prime}<1$. By the Theorem 1.1 (i) , we have $\rho_{\kappa}^{L} \leq \rho_{\kappa^{\prime}}^{L}$. By the sandwich theorem, it suffices to show that
(a) $\rho_{\kappa^{\prime}}^{L} \leq\left(\rho_{\kappa}^{L}\right)_{\kappa_{1}\left(\kappa^{\prime}\right)}^{S}$;
(b) $\lim _{\kappa^{\prime} \downarrow \kappa}\left(\rho_{\kappa}^{L}\right)_{\kappa_{1}\left(\kappa^{\prime}\right)}^{S}=\rho_{\kappa}^{L}$,
where $\kappa_{1}\left(\kappa^{\prime}\right):=\left(\kappa^{\prime}-\kappa\right) /(1-\kappa)$. It is clear that (b) follows from $\lim _{\kappa^{\prime} \downarrow \kappa} \kappa_{1}\left(\kappa^{\prime}\right)=0$ and (22). Noticing $\left(P_{\kappa^{\prime}}^{L}\right)_{\kappa_{1}}^{S}=\left(P_{\kappa^{\prime}}^{L}\right)_{\kappa_{1}}^{S \backslash L}$, and Theorem 1.1 (ii), we have (四) by $\left(\rho_{\kappa^{\prime}}^{L}\right)_{\kappa_{1}}^{S}=\left(\rho_{\kappa^{\prime}}^{L}\right)_{\kappa_{1}}^{S \backslash L} \geq \rho_{\kappa^{\prime}}^{L}$.

Secondly, we prove the left continuity. For $0 \leq \kappa^{\prime \prime}<\kappa<1$, to show $\lim _{\kappa^{\prime \prime}} \uparrow \kappa \rho_{\kappa^{\prime \prime}}^{L}=\rho_{\kappa}^{L}$, as $\rho_{\kappa^{\prime \prime}}^{L} \leq \rho_{\kappa}^{L}$, it suffices to show that $-\kappa_{2}+\left(1+\kappa_{2}\right) \rho_{\kappa}^{L} \leq \rho_{\kappa^{\prime \prime}}^{L}$ where $\kappa_{2}:=\left(\kappa-\kappa^{\prime \prime}\right) /(1-\kappa)$. By (2), it suffices to prove that

$$
\rho_{\kappa}^{L} \leq \frac{\rho_{\kappa^{\prime \prime}}^{L}+\kappa_{2}}{1+\kappa_{2}}=\left(\rho_{\kappa^{\prime \prime}}^{L}\right)_{\frac{\kappa_{2}}{1+\kappa_{2}}}^{S^{\prime}},
$$

which can be derived from $\left(P_{\kappa^{\prime \prime}}^{L}\right)_{\frac{\kappa_{2}}{1+\kappa_{2}}}^{S_{1}}=\left(P_{\kappa}^{L}\right)_{\frac{\kappa_{2}}{1+\kappa_{2}}}^{S \backslash L}$ and Theorem 1.1 (ii).
(iii). If $\rho=1$, by Theorem 1.1 (i), $\rho_{\kappa}^{L} \geq 1$. By (11), we have $\rho_{\kappa}^{L} \leq 1$. We obtain the conclusion. If $\rho_{\kappa}^{L}=1$, by (2) and Theorem (1.1) (ii), $\kappa+(1-\kappa) \rho=\rho_{\kappa}^{S} \geq \rho_{\kappa}^{L}=1$. Hence $\rho \geq 1$. From (1), we are done.
(iv). As $L \neq \varnothing$, choose $x \in L$. By Theorem (1.1 (ii) and (11), $\rho_{\kappa}^{\{x\}} \leq \rho_{\kappa}^{L} \leq 1$. Then by the sandwich theorem it suffices to prove that $\lim _{\kappa \uparrow 1} \rho_{\kappa}^{\{x\}}=1$. By (11), it suffices to show that $\forall \epsilon \in(0,1), \exists K \in[0,1), \forall \kappa \in(K, 1), 1-\rho_{k}^{\{x\}}<\epsilon$. By Lemma 2.1 (iiii), $1-\rho_{\kappa}^{\{x\}}<\epsilon$ is equivalent to $U_{\kappa}^{\{x\}}(x, x \mid 1 /(1-\epsilon))>1$. By Lemma [2.4 (ii), $U_{\kappa}^{\{x\}}(x, x \mid 1 /(1-\epsilon))=$ $\kappa /(1-\epsilon)+(1-\kappa) U(x, x \mid 1 /(1-\epsilon))$. As $\lim _{\kappa \uparrow 1} U_{\kappa}^{\{x\}}(x, x \mid 1 /(1-\epsilon))=1 /(1-\epsilon)>1$, we can find $K$ to obtain the desired result.

