WRIGHT-FISHER STOCHASTIC HEAT EQUATIONS WITH IRREGULAR DRIFTS

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Abstract. Consider $[0,1]$-valued solutions to the one-dimensional stochastic heat equation

$$\partial_t u_t = \frac{1}{2}\Delta u_t + b(u_t) + \sqrt{u_t(1-u_t)}\dot{W}$$

where $b(1) \leq 0 \leq b(0)$ and $\dot{W}$ is a space-time white noise. In this paper, we establish the weak existence and uniqueness of the above equation for a class of drifts $b(u)$ that may be irregular at the points where the noise is degenerate, that is, at $u = 0$ or $u = 1$. This class of drifts includes non-Lipschitz drifts like $b(u) = u^q(1-u)$ for every $q \in (0,1)$, and some discontinuous drifts like $b(u) = \mathbf{1}_{(0,1]}(u) - u$. This proves weak uniqueness for stochastic reaction-diffusion equations with Wright-Fisher noise and irregular drifts at zero, and demonstrates a regularization effect of the multiplicative space-time white noise without assuming the standard assumption that the noise coefficient is Lipschitz and non-degenerate.

The method we apply is a further development of a moment duality technique that uses branching-coalescing Brownian motions as the dual particle system. To handle an irregular drift in the above equation, particles in the dual system are allowed to have a number of offspring with infinite expectation, even an infinite number of offspring with positive probability. We show that, even though the branching mechanism with infinite number of offspring causes explosions in finite time, immediately after each explosion the total population comes down from infinity due to the coalescing mechanism. Our results on this dual particle system are of independent interest.

1. Introduction

1.1. **Motivation.** In this paper, we consider the $[0,1]$-valued continuous random field solution $(u_t(x))_{t\geq 0, x\in\mathbb{R}}$ to the stochastic partial differential equation (SPDE)

$$\begin{cases} \partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + b(u_t(x)) + \sigma(u_t(x))\dot{W}_{t,x}, & t>0, x\in\mathbb{R}, \\ u_0(x) = f(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with Wright-Fisher noise coefficient $\sigma(z) := \sqrt{z(1-z)}$ and drift

$$b(z) = \sum_{k\in\bar{\mathbb{N}}} b_k z^k = \sum_{k=0}^{\infty} b_k z^k + b_\infty \mathbf{1}_{\{1\}}(z), \quad z\in[0,1], \tag{1.2}$$

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satisfying $b(0) \geq 0 \geq b(1) > -\infty$. Here, $\dot{W}$ is a space-time white noise; f is a $[0,1]$-valued continuous function on $\mathbb{R}$; $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$; $(b_k)_{k\in\bar{\mathbb{N}}}$ is a family of real numbers; and $z^\infty := \mathbf{1}_{\{1\}}(z)$ for every $z \in [0,1]$.

We are going to show that there exists a unique in law solution to the above equation (1.1) provided there exists an $R \geq 1$ such that

$$b_1 \leq -\sum_{k\in\bar{\mathbb{N}}\setminus\{1\}} |b_k| R^{k-1}. \tag{1.3}$$

We will see later that under this assumption the drift b can be non-Lipschitz, and sometimes even discontinuous. In particular, we can handle one-dimensional equations like

$$\partial_t u_t = \frac{1}{2}\Delta u_t + u_t^q(1-u_t) + \sqrt{u_t(1-u_t)}\dot{W}$$

for every $q \in (0,1)$, and

$$\partial_t u_t = \frac{1}{2}\Delta u_t + \mathbf{1}_{(0,1]}(u_t) - u_t + \sqrt{u_t(1-u_t)}\dot{W}.$$

In the former case, this gives uniqueness in law for the stochastic reaction-diffusion equation with Wright-Fisher noise and Hölder drift near zero (example 1.3.1 below). In the latter case, the drift coefficient $b(\cdot)$ is discontinuous at $u = 0$, exactly the place where the noise coefficient $\sigma(u) = \sqrt{u(1-u)}$ is degenerate—the well-posedness results for this type of stochastic partial differential equations are rare. In fact, as we will discuss later with more details, the corresponding stochastic differential equation (SDE)

$$\mathrm{d}X_t = \big(\mathbf{1}_{(0,1]}(X_t) - X_t\big)\mathrm{d}t + \sqrt{X_t(1-X_t)}\mathrm{d}B_t$$

is ill-posed.

Our interest in investigating the equation (1.1) arises from an enormous body of literature studying the SPDE (1.1) with $b(\cdot)$ belonging to a class of certain smooth functions. Such equation is sometimes called the heat equation with Wright-Fisher noise and it arises as the scaling limits of the stepping stone model in population genetics (see [40]) and other important particle systems (see e.g. [34], [13]). In the above papers the drift belongs to a particular class of smooth functions, however, more general models may give rise to more general drifts (see [33] for an interesting discussion on non-spatial models), whose corresponding equations have been also studied (see [5] and [32]). This justifies studying well-posedness for (1.1) in presence of various types of drifts. The weak existence of (1.1) is standard for some continuous drift $b(\cdot)$ (see [39] and [32]). As for uniqueness of solutions, let us mention that the pathwise and strong uniqueness is still not resolved for (1.1) even in the case of zero drift. Thus, as we have mentioned above, we will concentrate on deriving weak existence/uniqueness of (1.1) for a class of irregular drifts.

First note, the weak uniqueness for (1.1) with a smooth drift of the form

$$b(z) = c_1(1-z) - c_2 z + c_3 z(1-z), \quad z \in [0,1],$$

(for $c_1, c_2 \geq 0, c_3 \in \mathbb{R}$) has been derived in [40] via a duality argument where the dual process is a system of coalescing Brownian motions with binary branching. Then, great progress was made in [4] where the weak uniqueness has been verified for a class of Lipschitz drifts that can be expressed in terms of power series whose coefficients satisfy certain assumptions. Note that noise coefficients more general than $\sigma(u) = \sqrt{u(1-u)}$ are allowed in [4], and duality with self-catalytic branching Brownian motions (including the branching-coalescing Brownian motions as one of the special cases) is again used for the proof of weak uniqueness. However, one should keep in mind, that in [4] only branching with finite mean is allowed in the dual model, which imposes certain restrictions on the drift coefficient b, such as the Lipschitz assumption among others.

Then an immediate question arises: is it possible to show well-posedness for (1.1) with drifts that are not-necessarily Lipschitz? Here we should mention another technique that is often used for resolving the weak uniqueness for stochastic equations. Namely, the Girsanov theorem (sometimes its version applied to SPDEs is called Dawson-Girsanov theorem). The Girsanov theorem was used in [32] to derive weak uniqeness for (1.1) with the drift bounded as follows:

$$|b(z)| \leq K\sqrt{z(1-z)}, \quad z \in [0,1],$$

which, at points $z = 0$ and $z = 1$, are Hölder continuous with exponent $1/2$. This leads us to a further question: *Does weak-uniqueness hold for* (1.1) *with a drift whose Hölder exponent is less than 1/2 at the points where the noise coefficient degenerates?*

As we have mentioned above, this paper gives an affirmative answer to this question. To prove it we use a modification of the duality method used in [4]. This modification is by no means trivial. It requires construction of the dual branching-coalescing Brownian motions with branching mechanism not-necessarily having a finite first moment. Even more than that, to treat some irregular drifts, the particles are asked to possibly have an *infinite number* of children at its branching time! In what follows we call this "infinite" branching. We establish a set of novel results for the branching-coalescing particle system with infinite branching, including its construction, which we believe are of independent interest. As we will show below, the total number of alive particles in this system is "reflecting from infinity"; the expected number of alive particles at any *fixed positive* time is almost surely finite; and the expectation of the number of births in the process over any finite time interval is also finite (see Theorem 1.4 below). The similar phenomenon of "reflecting from infinity" is also observed in other branching-coalescing type models, see [29] and [15]. In order to handle infinite branching, we used techniques from our previous work [6], where we prove a "coming down from infinity" result for coalescing Brownian motions, where we give necessary and sufficient conditions for an initially infinite collection of coalescing Brownian motions to collapse down to a finite number.

Another motivation for this work comes from the so-called *regularization by noise* area (see [14]) which is flourishing nowadays. In regularization by noise, one addresses the following question: does adding noise transform an ill-posed deterministic differential equations into a well-posed equation? In this context, the following ordinary SDE has been extensively studied in the literature:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t, \quad X_0 = x_0 \in \mathbb{R}^d, \tag{1.4}$$

where B is a d-dimensional Brownian motion and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a possibly irregular drift, see for example [27,43,46]. Also in many cases, well-posedness has been established for (1.4) for the drift b being a generalized function. For example, in [7,18,19,30] strong existence and uniqueness of solutions to (1.4) has been established for different types of distributional drifts. Weak existence and uniqueness of solutions to (1.4) has been also established in a number of papers, see for example [45]. Let us note that regularization by additive noise for ordinary differential equations has been studied for other noises as well, (Lévy processes, fractional Brownian processes), see for example [2,9,10,26,28,37,41].

As for regularization by additive noise for partial differential equations, the prominent example here is the stochastic heat equation driven by the additive space-time white noise, that is, one takes $\sigma \equiv 1$ in (1.1). One of the first results on strong existence and uniqueness for such SPDEs with irregular function-valued drifts b, have been obtained in [21] and [22]. Recently, there have been results on strong existence and uniqueness for such SPDE with b being a generalized function in a certain class (see e.g. [3]).

The well-posedness for equations with irregular drift driven by muliplicative noise has been studied mainly in the case of non-degenerate and Lipschitz noise coefficients: see, for example, [43] and [45] in the SDE setting and [20] in the SPDE setting. As for the well-posedness of equations with degenerate noise coefficients and irregular drifts see, for example, [12] for the SDE setting. For SPDEs driven by noises with degenerate non-Lipschitz coefficients, the results are not that rich. Strong well-posedness has been proved in [35] for an SPDE in the form (1.1) with σ belonging to Hölder continuous functions with exponent greater than 3/4 and Lipschitz drift b. Weak well-posedness has been recently given in [16] with some non-degenerate σ belonging to Hölder continuous functions with exponent greater than 3/4 and Hölder drift b. As for the SPDE (1.1) with σ being a Hölder function with exponent less than or equal to 3/4, only weak uniqueness for very particular noise coefficients σ (such as $\sigma(u) = u^\gamma$ with $\gamma \geq 1/2$, or $\sigma(u) = \sqrt{u(1-u)}$) and "nice" drifts b is known. By "nice" drift we mean either it satisfies conditions of Girsanov theorem or it is suitable for the duality technique (see [32] and [4] which are already mentioned above).

Thus, the goal of this paper is to extend the class of drifts in the stochastic heat equation driven by the Wright-Fisher space-time white noise for which the weak well-posedness holds.

1.2. **Main Results.** Before we introduce our main results, let us first discuss the rigorous definition of the solution to (1.1). Denote by $\mathcal{C}(\mathbb{R}, [0,1])$ the collection of $[0,1]$-valued continuous functions on $\mathbb{R}$, equipped with the topology of uniform convergence on compact sets. If there exists a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}_f)$, and on this space, an adapted $\mathcal{C}(\mathbb{R}, [0,1])$-valued continuous process $(u_t)_{t\geq 0}$, and an adapted space-time white noise $\dot{W}$, satisfying $u_0 = f$, and that for every $(t,x) \in (0,\infty) \times \mathbb{R}$ almost surely

$$u_t(x) = \int p_t(x-y) f(y) \mathrm{d}y + \iint_0^t p_{t-s}(x-y) b(u_s(y)) \mathrm{d}s\mathrm{d}y + \iint_0^t p_{t-s}(x-y)\sigma(u_s(y)) W(\mathrm{d}s\mathrm{d}y), \tag{1.5}$$

then we say $(u_t)_{t\geq 0}$ is a solution to the SPDE (1.1). Here,

$$p_t(x) := e^{-x^2/(2t)}/\sqrt{2\pi t}, \quad (t,x) \in (0,\infty)\times\mathbb{R} \tag{1.6}$$

is the heat kernel, and the third term on the right hand side of (1.5) is Walsh's stochastic integral driven by the space-time white noise [44]. Equation (1.5) is also known as the mild form of the SPDE (1.1).

Let us be more precise about the existence of the solutions. In this paper, we will be considering the weak existence. By that, we mean the existence of a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}_f)$, a random field $(u_t(x))_{t\geq 0, x\in\mathbb{R}}$, as well as a space-time white noise W, satisfying all the requirements above. If $b_\infty = 0$, then the drift coefficient $b(\cdot)$ is continuous; in this case, the weak existence of SPDE (1.1) is standard (see [39, Theorem 2.6] and [32, Section 2.1]). However, if $b_\infty \neq 0$, then the drift coefficient $b(\cdot)$ is discontinuous; in this case, the weak existence of SPDE (1.1) is not trivial, and will be part of our main result.

Let us also be more precise about the uniqueness of the solutions. Recall two uniqueness concepts—the pathwise uniqueness and the weak uniqueness. We say that pathwise uniqueness holds for the SPDE (1.1) if any two solutions on the same probability space driven by the same white noise are indistinguishable, i.e. they are equal for all time, almost surely. We say weak uniqueness holds for the SPDE (1.1) if any two solutions sharing the same initial value, not necessarily living in the same probability space nor driven by the same white noise, induce the same law in the path space $\mathcal{C}([0,\infty), \mathcal{C}(\mathbb{R},[0,1]))$.

As it has been mentioned in the first subsection, the pathwise uniqueness for the SPDE (1.1) is still open even in the case of zero drift. The weak uniqueness results are established in [4, 40] and [32] for a class of "nice" drifts. In what follows we will say that weak well-posedness holds for the SPDE (1.1) if both weak existence and weak uniqueness hold for it. Let us now present our main result.

Theorem 1.1. *Let $f \in \mathcal{C}(\mathbb{R},[0,1])$ be arbitrary. Let $(b_k)_{k\in\bar{\mathbb{N}}}$ be a family of real numbers. Let the function $b: z \mapsto b(z)$ be given as in* (1.2) *satisfying $b(0) \geq 0 \geq b(1) > -\infty$. If there exists an $R \geq 1$ such that* (1.3) *holds, then weak well-posedness holds for the SPDE* (1.1).

1.3. **Examples.** As we have mentioned above, Theorem 1.1 provides weak existence and weak uniqueness of SPDE (1.1) for a set of more singular drifts b than is obtained in [4] and [32]. Let us discuss some examples.

1.3.1. *Examples with non-Lipschitz drifts.* From the generalized binomial theorem, we have

$$(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \frac{q(q-1)(q-2)}{3!}x^3 + \cdots = \sum_{k=0}^{\infty}\binom{q}{k}x^k$$

for any $q > 0$ and $|x| \leq 1$, where

$$\binom{q}{0} := 1; \quad \text{and} \quad \binom{q}{n} := \frac{q(q-1)\dots(q-n+1)}{n!}, \quad n \geq 1.$$

Therefore, for $q \in (0,1)$, setting

$$b_0 := 0; \quad b_k := (-1)^k \binom{q}{k-1}, \quad k \in \mathbb{N}; \quad \text{and} \quad b_\infty := 0,$$

we have $b(z) = -(1-z)^q z$ for $z \in [0,1]$. One can also verify that (1.3) holds with $R = 1$. Therefore, by Theorem 1.1, there exists a unique in law solution u to the SPDE (1.1) with this drift and arbitrary $f \in C(\mathbb{R}, [0,1])$. Now $w = 1 - u$ will be the unique in law solution to the SPDE

$$\begin{cases} \partial_t w_t = \frac{1}{2}\Delta w_t + w_t^q(1-w_t) + \sqrt{w_t(1-w_t)}\dot{W}, & t > 0, x \in \mathbb{R}, \\ w_0(x) = 1 - f(x), & x \in \mathbb{R}. \end{cases} \tag{1.7}$$

The weak uniqueness of the SPDE (1.7), which confirms a conjecture we made in [5], does not follow from the result in [4], because the drift term is not Lipschitz. It also extends the weak well-posedness result covered in [32] to allow Hölder drift exponents $q \in (0, 1/2)$.

1.3.2. *Examples with discontinuous drifts.* Assume that $b_k = 0$ for every finite $k \neq 1$, $b_1 = -1$, and $b_\infty \in [-1, 1]$. Let $f \in C(\mathbb{R}, [0,1])$ be arbitrary. Then, by Theorem 1.1, there exists a unique in law solution $(u_t(x))_{t \geq 0, x \in \mathbb{R}}$ to the SPDE

$$\begin{cases} \partial_t u_t = \frac{1}{2}\Delta u_t - u_t + b_\infty \mathbf{1}_{\{1\}}(u_t) + \sqrt{u_t(1-u_t)}\dot{W}, \\ u_0 = 1 - f \in C(\mathbb{R}, [0,1]). \end{cases}$$

By defining $w_t = 1 - u_t$, we obtain a unique in law solution $(w_t(x))_{t \geq 0, x \in \mathbb{R}}$ to the SPDE

$$\begin{cases} \partial_t w_t = \frac{1}{2}\Delta w_t + 1 - w_t - b_\infty \mathbf{1}_{\{0\}}(w_t) + \sqrt{w_t(1-w_t)}\dot{W}, \\ w_0 = f. \end{cases} \tag{1.8}$$

We found (1.8) of a particular interest, since it shows the well-posedness of Wright-Fisher SPDEs with drifts that differ only by their values at the point $w = 0$. Also, one can check that solutions corresponding to different b_∞ have different distributions: this result is stated in the following lemma, whose proof is delayed to Section 6. In what follows we say $f \not\equiv 1$ if there exists $x \in \mathbb{R}$ such that $f(x) \neq 1$.

Lemma 1.2. *Let $f \in C(\mathbb{R}, [0,1])$ and let $f \not\equiv 1$. Fix arbitrary $b_\infty^{(1)}, b_\infty^{(2)} \in [-1, 1]$ with $b_\infty^{(1)} \neq b_\infty^{(2)}$. For $i = 1, 2$, let $w^{(i)}$ be the unique in law solution to (1.8) with $b_\infty = b_\infty^{(i)}$. Then $w^{(1)}$ and $w^{(2)}$ induce different laws on the path space $\mathcal{C}([0,\infty), \mathcal{C}(\mathbb{R}, [0,1]))$.*

Now, let us consider the SDE analogue of (1.8) with $b_\infty \in [-1, 1]$:

$$\mathrm{d}X_t = \big((1 - X_t) - b_\infty \mathbf{1}_{\{0\}}(X_t)\big)\mathrm{d}t + \sqrt{X_t(1-X_t)}\mathrm{d}B_t, \quad X_0 = x \in [0,1], \tag{1.9}$$

where B is a Brownian motion. In the next lemma we will show that the situation for (1.9) differs drastically from its SPDE counterpart.

Lemma 1.3.

(i) *Let $b_\infty = 1$. Then weak uniqueness does not hold for (1.9).*

(ii) *For any* $b_\infty \in [-1,1)$*, there is a pathwise unique solution to* (1.9) *which solves the equation*

$$\mathrm{d}X_t = (1-X_t)\mathrm{d}t + \sqrt{X_t(1-X_t)}\mathrm{d}B_t, \quad X_0 = x \in [0,1]. \tag{1.10}$$

The proof of the above lemma is simple and is delayed to Section 6. As we see from the above lemma, the Wright-Fisher noise has a very different regularizing effect in the SPDE setting compare to the SDE setting. In the case of $b_\infty = 1$, the well-posedness holds for the SPDE but not for the corresponding SDE. As for the case of $b_\infty \in [-1,1)$, in the SPDE setting there is a whole family of unique in law solutions corresponding to different b_∞, while in the SDE setting all the solutions are the same and the value of b_∞ does not play any role.

1.4. **The dual particle system.** To prove Theorem 1.1, we establish the moment duality between the SPDE (1.1) and a branching-coalescing Brownian particle system complimenting the previous results [40, Theorem 5.2] and [4, Theorem 1]. This particle system has three parameters:

- the branching rate $\mu > 0$;
- the offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$, which is a probability measure on $\bar{\mathbb{N}}$; and
- the initial configuration $\mathbf{x}_0 = (x_i)_{i=1}^n$, which is a (possibly infinite) list of real numbers. Here $n \in \mathbb{N} \cup \{\infty\}$. If $n < \infty$, then $(x_i)_{i=1}^n$ is a finite list; and if $n = \infty$, then $(x_i)_{i=1}^\infty$ is an infinite sequence. By our convention we denote $\mathrm{supp}(\mathbf{x}_0) = \{x_i\}_{i=1}^n \subset \mathbb{R}$ as the set of unique values in $\mathbf{x}_0$, which is the set of different initial locations.

Let us give an informal description of the branching-coalescing Brownian particle system, with the above parameters, through (1.11)–(1.14) below.

(1.11) At time 0, there are n many initial particles. For each finite integer $i \le n$, the i-th initial particle is located at position x_i.

(1.12) The particles in the system move as independent one-dimensional Brownian motions unless one of the events in the following steps occur.

(1.13) Each particle in the system induces a branching event according to an independent rate μ exponential clock. At each branching event, the corresponding particle (referred to as the parent) will be killed and replaced by a random number of new particles (referred to as the children) at the location where the parent is killed. The number of the children is independently sampled according to the offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$.

(1.14) Given the pairwise intersection local times of the particles in the system, each (unordered) pair of particles induces a coalescing event according to an independent rate 1/2 exponential clock with respect to their intersection local time; and at this coalescing event, one of the particle in that pair will be killed.

We want to mention that (1.11)–(1.14) does not give a rigorous definition of a particle system yet, due to a problem that, if the total population reaches ∞ at some finite time, then it is not clear how the pairwise dynamic (1.14) will work afterwards. Notice that this explosion of the total population would occur in finite time if either there are infinitely many initial particles already, or if one assumes that $p_\infty > 0$. And as it will be made clear later, to handle the discontinuous drifts $b(\cdot)$ with $b_\infty \neq 0$, we must handle the case of infinite branching.

This is why we will give a more rigorous construction of the branching-coalescing Brownian particle system in Section 2. In that detailed definition, the trajectory of each particle is constructed using an inductive procedure which allows us to be precise about the meaning of the pairwise dynamic (1.14) even after the total population explodes. A similar construction for the coalescing Brownian particle system (without branching) appeared in [42], and was employed already in our recent work [6] where the number of the initial particles is allowed to be infinity.

For the sake of discussing some of our main results for the dual particle system, let us introduce some notation. We will use X_t^α, an $\mathbb{R} \cup \{\dagger\}$-valued random variable, to represent the location of a particle labeled by $\alpha \in \mathcal{U}$ at time $t \geq 0$. Here the cemetery state $\dagger$ is an element not contained in $\mathbb{R}$, and

$$\mathcal{U} := \bigcup_{k=1}^{\infty} \mathbb{N}^k$$

is the space of the Ulam-Harris labels. We will label the particles in the system using the Ulam-Harris labels in a way that suggests their lineages: the initial particles are labeled with integers $\{i \in \mathbb{N} : i \leq n\}$, and if a particle has the label $\alpha = (\alpha_1, \cdots, \alpha_{m-1}, \alpha_m)$, then it is the α_m-th child created in the branching event induced by the particle $\overleftarrow{\alpha} := (\alpha_1, \cdots, \alpha_{m-1})$.

For every $t \geq 0$, let us also denote by $I_t := \{\alpha \in \mathcal{U} : X_t^\alpha \in \mathbb{R}\}$ the collection of labels of the particles alive at time t; and by J_t the collection of labels of the particles who induced a branching event up to time t. The cardinality of a given set I will be denoted by $|I|$. For some technical reason, we always assume that the set of the initial locations $\mathrm{supp}(\mathbf{x}_0)$ has finite cardinality. That is, the number of initial particles may be infinite but the set of their locations is finite. The probability space for the dual particle system will be denoted by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, which is not necessarily the same probability space corresponding to the SPDE (1.1).

Our main result on the branching-coalescing Brownian particle system, which is of the independent interest, is given in the next theorem. Note that $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ in this theorem denotes a branching-coalescing Brownian particle system which will be formally constructed in Section 2.

Theorem 1.4. *Suppose that $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ is a branching-coalescing Brownian particle system with arbitrary branching rate $\mu > 0$, offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$, and initial configuration $\mathbf{x}_0$. Suppose that the number of initial locations is finite, that is $|\mathrm{supp}(\mathbf{x}_0)| < \infty$. Then, the following statements hold.*

(1) For every $t > 0$, $\tilde{\mathbb{E}}[|I_t|] < \infty$.
(2) For every $t \geq 0$,

$$\tilde{\mathbb{E}}[|J_t|] = \mu \tilde{\mathbb{E}}\left[\int_0^t |I_s| \mathrm{d}s\right] < \infty.$$

Remark 1.5. Recall that we allow infinite offspring with positive probability, i.e. it is possible that $p_\infty > 0$. We want to mention some immediate corollaries of Theorem 1.4. Note that by this theorem, there are $|J_t| < \infty$ many branching events up to the finite time t. Since they are induced by independent exponential clocks, those branching events happens at different times. Let us denote the times of those branching events by

$$0 < \tau_1 < \tau_2 < \cdots < \tau_{|J_t|} < t$$

and define $\tau_0 = 0$ for convention. For each positive integer $k \leq |J_t|$, note that $t \mapsto |I_t|$ is non-increasing in the interval (τ_{k-1}, τ_k) due to the coalescing of the particles; also note that Theorem 1.4 (2) implies that almost surely $\int_{\tau_k}^{\tau_{k+1}} |I_s| \mathrm{d}s < \infty$. So it must be the case that, almost surely, $|I_s| < \infty$ for every $s \in (\tau_k, \tau_{k+1})$. In other word, the total population is reflecting from infinity, and can only reach infinity at the times $\{\tau_k : k = 0, 1, 2, \dots\}$.

Our duality formula between the SPDE (1.1) and its dual branching-coalescing Brownian particle system is a natural generalization of [4, Theorem 1], and will be presented later in Section 2. The weak uniqueness of (1.1) is a standard corollary of this duality formula. In the proof of the weak existence of (1.1), the duality formula will also play a crucial role when the drift is discontinuous.

1.5. **Organization of the paper.** In Section 2, we construct the dual particle system, state the crucial duality formula in Proposition 2.2, and give the proof of the weak uniqueness part of Theorem 1.1 using the duality. In Section 3, we prove several key properties for the dual particle system, in particular, Theorem 1.4 . In Section 4, we give the proof of the duality formula Proposition 2.2 using the results we proved in Section 3. In Section 5, we give the proof of the weak existence part of Theorem 1.1. In the Appendix, we collect the proofs of several technical lemmas.

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2. Duality

2.1. **The construction of the branching-coalescing Brownian particle system.** In this subsection, we give the formal construction of the branching-coalescing Brownian particle system. Recall from Subsection 1.4 that this model has three parameters: the branching rate μ, the offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$ and the initial configuration $(x_i)_{i=1}^n$. Again, the initial number of the particles $n \in \mathbb{N} \cup \{\infty\}$ is allowed to be infinite; and for every finite integer $i \leq n$, x_i is the location of the i-th initial particles. Also recall that $\mathcal{U}$ is the collection of the Ulam-Harris labels.

The Ulam-Harris labeling system is commonly used in the study of the branching particle systems, see [36] for one of its early appearances. A different labeling system, using the prime factorization, is proposed in [1] for self-catalytic branching Brownian motions. It was mentioned in both [1] and [4] that the particular labeling convention is not crucial to the duality method. However, in this paper, the Ulam-Harris labeling will help us be precise about the pairwise dynamic given by (1.14). In fact, as it will be made more clear later, when two particles coalesce, we will always remove the one with the larger label according to a total order $\prec$ of the space $\mathcal{U}$. This order is defined such that for any $\alpha \in \mathcal{U}$ and $\beta \in \mathcal{U}$, $\alpha \prec \beta$ if and only if one of the following three statements holds:

(i) $\|\alpha\|_\infty < \|\beta\|_\infty$;
(ii) $\|\alpha\|_\infty = \|\beta\|_\infty$ and $|\alpha| < |\beta|$;
(iii) $\|\alpha\|_\infty = \|\beta\|_\infty$, $|\alpha| = |\beta|$, and there exists an integer $m > 1$ such that $\alpha_m < \beta_m$ and $\alpha_k = \beta_k$ for every $k < m$.

Here, $|\alpha| := k$ and $\|\alpha\|_\infty := \max\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ for every $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathcal{U}$. We want to mention that different orders for the labeling space $\mathcal{U}$ are possible for the construction of the branching-coalescing Brownian motions. However, this order $\prec$ is particularly designed so that some technical lemmas (Lemmas 3.2, 5.1 and 5.2 below) hold.

Let us be precise about some building blocks for the construction. Let $\{(B_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ be a family of one-dimensional independent standard Brownian motions initiated at position 0.

(2.1) Let $\mathfrak{N}$ be a Poisson random measure on the space $(0,\infty) \times \mathcal{U} \times \bar{\mathbb{N}}$ with intensity $\hat{\mathfrak{N}}$ given so that $\hat{\mathfrak{N}}((0,t] \times \{\alpha\} \times \{k\}) = \mu t p_k$ for $t > 0, \alpha \in \mathcal{U}, k \in \bar{\mathbb{N}}$.

We assume that both $\{(B_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ and $\mathfrak{N}$ are defined on the same complete probability space, and are independent from each other. As mentioned in Subsection 1.4, this probability space will be denoted by $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$.

Inductively for each $\beta \in \mathcal{U}$, we construct random elements

$$\mathcal{X}_\beta := \Big(\xi_\beta, (\tilde{X}_t^\beta)_{t\geq 0}, (L_t^{\alpha,\beta})_{\alpha\prec\beta, t\geq 0}, (\mathfrak{M}(\cdot \times \{(\alpha,\beta)\}))_{\alpha\prec\beta}, (\zeta_{\alpha,\beta})_{\alpha\preceq\beta}, \zeta_\beta, (X_t^\beta)_{t\geq 0}, Z_\beta\Big)$$

according to the following rules (2.3)–(2.9) assuming that the random elements

$$\{\mathcal{X}_\alpha : \alpha \in \mathcal{U}, \alpha \prec \beta\} \tag{2.2}$$

are already constructed.

(2.3) Define a $\mathbb{R}_+$-valued random variable ξ_β and Brownian motion $(\tilde{X}_t^\beta)_{t\geq 0}$ so that
(i) if $|\beta| = 1$ and $\beta \leq n$, then $\xi_\beta := 0$ and $\tilde{X}_t^\beta := x_\beta + B_t^\beta$ for every $t \geq 0$;
(ii) if $|\beta| = 1$ and $\beta > n$, then $\xi_\beta := 0$ and $\tilde{X}_t^\beta := B_t^\beta$ for every $t \geq 0$;
(iii) if $|\beta| > 1$, then $\xi_\beta := \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}$ and

$$\tilde{X}_t^\beta := \begin{cases} \tilde{X}_t^{\overleftarrow{\beta}}, & t \in [0, \xi_\beta), \\ \tilde{X}_{\xi_\beta}^{\overleftarrow{\beta}} + B_t^\beta - B_{\xi_\beta}^\beta, & t \in [\xi_\beta, \infty). \end{cases}$$

(2.4) For each $\alpha \in \mathcal{U}$ with $\alpha \prec \beta$, $t \geq 0$ and $z \in \mathbb{R}$, define $L_{t,z}^{\alpha,\beta}$ to be the local time of the process $\tilde{X}_\cdot^\alpha - \tilde{X}_\cdot^\beta$ at position z up to time t, i.e.

$$L_{t,z}^{\alpha,\beta} := \lim_{h\downarrow 0} \frac{1}{h} \int_0^t \mathbf{1}_{\{\tilde{X}_s^\alpha - \tilde{X}_s^\beta \in [z,z+h)\}} \mathrm{d}\Big\langle \tilde{X}_\cdot^\alpha - \tilde{X}_\cdot^\beta \Big\rangle_s;$$

and write $L_t^{\alpha,\beta} := L_{t,0}^{\alpha,\beta}$. Here, $\langle \tilde{X}^\alpha_\cdot - \tilde{X}^\beta_\cdot \rangle$ is the quadratic variation of the process $\tilde{X}^\alpha_\cdot - \tilde{X}^\beta_\cdot$. Without loss of generality, we assume that almost surely, $(t,z) \mapsto L_{t,z}^{\alpha,\beta}$ is continuous, c.f. [38, Corollary 1.8 Chapter VI].