## 4. Proof of Theorem 1.2 (ii), (iii) and Theorem 1.4 (iiii)

Lemma 4.1. Let $0<\# L<\infty$ and $0 \leq \kappa_{1}<\kappa_{2}<1$,
(i) If $P_{\kappa_{2}}^{L}$ is rho-transient, then $\rho=\rho_{\kappa_{1}}^{L}=\rho_{\kappa_{2}}^{L}$, and $P_{\kappa_{1}}^{L}$ is rho-transient. As a consequence, if $P_{\kappa_{1}}^{L}$ is rho-recurrent, then $P_{\kappa_{2}}^{L}$ is rho-recurrent.
(ii) If $P_{\kappa_{1}}^{L}$ is rho-recurrent and $\rho_{\kappa_{1}}^{L}<1$, then $\rho_{\kappa_{1}}^{L}<\rho_{\kappa_{2}}^{L}$.

Proof. We first show (ii) under the condition that $L=\{x\}$. As $P_{\kappa_{2}}^{\{x\}}$ is rho-transient, by (4), Theorem 1.1 (i), Lemma 2.3 and Corollary [2.5, we have

$$
r(U \mid x, x) \geq 1 / \rho \geq 1 / \rho_{\kappa_{1}}^{\{x\}} \geq 1 / \rho_{\kappa_{2}}^{\{x\}}=r\left(U_{\kappa_{2}}^{\{x\}} \mid x, x\right)=r(U \mid x, x) .
$$

Then $\rho=\rho_{k_{1}}^{\{x\}}=\rho_{k_{2}}^{\{x\}}$. Thus, by using Lemma 2.1 (iiii) and Lemma 2.4 (i), we have that

$$
\begin{equation*}
1 \geq U_{\kappa_{1}}^{\{x\}}\left(x, x \mid 1 / \rho_{\kappa_{1}}^{\{x\}}\right)=U_{\kappa_{1}}^{\{x\}}\left(x, x \mid 1 / \rho_{\kappa_{2}}^{\{x\}}\right)=\kappa_{1} / \rho_{\kappa_{2}}^{\{x\}}+\left(1-\kappa_{1}\right) U\left(x, x \mid 1 / \rho_{k_{2}}^{\{x\}}\right) . \tag{9}
\end{equation*}
$$

By (11), $1 / \rho_{\kappa_{2}}^{\{x\}} \geq 1$. Then by (91), $U\left(x, x \mid 1 / \rho_{k_{2}}^{\{x\}}\right) \leq 1$. Thus, by Lemma 2.4 (i) and Corollary 2.2, we have that

$$
U_{\kappa_{1}}^{\{x\}}\left(x, x \mid 1 / \rho_{\kappa_{1}}^{\{x\}}\right) \leq \kappa_{2} / \rho_{\kappa_{2}}^{\{x\}}+\left(1-\kappa_{2}\right) U\left(x, x \mid 1 / \rho_{\kappa_{2}}^{\{x\}}\right)=U_{\kappa_{2}}^{\{x\}}\left(x, x \mid 1 / \rho_{\kappa_{2}}^{\{x\}}\right)<1 .
$$

Then by Corollary 2.2, $P_{\kappa_{1}}^{\{x\}}$ is also rho-transient as desired.
We then show (iii) under the condition that $L=\{x\}$. Using Lemma 2.4 (ii) twice and Corollary 2.2,

$$
U_{\kappa_{2}}^{\{x\}}\left(x, x \mid 1 / \rho_{\kappa_{1}}^{\{x\}}\right)=\frac{\kappa_{2}-\kappa_{1}}{1-\kappa_{1}} / \rho_{\kappa_{1}}^{\{x\}}+\frac{1-\kappa_{2}}{1-\kappa_{1}} U_{\kappa_{1}}^{\{x\}}\left(x, x \mid 1 / \rho_{\kappa_{1}}^{\{x\}}\right)=\frac{\kappa_{2}-\kappa_{1}}{1-\kappa_{1}} / \rho_{\kappa_{1}}^{\{x\}}+\frac{1-\kappa_{2}}{1-\kappa_{1}} .
$$

By $\frac{1-\kappa_{2}}{1-\kappa_{1}} \in(0,1)$ and $1 / \rho_{\kappa_{1}}^{\{x\}}>1$, we know that $U_{\kappa_{2}}^{\{x\}}\left(x, x \mid 1 / \rho_{\kappa_{1}}^{\{x\}}\right)>1$. By Lemma 2.1 (iiii), we know that $1 / \rho_{\kappa_{1}}^{\{x\}}>1 / \rho_{\kappa_{2}}^{\{x\}}$ as desired.

Finally, let us show both (ii) and (iii) when $L=\left\{x_{1}, \cdots, x_{m}\right\}$ with $m \geq 2$. In this case $P_{\kappa_{2}}^{L}=\left(\cdots\left(\left(P_{\kappa_{1}}^{L}\right)_{\kappa}^{\left\{x_{1}\right\}}\right)_{\kappa}^{\left\{x_{2}\right\}} \cdots\right)_{k}^{\left\{x_{m}\right\}}$ with $\kappa=\left(\kappa_{2}-\kappa_{1}\right) /\left(1-\kappa_{1}\right)$. It is clear that we can obtain the desired result of (i) by induction. For (iii), by what we have already proved and Theorem 1.1 (ili), we have $\rho_{\kappa_{1}}^{L}<\left(\rho_{\kappa_{1}}^{L}\right)_{\kappa_{1}}^{\left\{x_{1}\right\}} \leq \rho_{\kappa_{2}}^{L}$ as desired.

Proof of Theorem 1.2 (ii). Step 1. We only have to consider the existence part since the uniqueness is trivial. Define $\kappa_{c}(L):=\sup \left\{\kappa \in[0,1): P_{\kappa}^{L}\right.$ is rho-transient $\}$. As $P$ is rhotransient, we know $\kappa_{c}(L) \in[0,1]$. By Lemma4.1(i) , if $\kappa_{c}(L)>0$ and $\kappa \in\left[0, \kappa_{c}(L)\right)$, then $P_{\kappa}^{L}$ is rho-transient and $\rho=\rho_{\kappa}^{L}$; if $\kappa_{c}(L)<1$ and $\kappa \in\left(\kappa_{c}(L), 1\right)$, then $P_{\kappa}^{L}$ is rho-recurrent. Also in the latter case, since $\rho_{\kappa}^{L}<1$, by Theorem 1.1 (iiii) and Lemma 4.1 (iii), we have that $\rho_{\kappa}^{L}$ increases strictly in $\kappa \in\left[\kappa_{c}(L), 1\right)$.

Step 2. Let us prove that $\kappa_{c}(L)<1$ and $\rho_{\kappa_{c}(L)}^{L}=\rho$. For the sake of contradiction, assume that $\kappa_{c}(L)=1$. Then by Step 1 , for $\kappa \in[0,1), \rho_{\kappa}^{L}=\rho$. As $\rho<1$, it contradicts Theorem 1.1 (iv). Hence $\kappa_{c}(L)<1$. Now by Theorem 1.1 (iii) and Step 1, we have $\rho_{\kappa_{c}(L)}^{L}=\rho$.

Step 3. Let us show that $P_{\kappa_{c}(L)}^{L}$ is rho-recurrent and $\kappa_{c}(L)>0$ when $L=\{x\}$. By Lemma 2.4 (ii),

$$
\begin{equation*}
U_{\kappa}^{\{x\}}\left(x, x \mid 1 / \rho_{\kappa}^{\{x\}}\right)=\kappa / \rho_{\kappa}^{\{x\}}+(1-\kappa) U\left(x, x \mid 1 / \rho_{\kappa}^{\{x\}}\right), \quad \kappa \in[0,1) . \tag{10}
\end{equation*}
$$

It can be verified that the right hand side of (10) is continuous in $\kappa$ by Theorem 1.1 (ii), (iii), Corollary [2.2, and the monotone convergence theorem. Hence, by Step 1, 2 and

Corollary 2.2, $P_{\kappa_{c}(\{x\})}^{\{x\}}$ is rho-recurrent. Therefore, $\kappa_{c}(\{x\})>0$ because otherwise it would contradict the condition that $P$ is rho-transient.