(2.5) Conditioned on $\{(B_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$, $\mathfrak{N}$, and the random elements in (2.2), for every $\alpha \in \mathcal{U}$ with $\alpha \prec \beta$, let $\mathfrak{M}(\cdot \times \{(\alpha,\beta)\})$ be a Poisson random measure on $(0,\infty)$ with intensity $\hat{\mathfrak{M}}(\cdot \times \{(\alpha,\beta)\})$ given such that

$$\hat{\mathfrak{M}}((0,t] \times \{(\alpha,\beta)\}) = \frac{1}{2} L_t^{\alpha,\beta}, \quad t > 0.$$

(2.6) Define

$$\zeta_{\beta,\beta} := \inf\{t > \xi_\beta : \mathfrak{N}(\{t\} \times \{\beta\} \times \overline{\mathbb{N}}) = 1\},$$

and, for each $\alpha \in \mathcal{U}$ with $\alpha \prec \beta$,

$$\zeta_{\alpha,\beta} := \inf\{t > \xi_\beta \vee \xi_\alpha : \mathfrak{M}(\{t\} \times \{(\alpha,\beta)\}) = 1\}.$$

(2.7) Define a $\mathbb{R}_+$-valued random variable ζ_β so that

(i) if $|\beta| = 1$ and $\beta \leq n$, then

$$\zeta_\beta := \inf(\{\zeta_{\beta,\beta}\} \cup \{\zeta_{\alpha,\beta} : \alpha \in \mathcal{U}, \alpha \prec \beta, \zeta_{\alpha,\beta} \leq \zeta_\alpha\});$$

(ii) if $|\beta| > 1$, $\zeta_{\overleftarrow{\beta}} = \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}$ and $\beta_{|\beta|} \leq Z_{\overleftarrow{\beta}}$, then

$$\zeta_\beta := \inf(\{\zeta_{\beta,\beta}\} \cup \{\zeta_{\alpha,\beta} : \alpha \in \mathcal{U}, \alpha \prec \beta, \zeta_{\alpha,\beta} \leq \zeta_\alpha\});$$

(iii) if neither of the conditions in (i) nor (ii) hold, then $\zeta_\beta := \xi_\beta$.

(2.8) Define the $\mathbb{R} \cup \{\dagger\}$-valued process

$$X_t^\beta := \begin{cases} \dagger, & t \in [0, \xi_\beta), \\ \tilde{X}_t^\beta, & t \in [\xi_\beta, \zeta_\beta), \\ \dagger, & t \in [\zeta_\beta, \infty). \end{cases}$$

(2.9) Define a $\bar{\mathbb{N}}$-valued random variable Z_β to be the unique $z \in \bar{\mathbb{N}}$ such that $\mathfrak{N}$ has an atom at $(\zeta_{\beta,\beta}, \beta, z)$.

We will refer to the family of processes $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$, constructed through (2.3)–(2.9), as a branching-coalescing Brownian particle system with branching rate μ, offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$, and initial configuration $(x_i)_{i=1}^n$.

Let us give some further comments on the above construction. For each $\beta \in \mathcal{U}$, we call the random variables ξ_β, and ζ_β, the birth-time, and the death-time, of the particle $\beta \in \mathcal{U}$, respectively. For each $\beta \in \mathcal{U}$, if $\zeta_\beta = \zeta_{\beta,\beta}$ holds, then we say particle β induced a branching event at the time $\zeta_{\beta,\beta}$; and if there exists an $\alpha \in \mathcal{U}$ with $\alpha \prec \beta$ such that $\zeta_\beta = \zeta_{\alpha,\beta}$, then we say the particle pair (α,β) induced a coalescing event at time $\zeta_{\alpha,\beta}$. Note that $\mathfrak{M}$ is a random measure on $(0,\infty) \times \mathcal{R}$ where

$$\mathcal{R} := \{(\alpha,\beta) \in \mathcal{U}^2 : \alpha \prec \beta\}.$$

Intuitively speaking, the branching, and the coalescing, events are governed by the random measure $\mathfrak{N}$, and $\mathfrak{M}$, respectively. As have already been mentioned in Subsection 1.4, for every $t \geq 0$, we denote by

$$I_t := \{\alpha \in \mathcal{U} : X_t^\alpha \in \mathbb{R}\} = \{\alpha \in \mathcal{U} : \xi_\alpha \leq t < \zeta_\alpha\} \tag{2.10}$$

the collection of labels of the particles alive at time t; and by

$$J_t := \{\alpha \in \mathcal{U} : \zeta_{\alpha,\alpha} = \zeta_\alpha \leq t\} \tag{2.11}$$

the collection of labels of the particles who induced a branching event up to time t.

We say a branching-coalescing Brownian particle system is a *coalescing Brownian particle system* if its offspring distribution satisfies $p_1 = 1$; and say it is a *killing-coalescing Brownian particle system* if its offspring distribution satisfies $p_0 = 1$. In [6], we give a necessary and sufficient condition for the total population of a coalescing Brownian particle system to come down from infinity. We also identified all the coming down rates for different initial configurations. The proof of Theorem 1.4 heavily relies on those results in [6].

2.2. **The duality formula.** For the rest of this paper, let us assume without loss of generality that

$$\sum_{k=2}^{\infty} |b_k| + |b_\infty| > 0. \tag{2.12}$$

(Otherwise, $b(z) = b_0 + b_1 z$ for every $z \in [0,1]$; and the weakly well-posedness of (1.1) in this case is already given by [40] and [4].) To build a connection between the branching-coalescing Brownian particle system and the SPDE (1.1), we will be working with a specific branching rate μ and a specific offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$ given by

$$\mu = |b_0| + \sum_{k=2}^{\infty} |b_k| + |b_\infty| > 0 \tag{2.13}$$

and

$$p_k = \mu^{-1}|b_k|\mathbf{1}_{\{k\neq 1\}}, \quad k \in \bar{\mathbb{N}}. \tag{2.14}$$

We first give the finiteness of the expectation of a certain functional of the particle system that will be used in the presentation of the duality formula.

Proposition 2.1. *Let $(b_k)_{k\in\bar{\mathbb{N}}}$ be a family of real numbers satisfying* (2.12) *and* (1.3) *for some $R \geq 1$. Let $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ be a branching-coalescing Brownian particle system with branching rate μ given as in* (2.13)*, offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$ given as in* (2.14)*, and an initial configuration $(x_i)_{i=1}^n$ with finite n. Then, it holds that*

$$\tilde{\mathbb{E}}\big[e^{K_t}\big] < \infty, \quad t \geq 0$$

where

$$K_t := (\mu + b_1) \int_0^t |I_s| \mathrm{d}s, \quad t \geq 0. \tag{2.15}$$

The proof of Proposition 2.1 is postponed to Subsection 3.6.

Now we are ready to present the duality formula between the SPDE (1.1) and our branching-coalescing Brownian particle system.

Proposition 2.2. *Let $f \in \mathcal{C}(\mathbb{R},[0,1])$, $n \in \mathbb{N}$, and $(x_i)_{i=1}^n$ be a finite list of real numbers. Let $(b_k)_{k\in\bar{\mathbb{N}}}$ be a family of real numbers satisfying* (2.12) *and* (1.3) *for some $R \geq 1$. Suppose that the real-valued function $(b(z))_{z\in[0,1]}$, given as in* (1.2)*, satisfies $b(0) \geq 0 \geq b(1) > -\infty$. Suppose that the $\mathcal{C}(\mathbb{R},[0,1])$-valued process $(u_t)_{t\geq 0}$, on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}_f)$, is a solution to the SPDE* (1.1) *with initial value $u_0 = f$. Also, suppose that $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ is a branching-coalescing Brownian particle system, on a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$, with initial configuration $(x_i)_{i=1}^n$, branching rate $\mu > 0$ given as in* (1.13)*, and offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$ given as in* (2.14)*. Then it holds for every $T \geq 0$ that*

$$\mathbb{E}_f\left[\prod_{i=1}^n u_T(x_i)\right] = \tilde{\mathbb{E}}\left[(-1)^{|\tilde{J}_T|} e^{K_T} \prod_{\alpha\in I_T} f(X_T^\alpha)\right]. \tag{2.16}$$

Here, $(K_t)_{t\geq 0}$ is given as in (2.15) *and*

$$\tilde{J}_t := \{\alpha \in \mathcal{U} : \zeta_{\alpha,\alpha} = \zeta_\alpha \leq t, b_{Z_\alpha} < 0\}, \quad t \geq 0.$$

The proof of Proposition 2.2 is postponed to Section 4. As we have mentioned, Proposition 2.2 is a generalization of Theorem 1 of [4]. One of the main assumptions in [4] is the finiteness of the expected number of offspring in the branching mechanism that guarantees the non-explosion of the system. We do not make such an assumption, but still are capable (due to the coalescent mechanism) of showing the finiteness of the total population at almost every time, as well as the above duality formula.

The weak uniqueness part of Theorem 1.1 is a direct corollary of the duality formula Proposition 2.2.

Proof of the weak uniqueness part of Theorem 1.1. From Theorem 1.4 and 2.1 we know that the right hand side of (2.16) is well-defined and finite. The desired result now follows from Proposition 2.2 and [4, Lemma 1]. □

3. Analysis for the dual particle system

In this section, we prove several properties for the branching-coalescing Brownian particle system. In particular, we will prove Theorem 1.4, which implies that the total population is finite at almost every time. We will also give the proof of Proposition 2.1, and several other integrability results, which will be used in the proof of Proposition 2.2 in Section 4.

3.1. **The truncated particle system.** In this subsection, let us take a branching-coalescing Brownian particle system $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ with an arbitrary branching rate $\mu > 0$, initial configuration $(x_i)_{i=1}^n$, and offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$, constructed through (2.3)–(2.9). There is a natural coupling between this branching-coalescing Brownian particle system $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ and a branching Brownian particle system $\{(\bar{X}_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$, sharing the same initial configuration, spatial movement of the particles, and the number of offspring in each of their shared branching events. The difference is that in the latter system we remove the coalescing mechanism, so the particles will branch but no longer coalesce with each other. As a consequence, this new system dominates the original one, in the sense that, almost surely, for every $\alpha \in \mathcal{U}$ and $t \geq 0$, $X_t^\alpha \in \mathbb{R}$ (that is, X_t^α is not in the cemetary state) implies that $\bar{X}_t^\alpha = X_t^\alpha$. More precisely, this coupling is realized through (3.1) and (3.2) below.

(3.1) For each $\beta \in \mathcal{U}$, define a $\mathbb{R}_+$-valued random variable $\bar{\zeta}_\beta$ inductively so that
- (i) if $|\beta| = 1$ and $\beta \leq n$, then $\bar{\zeta}_\beta := \zeta_{\beta,\beta}$.
- (ii) if $|\beta| > 1$, $\bar{\zeta}_{\overleftarrow{\beta}} = \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}$ and $\beta_{|\beta|} \leq Z_{\overleftarrow{\beta}}$, then $\bar{\zeta}_\beta := \zeta_{\beta,\beta}$;
- (iii) if neither of the conditions in (i) nor (ii) hold, then $\bar{\zeta}_\beta := \xi_\beta$.

(3.2) For each $\beta \in \mathcal{U}$, define a $\mathbb{R} \cup \{\dagger\}$-valued process

$$\bar{X}_t^\beta := \begin{cases} \dagger, & t \in [0, \xi_\beta), \\ \tilde{X}_t^\beta, & t \in [\xi_\beta, \bar{\zeta}_\beta), \\ \dagger, & t \in [\bar{\zeta}_\beta, \infty). \end{cases}$$

Define

$$\bar{I}_t = \{\alpha \in \mathcal{U} : \bar{X}_t^\alpha \in \mathbb{R}\},$$

and

$$\bar{J}_t = \{\alpha \in \mathcal{U} : \zeta_{\alpha,\alpha} = \bar{\zeta}_\alpha \leq t\}$$

to be the labels of all living particles at time $t \geq 0$, and the labels of all particles who induced a branching event before time $t \geq 0$, respectively, for this branching Brownian particle system. The following lemma allows us to control the branching-coalescing Brownian particle systems using this coupling. It says that the set of labels of living particles in the system with coalescing is contained in the set of labels in the system without coalescing, and similarly for the set of labels of branching events.

We omit its proof, since it is elementary.

Lemma 3.1. *$I_t \subset \bar{I}_t$ and $J_t \subset \bar{J}_t$ for every $t \geq 0$ almost surely.*

We say the offspring distribution $(p_k)_{k \in \bar{\mathbb{N}}}$ is bounded, if there exists an $m \in \mathbb{N}$ such that $p_k = 0$ for every $k > m$ including $k = \infty$. Some elementary results for the branching-coalescing Brownian particle system with bounded offspring distribution and finite many initial particles are easy to obtain from the above lemma. For example, if the offspring distribution is bounded and the initial number of particles n is finite, we know that almost surely for every $t \geq 0$, $|I_t| \leq |\bar{I}_t| < \infty$ and $|J_t| \leq |\bar{J}_t| < \infty$; in this case, the $\mathcal{N}$-valued càdlàg Markov process

$$\bar{\mathbb{X}}_t := \sum_{\alpha \in \bar{I}_t} \delta_{\bar{X}_t^\alpha}, \quad t \geq 0$$

is called the branching Brownian motion; and we can verify that the process of the counting measures

$$\mathbb{X}_t := \sum_{\alpha \in I_t} \delta_{X_t^\alpha}, \quad t \geq 0 \tag{3.3}$$

is also an $\mathcal{N}$-valued càdlàg Markov process, where $\mathcal{N}$ is the space of finite $\mathbb{Z}_+$-valued measures on $\mathbb{R}$ equipped with the weak topology.

However, if there is no assumption made about $(p_k)_{k\in\bar{\mathbb{N}}}$ and n, then it is not a priori clear whether $|I_t|$ and $|J_t|$ are finite. To proof that this is indeed the case for almost every t, our strategy is to approximate them from below using a family of truncated branching-coalescing Brownian particle systems

$$\Big\{(X_t^{(l,m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\Big\}, \quad l,m \in \mathbb{N}\cup\{\infty\}.$$

Here, l is the truncation number for the initial particles, m is the truncation number for the branching mechanism, and $\{(X_t^{(l,m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ will be constructed as a branching-coalescing Brownian particle system with initial configuration $(x_i)_{i=1}^{n\wedge l}$ and some offspring distribution bounded by m. In fact, $\{(X_t^{(l,m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ will be constructed in the same probability space as the non-truncated particle system $\{(X_t^{\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ in a way that the truncated particle system is dominated by the original particle system. That is, $X_t^{(l,m),\alpha} \in \mathbb{R}$ (i.e. $X_t^{(l,m),\alpha} \neq \dagger$) implies that $X_t^{\alpha} = X_t^{(l,m),\alpha}$, for every $\alpha \in \mathcal{U}$ and $t \geq 0$ almost surely. More precisely, this truncated particle system is defined through (3.4)–(3.6) below.

(3.4) For each $\alpha \in \mathcal{U}$ and $m \in \mathbb{N}\cup\{\infty\}$, define $Z_\alpha^{(m)} := Z_\alpha \wedge m$. (Recall (2.9).)

(3.5) For each $l,m \in \mathbb{N}\cup\{\infty\}$, define a family of $\mathbb{R}_+$-valued random variables $\{\zeta_\beta^{(l,m)} : \beta \in \mathcal{U}\}$ inductively so that for each $\beta \in \mathcal{U}$,

(i) if $|\beta| = 1$ and $\beta \leq n\wedge l$, then

$$\zeta_\beta^{(l,m)} := \inf\big(\{\zeta_{\beta,\beta}\} \cup \big\{\zeta_{\alpha,\beta} : \alpha \in \mathcal{U}, \alpha \prec \beta, \zeta_{\alpha,\beta} \leq \zeta_\alpha^{(l,m)}\big\}\big);$$

(ii) if $|\beta| > 1$, $\zeta_{\overleftarrow{\beta}}^{(l,m)} = \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}$ and $\beta_{|\beta|} \leq Z_{\overleftarrow{\beta}}^{(m)}$, then

$$\zeta_\beta^{(l,m)} := \inf\big(\{\zeta_{\beta,\beta}\} \cup \big\{\zeta_{\alpha,\beta} : \alpha \in \mathcal{U}, \alpha \prec \beta, \zeta_{\alpha,\beta} \leq \zeta_\alpha^{(l,m)}\big\}\big);$$

(iii) if neither of the conditions in (i) nor (ii) hold, then $\zeta_\beta^{(l,m)} := \xi_\beta$.

(3.6) For each $l,m \in \mathbb{N}\cup\{\infty\}$ and $\alpha \in \mathcal{U}$, define $\mathbb{R}\cup\{\dagger\}$-valued process

$$X_t^{(l,m),\alpha} := \begin{cases} \dagger, & t \in [0,\xi_\alpha), \\ \tilde{X}_t^\alpha, & t \in [\xi_\alpha, \zeta_\alpha^{(l,m)}), \\ \dagger, & t \in [\zeta_\alpha^{(l,m)}, \infty). \end{cases}$$

It is not hard to verify that $\{(X_t^{(l,m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ is a branching-coalescing Brownian particle system with branching rate μ, initial configuration $(x_i)_{i=1}^{n\wedge l}$, and offspring distribution $(p_k^{(m)})_{k\in\bar{\mathbb{N}}}$ such that for every $k \in \bar{\mathbb{N}}$,

$$p_k^{(m)} := \begin{cases} p_k, & k < m, \\ \sum_{j\in\bar{\mathbb{N}}, j\geq m} p_j, & k = m, \\ 0, & k > m. \end{cases}$$

We call $\{(X_t^{(l,m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ the (l,m)-truncated version of $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$. Also note that $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ is the (∞,∞)-truncated version of itself.