Step 4. Let us show that $\kappa_{c}(L)>0$ when $L=\left\{x_{1}, \cdots, x_{m}\right\}$ with $m \geq 2$. Note that $P_{\kappa}^{L}=\left(\cdots\left((P)_{\kappa}^{\left\{x_{1}\right\}}\right)_{\kappa}^{\left\{x_{2}\right\}} \cdots\right)_{\kappa}^{\left\{x_{m}\right\}}$. By Step 1, 3 and Theorem 1.1 (iiii), we know that for any $x \in S$,
(11) if $Q$ is a rho-transient irreducible transition probability on $S$ with spectral radius $<1$, then so is $Q_{\kappa}^{\{x\}}$ for some $\kappa \in(0,1)$.
Now repeating using this, we can verify that $\left(\cdots\left((P)_{\kappa_{1}}^{\left\{x_{1}\right\}}\right)_{\kappa_{2}}^{\left\{x_{2}\right\}} \cdots\right)_{\kappa_{m}}^{\left\{x_{m}\right\}}$ is rho-transient for some $\kappa_{1}, \kappa_{2}, \cdots, \kappa_{m} \in(0,1)$. From Lemma 4.1 (i), we can verify that $P_{\min \left\{\kappa_{i}: i=1, \cdots, m\right\}}^{L}$ is rho-transient. Then $\kappa_{c}(L) \geq \min \left\{\kappa_{i}: i=1, \cdots, m\right\}>0$ as desired.

Step 5. Finally, let us show that $P_{\kappa_{c}(L)}^{L}$ is rho-recurrent when $L=\left\{x_{1}, \cdots, x_{m}\right\}$ with $m \geq 2$. For the sake of contradiction, let us assume that $P_{\kappa_{c}(L)}^{L}$ is rho-transient. By Step 2, we know that $\rho_{\kappa_{c}(L)}^{L}=\rho<1$. From Steps 1,4 and Theorem 1.1 (iiii), we have that (11) holds with $\{x\}$ being replaced by $L$. Applying this to $P_{\kappa_{c}(L)}^{L}$, we know that $\left(P_{\kappa_{c}(L)}^{L}\right)_{\kappa}^{L}$ is rho-transient for some $\kappa \in(0,1)$. This contradicts how $\kappa_{c}(L)$ is defined in Step 1. We are done.

Proof of Theorem 1.2 (iii). By $\rho<1$ and Theorem 1.1 (iiii), $\rho_{\kappa}^{L}<1$ for every $\kappa \in[0,1)$. As $P$ is rho-recurrent, by Lemma 4.1 (il), $P_{\kappa}^{L}$ is rho-recurrent for every $\kappa \in[0,1)$. Now by Lemma 4.1 (iii), $\rho_{\kappa}^{L}<\rho_{\kappa^{\prime}}^{L}$ for every $0 \leq \kappa<\kappa^{\prime}<1$.
Proof of Theorem 1.4 (iii). If $L=\varnothing$, then $\rho_{\kappa}^{L}=\rho<\rho_{\kappa}^{S}$ by (21). If $L \neq \varnothing$ and $P_{\kappa}^{L}$ is rhorecurrent, then by taking $x \in S \backslash L$, Theorem 1.2 (iii) and Theorem 1.1) (ii), $\rho_{\kappa}^{L}<\left(\rho_{\kappa}^{L}\right)_{\kappa}^{\{x\}} \leq$ $\rho_{\kappa}^{S}$. If $L \neq \varnothing$ and $P_{\kappa}^{L}$ is rho-transient, then, by Lemma 4.1 (ii) and (2), $\rho_{\kappa}^{L}=\rho<\rho_{\kappa}^{S}$.