For each $l, m \in \mathbb{N} \cup \{\infty\}$ and $t \geq 0$, define

$$I_t^{(l,m)} := \{\alpha \in \mathcal{U} : X_t^{(l,m),\alpha} \in \mathbb{R}\} = \{\alpha \in \mathcal{U} : \xi_\alpha \leq t < \zeta_\alpha^{(l,m)}\} \tag{3.7}$$

as the collection of labels of alive particles at time t in the (l, m)-truncated particle system; and

$$J_t^{(l,m)} := \{\alpha \in \mathcal{U} : \zeta_{\alpha,\alpha} = \zeta_\alpha^{(l,m)} \leq t\} \tag{3.8}$$

as the collection of labels of particles who induced a branching event up to time t in the (l, m)-truncated particle system. Note that, if $l < \infty$ and $m < \infty$, then the explosion won't happen for the (l, m)-truncated version of the branching-coalescing Brownian particle system, since its initial number of particles is bounded by l and its offspring distribution is bounded by m; in other words, almost surely for every $t \geq 0$, $|I_t^{(l,m)}|$ and $|J_t^{(l,m)}|$ are finite.

We often truncate the initial number of the particles and the offspring distribution using the same number, that is, $l = m$. To simplify notations in this case, we write $\zeta_\alpha^{(m)} := \zeta_\alpha^{(m,m)}$, $X_t^{(m),\alpha} := X_t^{(m,m),\alpha}$, $I_t^{(m)} := I_t^{(m,m)}$ and $J_t^{(m)} := J_t^{(m,m)}$ for every $m \in \mathbb{N} \cup \{\infty\}$, $\alpha \in \mathcal{U}$ and $t \geq 0$. The particle systems $\{(X_t^{(m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ will be referred to as the m-truncated version of the original branching-coalescing Brownian particle system.

For each $m \in \mathbb{N}$, from how the m-truncated particle system is constructed, one can easily verify that it is dominated by the original particle system in the sense that $X_t^{(m),\alpha} \in \mathbb{R}$ implies that $X_t^\alpha = X_t^{(m),\alpha}$ for every $\alpha \in \mathcal{U}$ and $t \geq 0$ almost surely. In particular, the set of labels $I_t^{(m)}$ is a subset of I_t for every $t \geq 0$ almost surely. It is also not hard to verify that $J_t^{(m)}$ is a subset of J_t for every $t \geq 0$ almost surely. This relationship is made more precise in the following lemma, which also serves as an alternative way of interpreting the truncation.

Lemma 3.2. *Almost surely, for each $m \in \mathbb{N}$ and $t \geq 0$, we have*

$$I_t^{(m)} = \{\alpha \in \mathcal{U} : \|\alpha\|_\infty \leq m, \alpha \in I_t\}$$

and

$$J_t^{(m)} = \{\alpha \in \mathcal{U} : \|\alpha\|_\infty \leq m, \alpha \in J_t\}.$$

In particular, for any $t \geq 0$, $|I_t^{(m)}|$ and $|J_t^{(m)}|$ increasingly converges to $|I_t|$ and $|J_t|$, respectively, as $m \uparrow \infty$.

The proof of Lemma 3.2 is technical, and is given in Appendix A.1.

3.2. **The point processes and their compensator.** In this subsection, let us recall some preliminary results on the stochastic integral of the point processes from [23]. Suppose that E is a Polish space, and $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual condition. We say $\mathfrak{G}$ is a (E-valued) point process if it is a random measure on $(0, \infty) \times E$ and, almost surely, there exists a countable $S \subset (0, \infty)$ and a map $g : S \to E$ such that

$$\mathfrak{G}((0, t] \times U) = |\{s \in S : s \leq t, g(s) \in U\}|, \quad t > 0, U \in \mathscr{B}(E).$$

We say a point process $\mathfrak{G}$ is adapted, if the process $\mathfrak{G}((0,\cdot]\times U)$ is adapted for each $U\in\mathscr{B}(E)$. With $\Gamma_{\mathfrak{G}}:=\{U\in\mathscr{B}(E):\mathbb{E}[\mathfrak{G}((0,t]\times U)]<\infty \text{ for all } t>0\}$, we say a point process $\mathfrak{G}$ is σ-finite, if there exists a sequence of $\{E_n\in\Gamma_{\mathfrak{G}}:n\in\mathbb{N}\}$ such that $E_n\uparrow E$. We say a random measure $\mathfrak{G}$ is of the class QL, if it is an adapted, σ-finite point process, and there exists a non-negative random measure $\hat{\mathfrak{G}}$ on $(0,\infty)\times\mathbb{E}$ such that for every $U\in\Gamma_{\mathfrak{G}}$,

- $\hat{\mathfrak{G}}((0,\cdot]\times U)$ is a continuous adapted process; and
- $\mathfrak{G}((0,\cdot]\times U)-\hat{\mathfrak{G}}((0,\cdot]\times U)$ is a martingale.

We call $\hat{\mathfrak{G}}$ the compensator of $\mathfrak{G}$. Denote by $\mathscr{L}$ the space of predictable random fields on $\mathbb{R}_+\times E$. For any non-negative random measure $\mathfrak{G}$ on $(0,\infty)\times E$ and $k\in\{1,2\}$, define

$$\mathscr{L}^k_{\mathfrak{G}}:=\left\{f\in\mathscr{L}:\mathbb{E}\left[\iint_0^t|f(s,y)|^k\mathfrak{G}(\mathrm{d}s,\mathrm{d}y)\right]<\infty,\forall t\geq 0\right\},$$

and

$$\mathscr{L}^{k,\mathrm{loc}}_{\mathfrak{G}}:=\left\{f\in\mathscr{L}:\iint_0^t|f(s,y)|^k\mathfrak{G}(\mathrm{d}s,\mathrm{d}y)<\infty,\forall t\geq 0,\text{a.s.}\right\}.$$

For each $k\in\{1,2\}$, denote by $\mathscr{M}^k$ the space of martingales $(m_t)_{t\geq 0}$ such that $\mathbb{E}[|m_t|^k]<\infty$ for every $t\geq 0$; and by $\mathscr{M}^{k,\mathrm{loc}}$ the space of processes $(m_t)_{t\geq 0}$ such that $(m_{t\wedge\sigma_n})_{t\geq 0}$ belongs to $\mathscr{M}^k$ for some sequence of stopping times $\sigma_n\uparrow\infty$.

For a QL point process $\mathfrak{G}$ with compensator $\hat{\mathfrak{G}}$, its compensated stochastic integral, denoted by

$$\mathcal{I}_{\tilde{\mathfrak{G}}}:f\mapsto\iint_0^\cdot f(s,y)\tilde{\mathfrak{G}}(\mathrm{d}s,\mathrm{d}y),$$

is constructed, for example, in [23]. We collect some basic facts about this integration in following lemma.

Lemma 3.3. *Let $\mathfrak{G}$ be a QL point process with compensator $\hat{\mathfrak{G}}$.*

(1) If $f\in\mathscr{L}^2_{\hat{\mathfrak{G}}}$ then $\mathcal{I}_{\tilde{\mathfrak{G}}}f\in\mathscr{M}^2$.
(2) If $f\in\mathscr{L}^{2,\mathrm{loc}}_{\hat{\mathfrak{G}}}$ then $\mathcal{I}_{\tilde{\mathfrak{G}}}f\in\mathscr{M}^{2,\mathrm{loc}}$.
(3) $\mathscr{L}^1_{\mathfrak{G}}=\mathscr{L}^1_{\hat{\mathfrak{G}}}$. Moreover, if $f\in\mathscr{L}^1_{\mathfrak{G}}$ then $\mathcal{I}_{\tilde{\mathfrak{G}}}f\in\mathscr{M}^1$.
(4) If $f\in\mathscr{L}^{1,\mathrm{loc}}_{\hat{\mathfrak{G}}}$ then $f\in\mathscr{L}^{1,\mathrm{loc}}_{\mathfrak{G}}$, $\mathcal{I}_{\tilde{\mathfrak{G}}}f\in\mathscr{M}^{1,\mathrm{loc}}$, and

$$\mathcal{I}_{\tilde{\mathfrak{G}}}f=\iint_0^\cdot f(s,y)\mathfrak{G}(\mathrm{d}s,\mathrm{d}y)-\iint_0^\cdot f(s,y)\hat{\mathfrak{G}}(\mathrm{d}s,\mathrm{d}y).$$

Let us now consider an arbitrary branching-coalescing Brownian particle system

$$\{(X^\alpha_t)_{t\geq 0}:\alpha\in\mathcal{U}\},$$

constructed in Section 2, with arbitrary branching rate $\mu>0$, initial configuration $(x_i)_{i=1}^n$, and offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$. Recall the definitions of $\mathfrak{N},\hat{\mathfrak{N}}$ and $\mathfrak{M},\hat{\mathfrak{M}}$ in (2.1) and (2.5), respectively. Denote by $(\tilde{\mathscr{F}}_t)_{t\geq 0}$ the smallest filtration of the probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$, satisfying the usual hypothesis, such that the following processes are $(\tilde{\mathscr{F}}_t)_{t\geq 0}$-adapted:

- $B^\alpha_\cdot$ for each $\alpha \in \mathcal{U}$;
- $\mathfrak{N}((0,\cdot] \times \{\alpha\} \times \{k\})$ for each $\alpha \in \mathcal{U}$, and $k \in \bar{\mathbb{N}}$; and
- $\mathfrak{M}((0,\cdot] \times \{(\alpha,\beta)\})$ for each $(\alpha,\beta) \in \mathcal{R}$.

With respect to this filtration $(\tilde{\mathscr{F}}_t)_{t\geq 0}$, we can verify that

- the processes $X^\alpha_\cdot$ and $\tilde{X}^\alpha_\cdot$ are adapted for each $\alpha \in \mathcal{U}$;
- the process $L^{\alpha,\beta}_{\cdot,z}$ are adapted for each $\alpha,\beta \in \mathcal{R}$ and $z \in \mathbb{R}$;
- the random variables ξ_α and ζ_α are stopping times for each $\alpha \in \mathcal{U}$; and
- the random variable Z_α is $\tilde{\mathscr{F}}_{\zeta_{\alpha,\alpha}}$-measurable for each $\alpha \in \mathcal{U}$.

The main message of this subsection is the following lemma, whose proof is straightforward, and therefore, omitted.

Lemma 3.4. *The random measures $\mathfrak{M}$ and $\mathfrak{N}$ are QL point processes with compensators $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{N}}$ respectively.*

Using the above result, we can verify the following lemma.

Lemma 3.5. *For every $t \geq 0$ it holds that*

$$\tilde{\mathbb{E}}[|J_t|] = \mu\tilde{\mathbb{E}}\left[\int_0^t |I_s|\mathrm{d}s\right].$$

Proof. Fix an arbitrary $t \geq 0$. Let us first assume that the number of the initial particles is finite, and the offspring distribution is bounded. Notice that for each $\alpha \in \mathcal{U}$, $\alpha \in J_t$ if and only if there exists a (unique) $s \in (0,t]$ such that $X^\alpha_{s-} \in \mathbb{R}$ and $\mathfrak{N}(\{s\}\times\{\alpha\}\times\bar{\mathbb{N}}) = 1$. Therefore, almost surely,

$$|J_t| = \int_{\mathcal{U}}\int_0^t \mathbf{1}_{\{X^\alpha_{s-}\in\mathbb{R}\}}\mathfrak{N}(\mathrm{d}s,\mathrm{d}\alpha,\bar{\mathbb{N}}).$$

From Lemma 3.1, $|J_t|$ is dominated by $|\bar{J}_t|$, the number of branching events of the coupling branching Brownian particle system. Therefore $\tilde{\mathbb{E}}[|J_t|] \leq \tilde{\mathbb{E}}[|\bar{J}_t|] < \infty$ since we assumed that the number of initial particles is finite and the offspring distribution is bounded. Now from Lemmas 3.3, 3.4, and Fubini's theorem we have

$$\tilde{\mathbb{E}}[|J_t|] = \mu\tilde{\mathbb{E}}\left[\sum_{\alpha\in\mathcal{U}}\int_0^t \mathbf{1}_{\{X^\alpha_{s-}\in\mathbb{R}\}}\mathrm{d}s\right] = \mu\tilde{\mathbb{E}}\left[\int_0^t |I_s|\mathrm{d}s\right]$$

as desired.

In the case the initial configuration is not finite or the offspring distribution is unbounded, we can first consider the m-truncated particle system where $m \in \mathbb{N}$. From what we have proved,

$$\tilde{\mathbb{E}}\left[\left|J^{(m)}_t\right|\right] = \mu\tilde{\mathbb{E}}\left[\int_0^t |I^{(m)}_s|\mathrm{d}s\right], \quad m \in \mathbb{N}.$$

Taking $m \uparrow \infty$, from Lemma 3.2 and the monotone convergence theorem, we obtain the desired result. □

3.3. **The embedded killing-coalescing Brownian motions.** In this subsection, we introduce a marking procedure for an arbitrarily given branching-coalescing Brownian particle system $\{(X^\alpha_t)_{t\ge 0} : \alpha \in \mathcal{U}\}$ with a bounded offspring distribution and finite many initial particles. This marking procedure marks out an embedded killing-coalescing Brownian particle system. (Recall that a killing-coalescing Brownian particle system is a branching coalescing Brownian particle system with $p_0 = 1$.) This embedded killing-coalescing Brownian particle system helps us to control the original particle system locally. It will be the main ingredient for the proof of Theorem 1.4.

For a given stopping time $\tau < \infty$ and a finite subset $\mathcal{A}$ of $\mathcal{U}$, by a $(\tau, \mathcal{A})$-marking procedure, we mean the following:

(3.9) At time τ, if α is the k-th smallest label in the set $I_\tau \cap \mathcal{A}$ according to the order $\prec$, then we mark the particle α with number k; if $\alpha \in I_\tau \cap \mathcal{A}^c$, then we mark that particle with number ∞.

(3.10) After time τ, each particle carries its mark unless a branching or a coalescing event happens.

(3.11) For each branching event after time τ, the children will be marked by the number ∞ no matter of the mark of the parent.

(3.12) For each coalescing event after time τ, if the two particles inducing the coalescing event are marked by the numbers a and b, then the survivor (i.e. the particle with the smaller Ulam-Harris label) will be marked by the number $\min\{a, b\}$.

Since we assumed that the offspring distribution is bounded and the number of the initial particles is finite, from Subsection 3.1, there are almost surely only finitely many branching/coalescing events up to any finite time; and thus, the above marking procedure is well-defined.

For any number $k \in \mathbb{N}$ and time $t \ge 0$, there exists at most one particle alive at time $\tau + t$ that is marked by the number k. Denote by $\psi(\tau, \mathcal{A}, t, k)$ the Ulam-Harris label of the particle carrying the mark k at time $\tau + t$, provided such particle exists; and set $\psi(\tau, \mathcal{A}, t, k) = \varnothing$ if such particle does not exist. Also define a process $X^\varnothing_t = \dagger$ for every $t \ge 0$. We will refer to the family of processes

$$\left\{ \left(X^{\psi(\tau,\mathcal{A},t,k)}_{\tau+t} \right)_{t\ge 0} : k \in \mathbb{N} \right\} \tag{3.13}$$

the $(\tau, \mathcal{A})$-embedded killing-coalescing Brownian particle system. Using the strong Markov property of the Brownian motions, it is straightforward to verify the following lemma.

Lemma 3.6. *Conditioned on $\tilde{\mathscr{F}}_\tau$, the $(\tau, \mathcal{A})$-embedded killing-coalescing Brownian particle system* (3.13) *is a killing-coalescing Brownian particle system whose initial configuration is $(X^{\alpha^{(k)}}_\tau)^N_{k=1}$. Here $N := |I_\tau \cap \mathcal{A}|$, $\{\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(N)}\} = I_\tau \cap \mathcal{A}$, and $\alpha^{(1)} \prec \alpha^{(2)} \prec \cdots \prec \alpha^{(N)}$.*

Let us introduce some more notation related to this embedded killing-coalescing Brownian particle system that will be used later. For each $k \in \mathbb{N}$, define

$$\zeta^{(\tau,\mathcal{A})}_k := \tau + \sup\Big\{ t \ge 0 : X^{\psi(\tau,\mathcal{A},t,k)}_{\tau+t} \in \mathbb{R} \Big\}, \quad k \in \mathbb{N},$$

to be the death-time of the mark k; and if the particle with mark k induces a branching event at the time $\zeta_k^{(\tau,\mathcal{A})}$, then we say $\Theta^{(\tau,\mathcal{A})}(k) := 1$; otherwise we set $\Theta^{(\tau,\mathcal{A})}(k) := 0$. For every $t \geq 0$, define

$$I_{\tau+t}^{(\tau,\mathcal{A})} := \{k \in \mathbb{N} : X_{\tau+t}^{\psi(\tau,\mathcal{A},t,k)} \in \mathbb{R}\} \tag{3.14}$$

to be the collection of the marks that are carried by some alive particles at time $\tau + t$; and

$$J_{\tau+t}^{(\tau,\mathcal{A})} := \{k \in \mathbb{N} : \zeta_k^{(\tau,\mathcal{A})} \leq \tau + t, \Theta^{(\tau,\mathcal{A})}(k) = 1\} \tag{3.15}$$

to be the collection of the marks who deceased in a branching event up to time $\tau + t$.

For example, if we assume that $n < \infty$ and $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ is a coalescing Brownian particle system with initial configuration $(x_i)_{i=1}^n$, then its $(0, \{0, \dots, n\})$-embedded coalescing Brownian particle system is a killing-coalescing Brownian particle system who shares the same initial configuration $(x_i)_{i=1}^n$. This implies that the total population of a killing-coalescing Brownian particle system is stochastically dominated by that of a coalescing Brownian particle system. In our earlier paper [6], we established an upper bound for the the expectation of the total population of the coalescing Brownian particle system. Now it is clear that this upper bound also holds for the killing-coalescing Brownian particle system. In particular, from [6, Theorem 1.4 & Proposition 1.5] we have the following result.

Lemma 3.7 ([6, Theorem 1.4 & Proposition 1.5])**.** *Consider a killing-coalescing Brownian particle system with initial configuration $(x_i)_{i=1}^n$. Suppose that $n < \infty$ and define $N_0 := |\{x_i : i = 1, \dots, n\}|$. Denote by μ its branching rate and $|\hat{I}_t|$ the total population at time $t \geq 0$. Then there exists a time $T_1 > 0$ and a constant $C_1 > 0$ such that*

$$\tilde{\mathbb{E}}\left[\left|\hat{I}_t\right|\right] \leq \frac{C_1 N_0}{\sqrt{t}} \wedge n, \quad t \in [0, T_1].$$

Here T_1 and C_1 are independent of μ and n.

We will use both Lemmas 3.6 and 3.7 in the proof of Theorem 1.4.

3.4. **Some upper bounds provided the offspring distribution and the number of initial particles are bounded.** In this subsection, let us consider a branching-coalescing Brownian particle system $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ with a bounded offspring distribution and an initial configuration $(x_i)_{i=1}^n$ such that $n < \infty$. Define random variables

$$N_t := |\{x \in \mathbb{R} : \exists \alpha \in I_t \text{ s.t. } X_t^\alpha = x\}| < \infty, \quad t \geq 0.$$

In this subsection, we aim to give upper bounds for the expectation of the random variables

$$|I_t|, \quad |J_t|, \quad \text{and} \quad \int_0^t |I_s| \mathrm{d}s. \tag{3.16}$$

Lemma 3.8. *There exists a (deterministic) time $T_2(\mu) \geq 0$ such that for every $t \in [0, T_2(\mu)]$ it holds that $\tilde{\mathbb{E}}[|J_t|] \leq N_0$. Here, $T_2(\mu)$ is independent of the initial configuration and the offspring distribution.*

Proof. If a particle is labeled by an Ullam-Harris notation α with length $|\alpha| = j$, then we say it is in the j-th generation. For each $j \in \mathbb{N}$ and $t \geq 0$, denote by

$$J_t(j) := \{\alpha \in \mathcal{U} : \zeta_{\alpha,\alpha} = \zeta_\alpha \leq t, |\alpha| = j\}$$

the collection of the labels of particles in the j-th generation who induced a branching event before time t. Then almost surely we have the decomposition

$$|J_t| = \sum_{j=1}^{\infty} |J_t(j)|, \quad t \geq 0.$$

Let us take a deterministic time $T_2 := T_2(\mu) > 0$ small enough so that $T_2 \leq T_1$ and that

$$\mu \int_0^{T_2} \frac{C_1}{\sqrt{s}} \mathrm{d}s = 2\mu C_1 \sqrt{T_2} \leq \frac{1}{2}. \tag{3.17}$$

Here C_1 and T_1 are the constants introduced in Lemma 3.7. Notice that the choice of T_2 is independent of the initial configuration and the offspring distribution. We claim that

$$\tilde{\mathbb{E}}\Big[\big|J_{T_2}(j)\big|\Big] \leq \frac{N_0}{2^j}, \quad j \in \mathbb{N}. \tag{3.18}$$

From this claim we have

$$\tilde{\mathbb{E}}\Big[\big|J_{T_2}\big|\Big] = \sum_{j=1}^{\infty} \tilde{\mathbb{E}}\Big[\big|J_{T_2}(j)\big|\Big] \leq N_0,$$

as desired for this lemma.

Let us prove the claim (3.18) for $j = 1$ by using the $(\tau, \mathcal{A})$-marking procedure, given as in Subsection 3.3, with $\tau = 0$ and $\mathcal{A} = I_0$. From Lemma 3.6, the $(0, I_0)$-embedded killing-coalescing Brownian particle system

$$\left\{ \left(X_t^{\psi(0,I_0,t,k)} \right)_{t \geq 0} : k \in \mathbb{N} \right\}$$

is a killing-coalescing Brownian particle system with killing rate μ and initial configuration $(x_i)_{i=1}^n$. Recall that the sets of labels $I_t^{(0,I_0)}$ and $J_t^{(0,I_0)}$ are given by (3.14) and (3.15) for every $t \geq 0$. Observe that almost surely

$$\left| J_{T_2}(1) \right| \leq \left| J_{T_2}^{(0,I_0)} \right|$$

since any branching event induced by an initial particle is also a branching event of the $(0, I_0)$-embedded killing-coalescing Brownian particle system. Now, from Lemmas 3.5–3.7 and (3.17), we have

$$\tilde{\mathbb{E}}\Big[\big|J_{T_2}(1)\big|\Big] \leq \tilde{\mathbb{E}}\Big[\big|J_{T_2}^{(0,I_0)}\big|\Big] = \mu \tilde{\mathbb{E}}\left[\int_0^{T_2} |I_s^{(0,I_0)}| \mathrm{d}s\right] \leq \mu \int_0^{T_2} \frac{N_0 C_1}{\sqrt{s}} \mathrm{d}s \leq N_0/2$$

as desired.

We now prove the claim (3.18) by induction over $j \in \mathbb{N}$. For the sake of induction, let us assume that $\tilde{\mathbb{E}}[|J_{T_2}(j)|] \leq N_0/2^j$ for some $j \in \mathbb{N}$. For any Ullam-Harris label $\alpha \in \mathcal{U}$, let us denote by

$$J_{T_2}^{\alpha} := \left\{\beta \in \mathcal{U} : \overleftarrow{\beta} = \alpha, \zeta_\beta = \zeta_{\beta,\beta} \leq T_2\right\}$$

the collection of the labels of the children of the particle α that induced a branching event before time T_2. Then we have a decomposition

$$\left|J_{T_2}(j+1)\right| = \sum_{\alpha \in \mathcal{U}:|\alpha|=j} \left|J_{T_2}^{\alpha}\right|.$$

We claim that

$$\tilde{\mathbb{E}}\left[\left|J_{T_2}^{\alpha}\right| \Big| \tilde{\mathscr{F}}_{\rho_\alpha}\right] \leq \mathbf{1}_{\{\alpha \in J_{T_2}\}}/2 \tag{3.19}$$

for each $\alpha \in \mathcal{U}$, where the stopping time ρ_α is given by

$$\rho_\alpha := \begin{cases} \zeta_\alpha, & \text{if } \alpha \in J_{T_2}, \\ T_2, & \text{otherwise.} \end{cases}$$

Admitting the claim (3.19), we have

$$\begin{aligned} \tilde{\mathbb{E}}\left[\left|J_{T_2}(j+1)\right|\right] &= \tilde{\mathbb{E}}\left[\sum_{\alpha \in \mathcal{U}:|\alpha|=j} \mathbb{E}\left[\left|J_{T_2}^{\alpha}\right| \Big| \tilde{\mathscr{F}}_{\rho_\alpha}\right]\right] \\ &\leq \frac{1}{2}\tilde{\mathbb{E}}\left[\sum_{\alpha \in \mathcal{U}:|\alpha|=j} \mathbf{1}_{\{\alpha \in J_{T_2}\}}\right] = \frac{1}{2}\tilde{\mathbb{E}}\left[\left|J_{T_2}(j)\right|\right] \leq N_0/2^{j+1}. \end{aligned}$$

Now the desired (3.18) follows by induction.

We still needs to verify the claim (3.19) for an arbitrarily fixed $\alpha \in \mathcal{U}$. Since the offspring distribution are bounded, there exists an $m \in \mathbb{N}$ such that $p_k = 0$ for every $k \in \bar{\mathbb{N}}$ with $k > m$. Let us consider the $(\rho_\alpha, \mathcal{U}_\alpha)$-marking procedure given as in Subsection 3.3 for the particle system where

$$\mathcal{U}_\alpha := \{(\alpha, k) : k \in \mathbb{N}\} \tag{3.20}$$

is the collection of all the possible labels of the children of the particle α. From Lemma 3.6 we know that, conditioned on $\tilde{\mathscr{F}}_{\rho_\alpha}$, the $(\rho_\alpha, \mathcal{U}_\alpha)$-embedded killing-coalescing Brownian particle system

$$\left\{\left(X_{\rho_\alpha+t}^{\psi(\rho_\alpha,\mathcal{U}_\alpha,t,k)}\right)_{t\geq 0} : k \in \mathbb{N}\right\}$$

is a killing-coalescing Brownian particle system with killing rate μ and initial configuration $(X_{\rho_\alpha}^{(\alpha,k)})_{k=1}^{N_\alpha}$. Here, on the event $\{\alpha \in J_{T_2}\}$, $N_\alpha := Z_\alpha \leq m$ is the number of children of the particle α; and on the event $\{\alpha \notin J_{T_2}\} = \{\rho_\alpha = T_2\}$, $N_\alpha := 0$, i.e., there is no initial particle for the $(\rho_\alpha,\mathcal{U}_\alpha)$-embedded killing-coalescing Brownian particle system. Recall that

$$\left|J_{T_2}^{(\rho_\alpha,\mathcal{U}_\alpha)}\right| = \left|\left\{k\in\mathbb{N}:\zeta_k^{(\rho_\alpha,\mathcal{U}_\alpha)}\le T_2,\Theta^{(\rho_\alpha,\mathcal{U}_\alpha)}(k)=1\right\}\right|$$

is the number of the branching events of the $(\rho_\alpha,\mathcal{U}_\alpha)$-embedded killing-coalescing Brownian particle system up to time T_2. Observe that almost surely

$$\left|J_{T_2}^{\alpha}\right| \le \left|J_{T_2}^{(\rho_\alpha,\mathcal{U}_\alpha)}\right|$$

since any branching event induced by a child of the particle α is also a branching event of the $(\rho_\alpha,\mathcal{U}_\alpha)$-embedded killing-coalescing Brownian particle system. Now, from Lemmas 3.5 and 3.7, we have

$$\tilde{\mathbb{E}}\left[\left|J_{T_2}^{\alpha}\right|\middle|\tilde{\mathscr{F}}_{\rho_\alpha}\right] \le \tilde{\mathbb{E}}\left[\left|J_{T_2}^{(\rho_\alpha,\mathcal{U}_\alpha)}\right|\middle|\tilde{\mathscr{F}}_{\rho_\alpha}\right] = \mu\tilde{\mathbb{E}}\left[\int_0^{T_2-\rho_\alpha}\left|I_{\rho_\alpha+s}^{(\rho_\alpha,\mathcal{U}_\alpha)}\right|\mathrm{d}s\middle|\tilde{\mathscr{F}}_{\rho_\alpha}\right]$$
$$\le \mathbf{1}_{\{\alpha\in J_{T_2}\}}\mu\int_0^{T_2}\frac{C_1}{\sqrt{s}}\mathrm{d}s \le \mathbf{1}_{\{\alpha\in J_{T_2}\}}/2$$

as claimed. □

As a corollary of Lemmas 3.5 and 3.8, we have the following.

Corollary 3.9. *Let $T_2(\mu)\ge 0$ be given as in Lemma 3.8, then*

$$\mu\tilde{\mathbb{E}}\left[\int_0^t |I_s|\mathrm{d}s\right] = \tilde{\mathbb{E}}[|J_t|] \le N_0, \quad \forall t\in[0,T_2(\mu)].$$

Similarly, let us give an upper bound for $\tilde{\mathbb{E}}[|I_t|]$.

Lemma 3.10. *Let $T_2(\mu)\ge 0$ be given as in Lemma 3.8, then for any $t\in[0,T_2(\mu)]$ it holds that*

$$\tilde{\mathbb{E}}[|I_t|] \le \frac{C_1N_0}{\sqrt{t}}\wedge n + 2\mu C_1^2\pi N_0$$

where C_1 is the constant given as in Lemma 3.7.

Proof. Let us fix an arbitrary $T\in(0,T_2(\mu)]$. Notice that for every $\beta\in\mathcal{U}$ with $|\beta|>1$, $\beta\in I_T$ implies $\overleftarrow{\beta}\in J_T$. Therefore

$$|I_T| \le |I_T^{\varnothing}| + \sum_{\alpha\in J_T}|I_T^{\alpha}| \tag{3.21}$$

where

$$I_T^{\varnothing} := \left\{\beta\in\mathcal{U}:|\beta|=1, X_T^{\beta}\in\mathbb{R}\right\}$$

is the collection of the labels of the initial particles still alive at time T, and for every $\alpha\in\mathcal{U}$,

$$I_T^{\alpha} := \left\{\beta\in\mathcal{U}:\overleftarrow{\beta}=\alpha, X_T^{\beta}\in\mathbb{R}\right\}$$

is the collection of the labels of the children of particle α who are alive at time T.