## 5. Proof of Theorem [1.3, Theorem 1.4 (ii), (iii) and Theorem 1.2 (iii)

Proof of Theorem 1.3. Step 1. Assuming that $P$ is rho-recurrent, let us prove that $P_{\kappa}^{L}$ is strictly rho-recurrent for every $\kappa \in(0,1)$. By Lemma 4.1 (ii), $P_{\kappa}^{L}$ is rho-recurrent. Suppose that $P_{\kappa}^{L}$ is not strictly rho-recurrent. Thus for arbitrarily fixed $x \in L$, we have

$$
\begin{equation*}
r\left(U_{\kappa}^{L} \mid x, x\right)=1 / \rho_{\kappa}^{L} . \tag{12}
\end{equation*}
$$

As $\kappa>0$, there exists $\kappa^{\prime}, \kappa_{1} \in(0, \kappa)$ such that $\left(\left(P_{\kappa}^{L \backslash\{x\}}\right)_{\kappa^{\prime}}^{\{x\}}\right)_{\kappa_{1}}^{\{x\}}=P_{\kappa}^{L}$. Let $P_{(1)}:=$ $\left(P_{\kappa}^{L \backslash\{x\}}\right)_{\kappa^{\prime}}^{\{x\}}$. As $P$ is rho-recurrent, by Lemma 4.1 (ii), $P_{(1)}$ is rho-recurrent. Denote by $\rho_{(1)}, U_{(1)}$ and $r\left(U_{(1)} \mid x, x\right)$ the spectral radius, U-function, and the convergence radius of the U-function of $P_{(1)}$, receptively. Then by (4), Corollary 2.5, (12) and Theorem 1.1 (i), $1 / \rho_{(1)} \leq r\left(U_{(1)} \mid x, x\right)=r\left(U_{\kappa}^{L} \mid x, x\right)=1 / \rho_{\kappa}^{L} \leq 1 / \rho_{(1)}$. Thus we have $1 / \rho_{(1)}=1 / \rho_{\kappa}^{L}$. Since $P_{\kappa}^{L}$ is rho-recurrent and $\left(P_{(1)}\right)_{\kappa_{1}}^{\{x\}}=P_{\kappa}^{L}$, by Corollary 2.2 and Lemma 2.4 (ii),

$$
1=U_{\kappa}^{L}\left(x, x \mid 1 / \rho_{\kappa}^{L}\right)=\left(U_{(1)}\right)_{\kappa_{1}}^{\{x\}}\left(x, x \mid 1 / \rho_{\kappa}^{L}\right)=\kappa_{1} / \rho_{\kappa}^{L}+\left(1-\kappa_{1}\right) U_{(1)}\left(x, x \mid 1 / \rho_{\kappa}^{L}\right)
$$

where we denote by $\left(U_{(1)}\right)_{\kappa_{1}}^{\{x\}}$ the U-function of the $\left(\{x\}, \kappa_{1}\right)$-lazy version of $P_{(1)}$. By $\rho<1$ and Theorem 1.1 (iiii), $1 / \rho_{\kappa}^{L}>1$. Together with $\kappa_{1} \in(0,1)$, we have $1>U_{(1)}\left(x, x \mid 1 / \rho_{\kappa}^{L}\right)=$ $U_{(1)}\left(x, x \mid 1 / \rho_{(1)}\right)$. Now by Corollary [2.2, $P_{(1)}$ is rho-transient, which is a contradiction.

Step 2. Suppose that $P$ is rho-transient. By Theorem 1.2 (ii), $P_{\kappa_{c}(L)}^{L}$ is rho-recurrent. Replacing $P$ by $P_{\kappa_{c}(L)}^{L}$ in Step 1, we can verify that $P_{\kappa}^{L}$ is strictly rho-recurrent for $\kappa \in\left(\kappa_{c}(L), 1\right)$.

Step 3. It remains to show that $P_{\kappa_{c}(L)}^{L}$ is critically rho-recurrent when $P$ is rho-transient. As $P_{\kappa_{c}(L)}^{L}$ is rho-recurrent (Theorem 1.2 (iil)), we assume for the sake of contradiction that $P_{\kappa_{c}(L)}^{L}$ is strictly rho-recurrent. Then by (5), there exists $x \in S$ such that

$$
\begin{equation*}
r\left(U_{\kappa_{c}(L)}^{L} \mid x, x\right)>1 / \rho_{\kappa_{c}(L)}^{L} . \tag{13}
\end{equation*}
$$