Consider the $(\tau, \mathcal{A})$-marking procedure, given as in Subsection 3.3, with $\tau = 0$ and $\mathcal{A} = I_0$. From Lemma 3.6, the $(0, I_0)$-embedded killing-coalescing Brownian particle system

$$\left\{ \left(X_t^{\psi(0,I_0,t,k)} \right)_{t \geq 0} : k \in \mathbb{N} \right\}$$

is a killing-coalescing Brownian particle system with killing rate μ and initial configuration $(x_i)_{i=1}^n$. Recall that the sets of labels $I_T^{(0,I_0)}$ are given by (3.14) for every $T \geq 0$. Note that $I_T^{\varnothing}$ is a subset of $I_T^{(0,I_0)}$, since any initial particles that are alive at time T are also marked by a finite integer in the $(0, I_0)$-marking procedure. Therefore, by Lemma 3.7, we have

$$\tilde{\mathbb{E}}[|I_T^{\varnothing}|] \leq \tilde{\mathbb{E}}\left[\left|I_T^{(0,I_0)}\right|\right] \leq \frac{C_1 N_0}{\sqrt{T}} \wedge n.$$

Since the offspring distribution is bounded, there exists an $m \in \mathbb{N}$ such that $p_k = 0$ for every $k > m$. We claim that for every $\alpha \in \mathcal{U}$,

$$\tilde{\mathbb{E}}\left[|I_T^{\alpha}|\middle|\tilde{\mathscr{F}}_{\rho_\alpha}\right] \leq \frac{C_1}{\sqrt{T - \rho_\alpha}} \wedge m \tag{3.22}$$

where the stopping time ρ_α is defined by

$$\rho_\alpha := \begin{cases} \zeta_\alpha, & \text{if } \alpha \in J_T, \\ T, & \text{otherwise.} \end{cases}$$

From this claim and (3.21), we know that

$$\begin{aligned} \tilde{\mathbb{E}}[|I_T|] &\leq \frac{C_1 N_0}{\sqrt{T}} \wedge n + \tilde{\mathbb{E}}\left[\sum_{\alpha \in J_T} \tilde{\mathbb{E}}\left[|I_T^{\alpha}|\middle|\tilde{\mathscr{F}}_{\rho_\alpha}\right]\right] \\ &\leq \frac{C_1 N_0}{\sqrt{T}} \wedge n + \tilde{\mathbb{E}}\left[\sum_{\alpha \in J_T} \left(\frac{C_1}{\sqrt{T - \zeta_\alpha}} \wedge m\right)\right]. \end{aligned}$$

Notice that for each $\alpha \in \mathcal{U}$, $\alpha \in J_T$ if and only if there exists a (unique) $s \in (0, T]$ such that $X_{s-}^{\alpha} \in \mathbb{R}$ and $\mathfrak{N}(\{s\} \times \{\alpha\} \times \bar{\mathbb{N}}) = 1$; and in this case, it holds that $\zeta_\alpha = s$. Therefore, almost surely

$$\sum_{\alpha \in J_T} \left(\frac{C_1}{\sqrt{T - \zeta_\alpha}} \wedge m\right) = \int_{\mathcal{U}} \int_0^T \left(\frac{C_1}{\sqrt{T - s}} \wedge m\right) \mathbf{1}_{\{X_{s-}^{\alpha} \in \mathbb{R}\}} \mathfrak{N}(\mathrm{d}s, \mathrm{d}\alpha, \bar{\mathbb{N}}). \tag{3.23}$$

Notice that the left hand side of (3.23) is dominated by $m|\bar{J}_T|$ (see Lemma 3.1), which, under the assumption of the bounded offspring distribution and finite many initial particles, has finite first moment. Therefore from Lemmas 3.3 and 3.4, we have

$$\tilde{\mathbb{E}}\left[\sum_{\alpha \in J_T} \left(\frac{C_1}{\sqrt{T - \zeta_\alpha}} \wedge m\right)\right] = \mu \tilde{\mathbb{E}}\left[\sum_{\alpha \in \mathcal{U}} \int_0^T \left(\frac{C_1}{\sqrt{T - s}} \wedge m\right) \mathbf{1}_{\{X_{s-}^{\alpha} \in \mathbb{R}\}} \mathrm{d}s\right].$$

From this, and Fubini's theorem, we know that

$$\tilde{\mathbb{E}}[|I_T|] \leq \frac{C_1 N_0}{\sqrt{T}} \wedge n + \mu \int_0^T \frac{C_1}{\sqrt{T-s}} \tilde{\mathbb{E}}[|I_s|]\mathrm{d}s.$$

Since $T \in (0, T_2(\mu)]$ is arbitrary, we can iterate the above inequality and get from Fubini's theorem that

$$\begin{aligned}
\tilde{\mathbb{E}}[|I_T|] &\leq \frac{C_1 N_0}{\sqrt{T}} \wedge n + \mu \int_0^T \frac{C_1}{\sqrt{T-s}} \Big[\frac{C_1 N_0}{\sqrt{s}} \wedge n + \mu \int_0^s \frac{C_1}{\sqrt{s-r}} \tilde{\mathbb{E}}[|I_r|]\mathrm{d}r\Big]\mathrm{d}s \\
&\leq \frac{C_1 N_0}{\sqrt{T}} \wedge n + \mu N_0 C_1^2 \int_0^T \frac{1}{\sqrt{T-s}} \frac{1}{\sqrt{s}} \mathrm{d}s + \\
&\quad \mu^2 C_1^2 \int_0^T \tilde{\mathbb{E}}[|I_r|] \Big(\int_r^T \frac{1}{\sqrt{T-s}} \frac{1}{\sqrt{s-r}} \mathrm{d}s \Big) \mathrm{d}r \\
&= \frac{C_1 N_0}{\sqrt{T}} \wedge n + \mu C_1^2 \pi N_0 + \mu^2 C_1^2 \pi \int_0^T \tilde{\mathbb{E}}[|I_r|]\mathrm{d}r.
\end{aligned}$$

Here in the last step, we used the fact that, for every $r \in [0, T)$,

$$\int_r^T \frac{1}{\sqrt{T-s}} \frac{1}{\sqrt{s-r}} \mathrm{d}s = \int_0^{T-r} \frac{1}{\sqrt{T-r-l}} \frac{1}{\sqrt{l}} \mathrm{d}l = \int_0^1 \frac{1}{\sqrt{z(1-z)}} \mathrm{d}z = \pi.$$

Now, from Corollary 3.9 and (3.17), we have

$$\tilde{\mathbb{E}}[|I_T|] \leq \frac{C_1 N_0}{\sqrt{T}} \wedge n + 2\mu C_1^2 \pi N_0$$

as desired for this lemma.

We still need to verify the claim (3.22) for an arbitrarily fixed $\alpha \in \mathcal{U}$. Consider the $(\rho_\alpha, \mathcal{U}_\alpha)$-marking procedure for the particle system where $\mathcal{U}_\alpha$ is given as in (3.20). From Lemma 3.6 we know that, conditioned on $\tilde{\mathscr{F}}_{\rho_\alpha}$, the $(\rho_\alpha, \mathcal{U}_\alpha)$-embedded killing-coalescing Brownian particle system

$$\Big\{ \Big(X_{\rho_\alpha + t}^{\psi(\rho_\alpha, \mathcal{U}_\alpha, t, k)} \Big)_{t \geq 0} : k \in \mathbb{N} \Big\}$$

is a killing-coalescing Brownian particle system with killing rate μ and initial configuration $(X_{\rho_\alpha}^{(\alpha,k)})_{k=1}^{N_\alpha}$. Here, on the event $\{\alpha \in J_T\}$, $N_\alpha = Z_\alpha \leq m$ is the number of children of the particle α; and on the event $\{\alpha \notin J_T\} = \{\rho_\alpha = T\}$, we have $N_\alpha = 0$, i.e., there is no initial particle for the $(\rho_\alpha, \mathcal{U}_\alpha)$-embedded killing-coalescing Brownian particle system. Recall that

$$\Big| I_T^{(\rho_\alpha, \mathcal{U}_\alpha)} \Big| = \Big| \Big\{ k \in \mathbb{N} : X_T^{\psi(\rho_\alpha, \mathcal{U}_\alpha, T-\rho_\alpha, k)} \in \mathbb{R} \Big\} \Big|$$

is the number of particles of the $(\rho_\alpha, \mathcal{U}_\alpha)$-embedded killing-coalescing Brownian particle system at time T. Observe that

$$|I_T^\alpha| \leq \Big| I_T^{(\rho_\alpha, \mathcal{U}_\alpha)} \Big|, \quad \text{a.s.}$$

To see this, note that both sides of the above inequality equals 0 on the event $\{\alpha \notin J_T\} = \{\rho_\alpha = T\}$; and on the event $\{\alpha \in J_T\}$, any child of the particle α is always marked by a finite number in the $(\rho_\alpha, \mathcal{U}_\alpha)$-marking procedure. Now from Lemma 3.7 we have that

$$\tilde{\mathbb{E}}\Big[|I_T^\alpha|\Big|\tilde{\mathscr{F}}_{\rho_\alpha}\Big] \leq \tilde{\mathbb{E}}\Big[\Big|I_T^{(\rho_\alpha,\mathcal{U}_\alpha)}\Big|\Big|\tilde{\mathscr{F}}_{\rho_\alpha}\Big] \leq \frac{C_1}{\sqrt{T-\rho_\alpha}} \wedge m$$

as claimed. □

Corollary 3.9 and Lemma 3.10 give the upper bounds for the expectations of the random variables listed in (3.16) up to the time $T_2(\mu)$. Using the Markov property of the measure valued process $(\mathbb{X}_t)_{t\geq 0}$ given in (3.3), we can verify that for any $t \in (T_2, 2T_2]$,

$$\begin{aligned} \tilde{\mathbb{E}}[|I_t|] &\leq \tilde{\mathbb{E}}\left[\frac{C_1 N_{T_2}}{\sqrt{t-T_2}} \wedge |I_{T_2}| + 2\mu C_1^2 \pi N_{T_2}\right] \\ &\leq (1+2\mu C_1^2\pi)\tilde{\mathbb{E}}\Big[\Big|I_{T_2}\Big|\Big] \leq (1+2\mu C_1^2\pi)\left(\frac{C_1}{\sqrt{T_2}} + 2\mu C_1^2\pi\right) N_0 \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathbb{E}}[|J_t|] &= \mu\tilde{\mathbb{E}}\left[\int_0^t |I_s|\mathrm{d}s\right] = \mu\tilde{\mathbb{E}}\left[\int_0^{T_2} |I_s|\mathrm{d}s\right] + \mu\tilde{\mathbb{E}}\left[\int_{T_2}^t |I_s|\mathrm{d}s\right] \\ &\leq N_0 + \tilde{\mathbb{E}}[N_{T_2}] \leq N_0 + \tilde{\mathbb{E}}\Big[\Big|I_{T_2}\Big|\Big] \leq \left(1 + \frac{C_1}{\sqrt{T_2}} + 2\mu C_1^2\pi\right) N_0. \end{aligned}$$

Repeating this procedure inductively for $t \in (kT_2, (k+1)T_2]$ with $k \in \mathbb{N}$, one can verify the following result.

Corollary 3.11. *For every $t \geq 0$, there exist $C_2(\mu, t) > 0$ and $C_3(\mu, t) > 0$ such that*

$$\tilde{\mathbb{E}}[|I_s|] \leq \frac{C_1 N_0}{\sqrt{s}} \wedge n + C_2(\mu, t) N_0, \quad 0 \leq s \leq t$$

and

$$\tilde{\mathbb{E}}[|J_t|] = \mu\tilde{\mathbb{E}}\left[\int_0^t |I_s|\mathrm{d}s\right] \leq C_3(\mu, t) N_0.$$

Here, the constants $C_2(\mu, t)$ and $C_3(\mu, t)$ are independent of the initial configuration and the bounded offspring distribution.

3.5. **Uniform upper bound for arbitrary offspring distribution and initial configuration.** In this subsection, let us consider a branching-coalescing Brownian particle system $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ with an arbitrary offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$ and an arbitrary initial configuration $(x_i)_{i=1}^n$. Suppose that $N_0 := |\{x_i : i \in \mathbb{N}, i \leq n\}| < \infty$.

Proof of Theorem 1.4. For every $m \in \mathbb{N}$, denote by $\{(X_t^{(m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ the m-truncated version of the particle system $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ given as in Subsection 3.1. Denote $(I_t^{(m)})_{t\geq 0}$ and $(J_t^{(m)})_{t\geq 0}$ as in (3.7) and (3.8). It was known from Lemma 3.1 that for every $t \geq 0$, $|I_t^{(m)}|$ and $|J_t^{(m)}|$ monotonically increase to $|I_t|$ and $|J_t|$, respectfully, as $m \uparrow \infty$.

Since $\{(X_t^{(m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ is a branching-coalescing Brownian particle system with its offspring distribution and initial number of particles both bounded by m, we can conclude from Corollary 3.11 that for every $t \geq 0$,

$$\tilde{\mathbb{E}}\left[\left|I_t^{(m)}\right|\right] \leq \frac{C_1 N_0}{\sqrt{t}} \wedge m + C_2(\mu, t) N_0$$

and

$$\tilde{\mathbb{E}}\left[\left|J_t^{(m)}\right|\right] = \mu \tilde{\mathbb{E}}\left[\int_0^t |I_s^{(m)}| \mathrm{d}s\right] \leq C_3(\mu, t) N_0$$

where the constants $C_2(\mu, t) > 0$ and $C_3(\mu, t) > 0$ are independent of the initial configuration $(x_i)_{i=1}^n$ and the truncation number m. Taking $m \uparrow \infty$, the desired result now follows from the monotone convergence theorem. □

3.6. **The exponential term.** In this subsection, let $(b_k)_{k\in\bar{\mathbb{N}}}$ be a family of real numbers satisfying (2.12) and (1.3) for some $R \geq 1$. Let $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ be a branching-coalescing Brownian particle system with initial configuration $(x_i)_{i=1}^n$ such that $n < \infty$, the branching rate μ is given as in (2.13), and the offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$ is given as in (2.14). Recall from (3.7) that, for every $m \in \mathbb{N}$, $I_t^{(m)}$ is the labels of the particles in the m-truncated particle system living at time $t \geq 0$. Also recall that $(K_t)_{t\geq 0}$ is given as in (2.15). We will prove a result which is stronger than Proposition 2.1. This stronger result will be used later in the proof of Proposition 2.2.

Lemma 3.12. *For every $T \geq 0$, it holds that*

$$\sup_{0\leq t\leq T} \tilde{\mathbb{E}}\left[\left(1 + e^{K_t}\right)\left(1 + |I_t| + \left|I_t^{(m)}\right|^2\right)\right] < \infty.$$

We postpone the proof of Lemma 3.12 to Appendix A.2. The proof uses a supermartingale argument which is in a similar spirit to the proof of Lemma 3 of [4].

Proof of Proposition 2.1. The desired result is an immediate corollary of Lemma 3.12. □

We also need another expectation bound for the truncated particle system. It will be used in the proof of Proposition 2.2.

Lemma 3.13. *For every $m \in \mathbb{N}$ with $m \geq 2$ and $T \geq 0$, it holds that*

$$\tilde{\mathbb{E}}\left[\left(1 + e^{K_T^{(m)}}\right) \sum_{\alpha,\beta\in I_{[0,T]}^{(m)}:\alpha\prec\beta} \left(1 + \sup_{z\in\mathbb{R}} L_{T,z}^{\alpha,\beta}\right)\right] < \infty$$

where

$$K_t^{(m)} := \left(\mu + b_1 - \frac{1}{m}\right) \int_0^t |I_s^{(m)}| \mathrm{d}s, \quad t \geq 0$$

and

$$I_{[0,T]}^{(m)} := \bigcup_{s\in[0,T]} I_s^{(m)}$$

is the collection of labels of the particles born up until time T in the m-truncated branching-coalescing Brownian particle system.

We postpone the proof of Lemma 3.13 to Appendix A.2. In the proof, roughly speaking, we first analyze the moments of the exponential term and the local time terms separately, and then use Hölder's inequality. This is in a similar spirit to an argument in [4, p. 1725].

4. Proof of Proposition 2.2

In this section, we assume that the assumptions in Proposition 2.2 hold. More precisely, let $f \in \mathcal{C}(\mathbb{R}, [0,1])$, $n \in \mathbb{N}$ and $(x_i)_{i=1}^n$ be a finite list of real numbers. Let the real-valued function $(b(z))_{z\in[0,1]}$ satisfy $b(0) \geq 0 \geq b(1) > -\infty$, (1.2) and (1.3) for some $R \geq 1$. Suppose that the $\mathcal{C}(\mathbb{R}, [0,1])$-valued process $(u_t)_{t\geq 0}$, on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}_f)$, is a solution to the SPDE (1.1) with initial value $u_0 = f$. Let $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ be a branching-coalescing Brownian particle system, on a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$, with initial configuration $(x_i)_{i=1}^n$, branching rate $\mu > 0$ given by (1.13), and offspring distribution $(p_k)_{k\in\bar{\mathbb{N}}}$ given by (2.14).

Denote by $\mathbf{P}$ the product probability measure $\mathbb{P}_f \times \tilde{\mathbb{P}}$ on the product space $\Omega \times \tilde{\Omega}$. To establish the duality (Proposition 2.2), we consider

$$\Xi_{t,s}^{\epsilon,m} := \mathbf{E}\left[(-1)^{\left|\tilde{J}_s^{(m)}\right|} e^{K_s^{(m)}} \prod_{\alpha\in I_s^{(m)}} (P_\epsilon u_t)(X_s^\alpha)\right], \quad t,s \geq 0, \epsilon \geq 0, m \in \mathbb{N}. \tag{4.1}$$

Here, $(P_\epsilon)_{\epsilon\geq 0}$ is the one-dimensional heat semi-group, i.e. the transition semi-group of the one-dimensional Brownian motion; $I_s^{(m)}$ is the collections of the labels of the living particles at time $s \geq 0$ of the m-truncated branching-coalescing Brownian particle system given as in (3.7);

$$\tilde{J}_s^{(m)} := \{\alpha \in \mathcal{U} : \zeta_\alpha^{(m)} = \zeta_{\alpha,\alpha} \leq s, b_{Z_\alpha} < 0\}, \quad s \geq 0, m \in \mathbb{N};$$

and

$$K_s^{(m)} := (\mu + b_1 - \frac{1}{m}) \int_0^s |I_r^{(m)}| \mathrm{d}r, \quad s \geq 0, m \in \mathbb{N}.$$

Notice that, as $m \uparrow \infty$, the almost sure limit of the alternating term $(-1)^{|\tilde{J}_s^{(m)}|}$ in (4.1) is $(-1)^{|\tilde{J}_s|}$, since by Theorem 1.4 and Lemma 3.2, $|\tilde{J}_s^{(m)}| \uparrow |\tilde{J}_s| \leq |J_s| < \infty$, a.s. Also, observe that

$$\left|(-1)^{|\tilde{J}_s^{(m)}|} e^{K_s^{(m)}} \prod_{\alpha\in I_s^{(m)}} (P_\epsilon u_t)(X_s^\alpha)\right| \leq 1 + e^{K_s} \in L^1(\mathbf{P}), \quad s,t \geq 0, \epsilon \geq 0, m \in \mathbb{N}, \tag{4.2}$$

by Proposition 2.1. Therefore, the right hand side of (4.1) is well-defined and finite. Moreover, by Lemma 3.2 and the dominated convergence theorem, we have

$$\lim_{m\to\infty} \Xi_{t,s}^{\epsilon,m} = \Xi_{t,s}^{\epsilon,\infty} := \mathbf{E}\left[(-1)^{|\tilde{J}_s|} e^{K_s} \prod_{\alpha\in I_s} (P_\epsilon u_t)(X_s^\alpha)\right], \quad t,s,\epsilon \geq 0. \tag{4.3}$$

Now, the desired duality formula (2.16) can be written as $\Xi^{0,\infty}_{T,0} = \Xi^{0,\infty}_{0,T}$.

Our proof of Proposition 2.2 follows closely the strategy of the proof of Theorem 1 of [4]. There are mainly three steps:

- Step 1. By applying Ito's formula to $(-1)^{\left|\tilde{J}^{(m)}_s\right|} e^{K^{(m)}_s} \prod_{\alpha\in I^{(m)}_s} (P_\epsilon u_t)(X^\alpha_s)$ as a function of u_t, we obtain a decomposition for $\Xi^{\epsilon,m}_{t,s} - \Xi^{\epsilon,m}_{0,s}$.
- Step 2. By applying Ito's formula to $(-1)^{\left|\tilde{J}^{(m)}_s\right|} e^{K^{(m)}_s} \prod_{\alpha\in I^{(m)}_s} (P_\epsilon u_t)(X^\alpha_s)$ as a function of $\{(X^\alpha_s)_{s\geq 0} : \alpha \in \mathcal{U}\}$, we obtain a decomposition for $\Xi^{\epsilon,m}_{t,s} - \Xi^{\epsilon,m}_{t,0}$.
- Step 3. By inserting the two decompositions above in the equality
$$\int_0^T \big(\Xi^{\epsilon,m}_{r,0} - \Xi^{\epsilon,m}_{0,r}\big)\mathrm{d}r = \int_0^T \big(\Xi^{\epsilon,m}_{T-s,s} - \Xi^{\epsilon,m}_{0,s}\big)\mathrm{d}s - \int_0^T \big(\Xi^{\epsilon,m}_{t,T-t} - \Xi^{\epsilon,m}_{t,0}\big)\mathrm{d}t,$$
and then by taking the iterated limit as we first let $\epsilon \downarrow 0$ and then let $m \uparrow \infty$, we can verify that
$$\int_0^T \big(\Xi^{0,\infty}_{r,0} - \Xi^{0,\infty}_{0,r}\big)\mathrm{d}r = 0$$
for every T. This, and a continuity result of the maps $r \mapsto \Xi^{0,\infty}_{0,r}$ and $r \mapsto \Xi^{0,\infty}_{r,0}$ will finish the proof of Proposition 2.2.

Let us mention a crucial difference between our approach and the one in [4]. In [4], the particle system is stopped at a sequence of stopping times, instead of having truncated offspring at each of its branching events. This is partially due to the fact that the offspring distribution considered in [4] already have all finite moments, without the need of further truncation.

The precise statements of the three main steps above are given in the following three lemmas.

Lemma 4.1 (Step 1)**.** *For any $t, s \geq 0$, $\epsilon > 0$ and $m \in \mathbb{N}$, it holds that*
$$\Xi^{\epsilon,m}_{t,s} - \Xi^{\epsilon,m}_{0,s} = \Lambda^{\epsilon,m}_{t,s} + \Phi^{\epsilon,m}_{t,s} + \Psi^{\epsilon,m}_{t,s}$$
where
$$\Lambda^{\epsilon,m}_{t,s} := \mathbf{E}\left[(-1)^{|\tilde{J}^{(m)}_s|} e^{K^{(m)}_s} \int_0^t \left(\sum_{\alpha\in I^{(m)}_s} \Big(\frac{\Delta}{2} P_\epsilon u_r\Big)(X^\alpha_s) \prod_{\beta\in I^{(m)}_s\setminus\{\alpha\}} (P_\epsilon u_r)\big(X^\beta_s\big)\right) \mathrm{d}r\right],$$
$$\Phi^{\epsilon,m}_{t,s} := \mathbf{E}\left[(-1)^{|\tilde{J}^{(m)}_s|} e^{K^{(m)}_s} \int_0^t \left(\sum_{\alpha\in I^{(m)}_s} (P_\epsilon(b\circ u_r))(X^\alpha_s) \prod_{\beta\in I^{(m)}_s\setminus\{\alpha\}} (P_\epsilon u_r)\big(X^\beta_s\big)\right) \mathrm{d}r\right], \tag{4.4}$$
and
$$\Psi^{\epsilon,m}_{t,s} := \mathbf{E}\Bigg[(-1)^{|\tilde{J}^{(m)}_s|} e^{K^{(m)}_s} \times$$

$$\int_0^t \mathrm{d}r \int \sigma(u_r(y))^2 \left(\sum_{\alpha,\beta\in I_s^{(m)}:\alpha\prec\beta} p_\epsilon(y-X_s^\alpha)p_\epsilon(y-X_s^\beta) \prod_{\gamma\in I_s^{(m)}\setminus\{\alpha,\beta\}} (P_\epsilon u_r)(X_s^\gamma)\right)\mathrm{d}y\Bigg]$$

are all well-defined. Furthermore, for any $\epsilon>0$ and $m\in\mathbb{N}$,

$$\sup_{s,t\in[0,T]} \max\left\{\left|\Xi_{t,s}^{\epsilon,m}\right|,\left|\Lambda_{t,s}^{\epsilon,m}\right|,\left|\Phi_{t,s}^{\epsilon,m}\right|,\left|\Psi_{t,s}^{\epsilon,m}\right|\right\}<\infty,\quad T\geq 0.$$

Lemma 4.2 (Step 2)**.** *For any $t,s\geq 0$, $\epsilon>0$ and $m\in\mathbb{N}$, it holds that*

$$\Xi_{t,s}^{\epsilon,m}-\Xi_{t,0}^{\epsilon,m}=\tilde{\Lambda}_{t,s}^{\epsilon,m}+\tilde{\Phi}_{t,s}^{\epsilon,m}+\tilde{\Psi}_{t,s}^{\epsilon,m}$$

where

$$\tilde{\Lambda}_{t,s}^{\epsilon,m}:=\mathbf{E}\left[\int_0^s (-1)^{|\tilde{J}_r^{(m)}|}e^{K_r^{(m)}}\left(\sum_{\alpha\in I_r^{(m)}}\left(\frac{\Delta}{2}P_\epsilon u_t\right)(X_r^\alpha)\prod_{\beta\in I_r^{(m)}\setminus\{\alpha\}}(P_\epsilon u_t)\big(X_r^\beta\big)\right)\mathrm{d}r\right],$$

$$\tilde{\Phi}_{t,s}^{\epsilon,m}:=\mathbf{E}\left[\int_0^s (-1)^{|\tilde{J}_r^{(m)}|}e^{K_r^{(m)}}\left(\sum_{\alpha\in I_r^{(m)}}\big(b^{(m)}\circ(P_\epsilon u_t)\big)(X_r^\alpha)\prod_{\beta\in I_r^{(m)}\setminus\{\alpha\}}(P_\epsilon u_t)\big(X_r^\beta\big)\right)\mathrm{d}r\right], \tag{4.5}$$

and

$$\tilde{\Psi}_{t,s}^{\epsilon,m}:=\frac{1}{2}\mathbf{E}\Bigg[\int_0^s (-1)^{|\tilde{J}_r^{(m)}|}e^{K_r^{(m)}}\times \sum_{\alpha,\beta\in I_r^{(m)}:\alpha\prec\beta}(\sigma\circ(P_\epsilon u_t))(X_r^\alpha)^2\left(\prod_{\gamma\in I_r^{(m)}\setminus\{\alpha,\beta\}}(P_\epsilon u_t)(X_r^\gamma)\right)\mathrm{d}L_r^{\alpha,\beta}\Bigg] \tag{4.6}$$

are all well-defined. Here,

$$b^{(m)}(z):=\sum_{k\in\bar{\mathbb{N}}} b_k z^{k\wedge m}-\frac{1}{m}z,\quad z\in[0,1],m\in\mathbb{N}. \tag{4.7}$$

Furthermore, for any $\epsilon>0$ and $m\in\mathbb{N}$,

$$\sup_{s,t\in[0,T]} \max\left\{\left|\tilde{\Lambda}_{t,s}^{\epsilon,m}\right|,\left|\tilde{\Phi}_{t,s}^{\epsilon,m}\right|,\left|\tilde{\Psi}_{t,s}^{\epsilon,m}\right|\right\}<\infty,\quad T\geq 0.$$

The above two lemmas allows us to write down the following decomposition: For any $T\geq 0$, $\epsilon>0$ and $m\in\mathbb{N}$,

$$\begin{aligned}\int_0^T \big(\Xi_{r,0}^{\epsilon,m}-\Xi_{0,r}^{\epsilon,m}\big)\mathrm{d}r &= \int_0^T \big(\Xi_{T-s,s}^{\epsilon,m}-\Xi_{0,s}^{\epsilon,m}\big)\mathrm{d}s-\int_0^T \big(\Xi_{t,T-t}^{\epsilon,m}-\Xi_{t,0}^{\epsilon,m}\big)\mathrm{d}t\\ &= \int_0^T \Lambda_{T-s,s}^{\epsilon,m}\mathrm{d}s-\int_0^T \tilde{\Lambda}_{t,T-t}^{\epsilon,m}\mathrm{d}t+\int_0^T \Phi_{T-s,s}^{\epsilon,m}\mathrm{d}s-\int_0^T \tilde{\Phi}_{t,T-t}^{\epsilon,m}\mathrm{d}t+\\ &\quad \int_0^T \Psi_{T-s,s}^{\epsilon,m}\mathrm{d}s-\int_0^T \tilde{\Psi}_{t,T-t}^{\epsilon,m}\mathrm{d}t.\end{aligned} \tag{4.8}$$

Lemma 4.3 (Step 3)**.**

(1) For every $s,t\geq 0$ and $m\in\mathbb{N}$, it holds that

$$\lim_{\epsilon\downarrow 0}\Xi^{\epsilon,m}_{t,s}=\Xi^{0,m}_{t,s}.$$

(2) Both $r\mapsto\Xi^{0,\infty}_{r,0}$ and $r\mapsto\Xi^{0,\infty}_{0,r}$ are continuous functions on $[0,\infty)$.

(3) For every $T\geq 0$, $\epsilon>0$ and $m\in\mathbb{N}$, it holds that

$$\int_0^T\Lambda^{\epsilon,m}_{T-s,s}\mathrm{d}s=\int_0^T\tilde{\Lambda}^{\epsilon,m}_{t,T-t}\mathrm{d}t.$$

(4) For every $T\geq 0$ and $m\in\mathbb{N}$, it holds that

$$\lim_{\epsilon\downarrow 0}\bigg(\int_0^T\Psi^{\epsilon,m}_{T-s,s}\mathrm{d}s-\int_0^T\tilde{\Psi}^{\epsilon,m}_{t,T-t}\mathrm{d}t\bigg)=0.$$

(5) For every $T\geq 0$, it holds that

$$\lim_{m\uparrow\infty}\lim_{\epsilon\downarrow 0}\bigg(\int_0^T\Phi^{\epsilon,m}_{T-s,s}\mathrm{d}s-\int_0^T\tilde{\Phi}^{\epsilon,m}_{t,T-t}\mathrm{d}t\bigg)=0.$$

Let us first explain how Proposition 2.2 follows from Lemmas 4.1, 4.2, and 4.3.

Proof of Proposition 2.2. Let $T\geq 0$ be arbitrary. By (4.2) and Proposition 2.1, we know that

$$\sup_{0\leq s\leq T,t\geq 0,\epsilon\geq 0,m\in\mathbb{N}}|\Xi^{\epsilon,m}_{t,s}|\leq\tilde{\mathbb{E}}[1+e^{K_T}]<\infty.$$

Therefore, by Lemma 4.3 (1), (4.3), and the bounded convergence theorem, we have

$$\lim_{m\to\infty}\lim_{\epsilon\downarrow 0}\int_0^T\big(\Xi^{\epsilon,m}_{r,0}-\Xi^{\epsilon,m}_{0,r}\big)\mathrm{d}r=\int_0^T\big(\Xi^{0,\infty}_{r,0}-\Xi^{0,\infty}_{0,r}\big)\mathrm{d}r.$$

By taking $\epsilon\downarrow 0$ and then $m\uparrow\infty$ in (4.8), we get from Lemma 4.3 (3–5) that

$$\int_0^T\big(\Xi^{0,\infty}_{r,0}-\Xi^{0,\infty}_{0,r}\big)\mathrm{d}r=0.$$

Finally, since $T>0$ is arbitrary, from Lemma 4.3 (2), we get $\Xi^{0,\infty}_{T,0}=\Xi^{0,\infty}_{0,T}$ as desired. □

The rest of this section is devoted to the proofs of Lemmas 4.1, 4.2, and 4.3.

Proof of Lemma 4.1. Let us fix an arbitrary $\epsilon>0$.

Step 1. Recall that for any $g\in\mathcal{C}(\mathbb{R},[0,1])$ and $x\in\mathbb{R}$,

$$(P_\epsilon g)(x)=\int p_\epsilon(x-y)g(y)\mathrm{d}y$$

where $p_\epsilon(x)=e^{-x^2/(2\epsilon)}/\sqrt{2\pi\epsilon},(\epsilon,x)\in(0,\infty)\times\mathbb{R}$ is the heat kernel. It is standard to argue, see [39, p. 431] for example, that for any $t\geq 0$ and $x\in\mathbb{R}$, the following holds almost surely

$$(P_\epsilon u_t)(x)-(P_\epsilon u_0)(x)$$

$$= \int_0^t \frac{\Delta}{2}(P_\epsilon u_r)(x)\mathrm{d}r + \int_0^t \big(P_\epsilon(b\circ u_r)\big)(x)\mathrm{d}r + \iint_0^t \sigma(u_r(y))p_\epsilon(x-y)W(\mathrm{d}r\mathrm{d}y).$$

Applying Itô's formula, for any real-valued finite list $(x_i)_{i\in I}$ with $I = \{1,\cdots,n\}$ and $t\geq 0$, we have

$$\begin{aligned}
(4.9)\quad & \prod_{i=1}^n (P_\epsilon u_t)(x_i) - \prod_{i=1}^n (P_\epsilon u_0)(x_i) \\
&= \int_0^t \left(\sum_{i\in I}\left(\frac{\Delta}{2}P_\epsilon u_r\right)(x_i) \prod_{j\in I\setminus\{i\}}(P_\epsilon u_r)(x_j)\right)\mathrm{d}r + \\
&\int_0^t \left(\sum_{i\in I}(P_\epsilon(b\circ u_r))(x_i) \prod_{j\in I\setminus\{i\}}(P_\epsilon u_r)(x_j)\right)\mathrm{d}r + \\
&\iint_0^t \left(\sum_{i\in I}\sigma(u_r(y))p_\epsilon(x_i-y) \prod_{j\in I\setminus\{i\}}(P_\epsilon u_r)(x_j)\right)W(\mathrm{d}r\mathrm{d}y) + \\
&\iint_0^t \left(\sum_{(i,j)\in I^2: i<j}\sigma(u_r(y))^2 p_\epsilon(x_i-y)p_\epsilon(x_j-y) \prod_{k\in I\setminus\{i,j\}}(P_\epsilon u_r)(x_k)\right)\mathrm{d}r\mathrm{d}y.
\end{aligned}$$

Step 2. It is easy to see that the stochastic integral with respect to white noise at the right hand side of (4.9) is a maringale. Then, by taking expectation on both sides of (4.9) with respect to the measure $\mathbb{P}$, we can verify that for each $(x_i)_{i\in I}$ and $t\geq 0$,

(4.10)

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i\in I}(P_\epsilon u_t)(x_i)\right] - \mathbb{E}\left[\prod_{i\in I}(P_\epsilon u_0)(x_i)\right] \\
&= \mathbb{E}\left[\int_0^t \left(\sum_{i\in I}\left(\frac{\Delta}{2}P_\epsilon u_r\right)(x_i) \prod_{j\in I\setminus\{i\}}(P_\epsilon u_r)(x_j)\right)\mathrm{d}r\right] + \\
&\quad \mathbb{E}\left[\int_0^t \left(\sum_{i\in I}(P_\epsilon(b\circ u_r))(x_i) \prod_{j\in I\setminus\{i\}}(P_\epsilon u_r)(x_j)\right)\mathrm{d}r\right] + \\
&\quad \mathbb{E}\left[\iint_0^t \left(\sum_{(i,j)\in I^2: i<j}\sigma(u_r(y))^2 p_\epsilon(x_i-y)p_\epsilon(x_j-y) \prod_{k\in I\setminus\{i,j\}}(P_\epsilon u_r)(x_k)\right)\mathrm{d}r\mathrm{d}y\right].
\end{aligned}$$

Here, it is straightforward to verify that the two expectations on the left hand side of (4.10) are bounded by 1; the first, second, and the third, expectations on the right hand side of (4.10) are bounded by $\epsilon^{-1}|I|t$, $\|b\|_\infty|I|t$, and $\|\sigma\|_\infty^2\|p_\epsilon\|_\infty|I|^2t$, respectively.