Case a): $P_{\kappa_{c}(L)}^{L}(x, x)>0$. In this case, for every $\kappa_{2} \in\left[0, P_{\kappa_{c}(L)}^{L}(x, x)\right]$, observe that there exists transition probability $\tilde{P}\left(\kappa_{2}\right)$ such that $\left(\tilde{P}\left(\kappa_{2}\right)\right)_{\kappa_{2}}^{\{x\}}=P_{\kappa_{c}(L)}^{L}$. Denote by $\tilde{\rho}\left(\kappa_{2}\right), \tilde{U}\left(\kappa_{2}\right)$ and $r\left(\tilde{U}\left(\kappa_{2}\right) \mid x, x\right)$ the spectral radius, the U-function, and the convergence radius of the U-function of $\tilde{P}\left(\kappa_{2}\right)$, respectively. By Corollary 2.5, $r\left(\tilde{U}\left(\kappa_{2}\right) \mid x, x\right)=r\left(U_{\kappa_{c}(L)}^{L} \mid x, x\right)$. Note that

$$
\tilde{P}\left(\kappa_{2}\right)=\left(\tilde{P}\left(P_{\kappa_{c}(L)}^{L}(x, x)\right)\right)_{\frac{P_{k_{c}(L)}^{L}(x, x)-\kappa_{2}}{1-\kappa_{2}}}^{\{x\}} .
$$

Therefore, by Theorem 1.1 (iii), $1 / \tilde{\rho}\left(\kappa_{2}\right)$ is continuous in $\kappa_{2}$. As $\tilde{\rho}(0)=\rho_{\kappa_{c}(L)}^{L}$ and (13), we can choose some $\kappa_{2} \in\left(0, P_{\kappa_{c}(L)}^{L}(x, x)\right)$ small enough, such that $1 / \tilde{\rho}\left(\kappa_{2}\right)<r\left(\tilde{U}\left(\kappa_{2}\right) \mid x, x\right)$. Thus by Lemma 2.3, $\tilde{P}\left(\kappa_{2}\right)$ is rho-recurrent. As $\rho<1$, by Theorem 1.1 (iiii), $\tilde{\rho}\left(\kappa_{2}\right)<1$. Then by Lemma 4.1 (iii) and Theorem 1.2 (i),

$$
\begin{equation*}
\tilde{\rho}\left(\kappa_{2}\right)<\rho_{\kappa_{c}(L)}^{L}=\rho . \tag{14}
\end{equation*}
$$

This leads to the following contradiction. Observe that $\tilde{P}\left(\kappa_{2}\right)(y, y)>0$ for every $y \in L$ (If $x \neq y$, then $\tilde{P}\left(\kappa_{2}\right)(y, y)=P_{\kappa_{c}(L)}^{L}(y, y)>0$; If $x=y$, as $\kappa_{2}<P_{\kappa_{c}(L)}^{L}(x, x)$, $\tilde{P}\left(\kappa_{2}\right)(y, y)>0$ also holds). Thus there exists $\kappa_{3} \in(0,1)$ and transition probability $P_{(2)}$ such that $\left(P_{(2)}\right)_{\kappa_{3}}^{L}=\tilde{P}\left(\kappa_{2}\right)$. Let $\rho_{(2)}$ be the spectral radius of $P_{(2)}$. Hence by Theorem 1.1 (ii), we have that

$$
\begin{equation*}
\rho_{(2)} \leq \tilde{\rho}\left(\kappa_{2}\right) \tag{15}
\end{equation*}
$$

Observe that $P_{\kappa_{c}(L)}^{L}=\left(\left(P_{(2)}\right)_{\kappa_{3}}^{L}\right)_{\kappa_{2}}^{\{x\}}=\left(\left(P_{(2)}\right)_{\kappa_{2}}^{\{x\}}\right)_{\kappa_{3}}^{L}$. Using Theorem 1.2 (ii), we have that $\left(P_{(2)}\right)_{k_{2}}^{\{x\}}$ is rho-transient and its spectral radius is $\rho$. Then by Lemma 4.1 (ii), $\rho_{(2)}=\rho$. Together with (14) and (15) forms a contradiction.

Case b): $P_{\kappa_{c}(L)}^{L}(x, x)=0$. Fix an arbitrary $\kappa \in[0,1)$ and define $Q:=P_{\kappa}^{S}$. By Lemma 2.8 and our assumptions about $P$ and $P_{\kappa_{c}(L)}^{L}, Q$ is rho-transient and $Q_{\kappa_{c}(L)}^{L}=$ $\left(P_{\kappa_{c}(L)}^{L}\right)_{\kappa}^{S}$ is strictly rho-recurrent. By (21) and Lemma 2.8, we can verify that $\kappa_{c}(L)$ is also the critical value in Theorem 1.2 (i) with respect to the underlying transition probability $Q$ and lazy state $L$. Now since $Q_{\kappa_{c}(L)}^{L}(y, y)>0$ holds for every $y \in S$, we can argue similarly as in case a) and arrive at a contradiction.

Proof of Theorem 1.4 (ii). By Lemma 2.8, if $P$ is rho-transient or critically rho-recurrent, so is $P_{\kappa}^{S}$. Note that $P_{\kappa}^{S}=\left(P_{\kappa}^{L}\right)_{\kappa}^{S \backslash L}$. Thus $P_{\kappa}^{S}$ is the $(S \backslash L, \kappa)$-lazy version of $P_{\kappa}^{L}$. For $\kappa \in(0,1)$, if $P_{\kappa}^{S}$ is rho-transient or critically rho-recurrent, then by applying Theorem 1.3
and Theorem 1.2 (i), we have $\rho_{\kappa}^{L}=\rho_{\kappa}^{S}$. If $P_{\kappa}^{S}$ is strictly rho-recurrent, then by Theorem 1.3 and Theorem 1.2 (ii) and (iii), we have $\rho_{\kappa}^{L}<\rho_{\kappa}^{S}$.
Proof of Theorem 1.4 (iii). By Lemma 2.8, $P_{\kappa}^{S}$ is strictly rho-recurrent. As \# $\left.S \backslash L\right)>0$, we can choose $x \in S \backslash L$. By Theorem 1.1 (ii), $\rho_{\kappa}^{L} \leq \rho_{\kappa}^{S \backslash\{x\}}$. As $\left(P_{\kappa}^{S \backslash\{x\}}\right)_{\kappa}^{\{x\}}=P_{\kappa}^{S}$, by Theorem 1.3 and Theorem 1.2 (ii) and (iii), $\rho_{\kappa}^{S \backslash\{x\}}<\rho_{\kappa}^{S}$. We are done.
Proof of Theorem 1.2 (iiii). Take $x \in L$ and $0 \leq \kappa_{1}<\kappa_{2}<1$. Firstly assume that $P_{\kappa_{1}}^{L}$ is rho-recurrent. By $\rho<1$ and Theorem (1.1 (iiii), we have $\rho_{\kappa_{1}}^{L}<1$. By Lemma 4.1 (iii) and Theorem 1.1 (ii), we have $\rho_{\kappa_{1}}^{L}<\left(\rho_{\kappa_{1}}^{L}\right)_{\kappa}^{\{x\}} \leq \rho_{\kappa_{2}}^{L}$ where $\kappa:=\frac{\kappa_{2}-\kappa_{1}}{1-\kappa_{1}}$. Now assume that $P_{\kappa_{1}}^{L}$ is rho-transient. By Lemma 2.8, $\left(P_{\kappa_{1}}^{L}\right)_{\kappa}^{S}$ is also rho-transient. As $\left(P_{\kappa_{1}}^{L}\right)_{\kappa}^{S}=\left(P_{\kappa_{2}}^{L}\right)_{\kappa}^{S \backslash L}$, by (22), $\left(P_{\kappa_{2}}^{L}\right)_{\kappa}^{S \backslash L}$ is rho-transient and $\rho_{\kappa_{1}}^{L}<\left(\rho_{\kappa_{1}}^{L}\right)_{\kappa}^{S}=\left(\rho_{\kappa_{2}}^{L}\right)_{\kappa}^{S \backslash L}$. As \# $(S \backslash L)<\infty$, by Lemma 4.1 (ii), $\left(\rho_{\kappa_{2}}^{L}\right)_{\kappa}^{S \backslash L}=\rho_{\kappa_{2}}^{L}$. We are done.

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[^0]:    2020 Mathematics Subject Classification. 60J10, 37A30.
    Key words and phrases. lazy Markov chain, spectral radius, rho-recurrent/transient, phase transition.