Step 3. Let us replace the deterministic $(x_i)_{i\in I}$ in (4.10) by the random $(X^\alpha_s)_{\alpha\in I^{(m)}_s}$, and take expectations with respect to $\tilde{\mathbb{P}}$, after multiplied by $(-1)^{\tilde{J}^{(m)}_s} e^{K^{(m)}_s}$, for each of the terms in (4.10). This leads us to the desired result for this lemma after applying Fubini's theorem. To use Fubini's theorem, of course, we need to verify the integrable conditions for each term.

For instance, for the first term on the right hand side, it is sufficient to show that

$$\mathbf{E}\big[(1+e^{K_s})|I_s|\big]<\infty,\quad s\geq 0, \tag{4.11}$$

since for each $s\geq 0$, almost surely,

$$\int_0^t\left|(-1)^{|\tilde{J}^{(m)}_s|}e^{K^{(m)}_s}\left(\sum_{\alpha\in I^{(m)}_s}\Big(\frac{\Delta}{2}P_\epsilon u_r\Big)(X^\alpha_s)\prod_{\beta\in I^{(m)}_s\setminus\{\alpha\}}(P_\epsilon u_r)\big(X^\beta_s\big)\right)\right|\mathrm{d}r$$
$$\leq \epsilon^{-1}\int_0^t e^{K^{(m)}_s}|I^{(m)}_s|\mathrm{d}r\leq t(1+e^{K_s})\epsilon^{-1}|I_s|.$$

Note that (4.11) holds by Lemma 3.12, and furthermore,

$$\sup_{s,t\in[0,T]}|\Lambda^{\epsilon,m}_{t,s}|\leq\frac{T}{\epsilon}\sup_{0\leq s\leq T}\tilde{\mathbb{E}}\big[(1+e^{K_s})|I_s|\big]<\infty,\quad T\geq 0.$$

Similar arguments are valid for other terms of (4.10) as well, while replacing $(x_i)_{i\in I}$ with $(X^\alpha_s)_{\alpha\in I^{(m)}_s}$, multiplying each terms with $(-1)^{\tilde{J}^{(m)}_s}e^{K^{(m)}_s}$, and then taking expectations with respect to $\tilde{\mathbb{P}}$. We are done. □

Proof of Lemma 4.2. Step 1. Take an arbitrary $h\in\mathcal{C}^2_{\mathrm{b}}(\mathbb{R},[0,1])$, i.e. a $[0,1]$-valued twice continuously differentiable function h on $\mathbb{R}$ satisfying $\|h'\|_\infty<\infty$ and $\|h''\|_\infty<\infty$. Also define $h(\dagger):=1$. Recall that ξ_α and $\zeta^{(m)}_\alpha$ are respectively the birth-time and the death-time of the particle $\alpha\in\mathcal{U}$ in the m-truncated branching-coalescing Brownian particle system. Using Ito's formula, it is standard to verify that for any deterministic finite subset $\mathcal{I}$ of $\mathcal{U}$ and $s\geq 0$, we have almost surely

$$\begin{aligned}\prod_{\alpha\in\mathcal{I}}h(X^{(m),\alpha}_s)-\prod_{\alpha\in\mathcal{I}}h(X^{(m),\alpha}_0)-\sum_{r\leq s}\left(\mathbf{\Delta}\prod_{\alpha\in\mathcal{I}}h(X^{(m),\alpha}_r)\right)\\ =\int_0^s\sum_{\alpha\in\mathcal{I}}\left(\prod_{\beta\in\mathcal{I}\setminus\{\alpha\}}h(X^{(m),\beta}_{r-})\right)\mathbf{1}_{(\xi_\alpha,\zeta^{(m)}_\alpha]}(r)\frac{1}{2}h''(X^{(m),\alpha}_{r-})\mathrm{d}r+\\ \int_0^s\sum_{\alpha\in\mathcal{I}}\left(\prod_{\beta\in\mathcal{I}\setminus\{\alpha\}}h(X^{(m),\beta}_{r-})\right)\mathbf{1}_{(\xi_\alpha,\zeta^{(m)}_\alpha]}(r)h'(X^{(m),\alpha}_{r-})\mathrm{d}B^\alpha_r.\end{aligned} \tag{4.12}$$

Here, we write $\mathbf{\Delta}A_t:=A_t-A_{t-}$ for any $t\geq 0$ and càdlàg process $(A_t)_{t\geq 0}$.

Step 2. Let us fix the arbitrary $m\in\mathbb{N}$, and take a sequence of deterministic finite subset $\mathcal{I}_n$ of $\mathcal{U}$ such that $\mathcal{I}_n\uparrow\mathcal{U}$ as $n\uparrow\infty$. Recall that $I^{(m)}_r$ is the set of labels of the living particles at time $r \geq 0$ for the m-truncated branching-coalescing Brownian particle system. From Subsection 3.1, we have for each $s \geq 0$,

$$I_{[0,s]}^{(m)} := \bigcup_{r\in[0,s]} I_r^{(m)}$$

is a (random) finite subset of $\mathcal{U}$. Therefore, almost surely for every $s \geq 0$, there exists a large (random) $\tilde{N}_s \in \mathbb{N}$, such that for any $n \geq \tilde{N}_s$ it holds that $I_{[0,s]}^{(m)} \subset \mathcal{I}_n$. Using this, we can verify that, after replacing $\mathcal{I}$ by this sequence of $\mathcal{I}_n$ and then taking $n \uparrow \infty$, the second term on the right hand side of (4.12) is Cauchy sequence in $\mathscr{M}_c^2$, the space of continuous L^2-martingales; while all the other terms of (4.12) converges almost surely. This allows us to verify that almost surely for each $s \geq 0$,

$$\prod_{\alpha\in I_s^{(m)}} h(X_s^{(m),\alpha}) - \prod_{\alpha\in I_0^{(m)}} h(X_0^{(m),\alpha}) - \sum_{r\leq s}\left\{\mathbf{\Delta} \prod_{\alpha\in I_r^{(m)}} h(X_r^{(m),\alpha})\right\}$$
$$= \frac{1}{2}\int_0^s \sum_{\alpha\in I_{r-}^{(m)}} \left(\prod_{\beta\in I_{r-}^{(m)}\setminus\{\alpha\}} h(X_{r-}^{(m),\beta})\right) h''(X_{r-}^{(m),\alpha})\mathrm{d}r + M_s^{m,h}$$

where $M_\cdot^{m,h}$ is a continuous L^2-martingale with quadratic variation

$$\langle M_s^{m,h}\rangle = \int_0^s \sum_{\alpha\in I_{r-}^{(m)}} \left(\prod_{\beta\in I_{r-}^{(m)}\setminus\{\alpha\}} h(X_{r-}^{(m),\beta})\right)^2 h'(X_{r-}^{(m),\beta})^2\mathrm{d}r, \quad s \geq 0.$$

Step 3. Let us define the process

$$H_r^{m,h} := (-1)^{\tilde{J}_r^{(m)}} e^{K_r^{(m)}} \prod_{\alpha\in I_r^{(m)}} h\big(X_r^{(m),\alpha}\big), \quad r \geq 0.$$

Since there are only finitely many jumps for the process $H_\cdot^{m,h}$ up to any finite time, it is straightforward to verify, using Step 2 and Ito's formula, that almost surely for every $s \geq 0$,

$$\begin{aligned}(4.13)\quad & H_s^{m,h} - H_0^{m,h} \\ &= \int_0^s H_{r-}^{m,h} \sum_{\alpha\in I_{r-}^{(m)}} \left(\mu + b_1 - \frac{1}{m} + \frac{h''(X_{r-}^{(m),\alpha})}{2h(X_{r-}^{(m),\alpha})}\right)\mathrm{d}r + \\ &\int_0^s (-1)^{\tilde{J}_{r-}^{(m)}} e^{K_{r-}^{(m)}} \mathrm{d}M_r^{m,h} + \\ &\int_{(0,s]\times\mathcal{U}\times\bar{\mathbb{N}}} \mathbf{1}_{\{\alpha\in I_{r-}^{(m)}\}} H_{r-}^{m,h}\left((-1)^{\mathbf{1}_{\{b_k<0\}}}\frac{h(X_{r-}^{(m),\alpha})^{k\wedge m}}{h(X_{r-}^{(m),\alpha})} - 1\right)\mathfrak{N}(\mathrm{d}r,\mathrm{d}\alpha,\mathrm{d}k) +\end{aligned}$$

$$\int_{(0,s]\times\mathcal{R}} \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_{r-}\}} H^{m,h}_{r-}(h(X^{(m),\alpha}_{r-})^{-1}-1)\mathfrak{M}(\mathrm{d}r,\mathrm{d}(\alpha,\beta)).$$

Step 4. We want to take the expectation of (4.13). However, it is not clear whether the second term on the right hand side is a (true) martingale. Notice that its quadratic variation is given by

$$\left\langle \int_0^s (-1)^{\tilde{J}^{(m)}_{r-}} e^{K^{(m)}_{r-}} \mathrm{d}M^{m,h}_r \right\rangle = \int_0^s e^{2K^{(m)}_{r-}} \sum_{\alpha\in I^{(m)}_{r-}} \left(\prod_{\beta\in I^{(m)}_{r-}\setminus\{\alpha\}} h(X^{(m),\beta}_{r-}) \right)^2 h'(X^{(m),\alpha}_{r-})^2 \mathrm{d}r$$
$$\le \|h'\|^2_\infty \int_0^s e^{2K^{(m)}_{r-}} |I^{(m)}_{r-}| \mathrm{d}r, \quad s\ge 0.$$

Therefore, we can define a sequence of predictable stopping time $\tau_n \uparrow \infty$ by

$$\tau_n := \inf\left\{ t\ge 0 : \|h'\|^2_\infty \int_0^t e^{2K^{(m)}_{r-}} |I^{(m)}_{r-}| \mathrm{d}r = n \right\}, \quad n\in\mathbb{N},$$

which guarantees that, for each $n\in\mathbb{N}$,

$$s \mapsto \int_0^{s\wedge\tau_n} (-1)^{\tilde{J}^{(m)}_{r-}} e^{K^{(m)}_{r-}} \mathrm{d}M^{m,h}_r$$

is an L^2-martingale. Let us then take the expectation of (4.13) while replacing s by $s\wedge\tau_n$ and obtain, for each $n\in\mathbb{N}$ and $s\ge 0$,

(4.14)

$$\begin{aligned}
&\tilde{\mathbb{E}}\big[H^{m,h}_{s\wedge\tau_n}\big] - \tilde{\mathbb{E}}\Big[H^{m,h}_0\Big] \\
&= \tilde{\mathbb{E}}\left[\int_0^{s\wedge\tau_n} H^{m,h}_{r-} \sum_{\alpha\in I^{(m)}_{r-}} \left(\mu + b_1 - \frac{1}{m} + \frac{h''(X^{(m),\alpha}_{r-})}{2h(X^{(m),\alpha}_{r-})} \right) \mathrm{d}r \right] + \\
&\quad \tilde{\mathbb{E}}\left[\int_{(0,s\wedge\tau_n]\times\mathcal{U}\times\bar{\mathbb{N}}} \mathbf{1}_{\{\alpha\in I^{(m)}_{r-}\}} H^{m,h}_{r-} \left((-1)^{\mathbf{1}_{\{b_k<0\}}} \frac{h(X^{(m),\alpha}_{r-})^{k\wedge m}}{h(X^{(m),\alpha}_{r-})} - 1 \right) \hat{\mathfrak{N}}(\mathrm{d}r,\mathrm{d}\alpha,\mathrm{d}k) \right] + \\
&\quad \tilde{\mathbb{E}}\left[\int_{(0,s\wedge\tau_n]\times\mathcal{R}} \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_{r-}\}} H^{m,h}_{r-} (h(X^{(m),\alpha}_{r-})^{-1}-1) \hat{\mathfrak{M}}(\mathrm{d}r,\mathrm{d}(\alpha,\beta)) \right].
\end{aligned}$$

Here, we have replaced $\mathfrak{N}$, and $\mathfrak{M}$, by their compensators $\hat{\mathfrak{N}}$, and $\hat{\mathfrak{M}}$, respectively. This is allowed, due to Lemma 3.3 and the fact that for any $s\ge 0$ and $n\in\mathbb{N}$,

$$\int_{(0,s\wedge\tau_n]\times\mathcal{U}\times\bar{\mathbb{N}}} \left| \mathbf{1}_{\{\alpha\in I^{(m)}_{r-}\}} H^{m,h}_{r-} \left((-1)^{\mathbf{1}_{\{b_k<0\}}} \frac{h(X^{(m),\alpha}_{r-})^{k\wedge m}}{h(X^{(m),\alpha}_{r-})} - 1 \right) \right| \hat{\mathfrak{N}}(\mathrm{d}r,\mathrm{d}\alpha,\mathrm{d}k)$$

$$
\begin{aligned}
&\leq \int_0^s \sum_{\alpha\in I^{(m)}_{r-}} \sum_{k\in\bar{\mathbb{N}}} \left| \mathbf{1}_{\{\alpha\in I^{(m)}_{r-}\}} \frac{H^{m,h}_{r-}}{h(X^{(m),\alpha}_{r-})} \Big((-1)^{\mathbf{1}_{\{b_k<0\}}} h(X^{(m),\alpha}_{r-})^{k\wedge m} - h(X^{(m),\alpha}_{r-}) \Big) \right| \mu p_k \mathrm{d}r \\
&\leq 2\mu \int_0^s e^{K^{(m)}_{r-}} |I^{(m)}_{r-}| \mathrm{d}r \in L^1(\tilde{\mathbb{P}}), \quad \text{by Lemma 3.12}
\end{aligned}
$$

and that

$$
\begin{aligned}
(4.15)\qquad & 2\int_{(0,s\wedge\tau_n]\times\mathcal{R}} \left| \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_{r-}\}} H^{m,h}_{r-} (h(X^{(m),\alpha}_{r-})^{-1} - 1) \right| \hat{\mathfrak{M}}(\mathrm{d}r, \mathrm{d}(\alpha,\beta)) \\
&\leq \sum_{(\alpha,\beta)\in\mathcal{R}} \int_0^s \left| \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_{r-}\}} \frac{H^{m,h}_{r-}}{h(X^{(m),\alpha}_{r-})} (1 - h(X^{(m),\alpha}_{r-})) \right| \mathrm{d}L^{\alpha,\beta}_r \\
&\leq 2 \sum_{(\alpha,\beta)\in\mathcal{R}} \int_0^s \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_{r-}\}} e^{K^{(m)}_{r-}} \mathrm{d}L^{\alpha,\beta}_r \\
&\leq 2(1+e^{K^{(m)}_s}) \sum_{\alpha,\beta\in I^{(m)}_{[0,s]}:\alpha\prec\beta} L^{\alpha,\beta}_s \in L^1(\tilde{\mathbb{P}}), \quad \text{by Lemma 3.13.}
\end{aligned}
$$

Finally, observing that there are certain cancellations on the right hand side of (4.14), we obtain that for any $s \geq 0$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
(4.16)\qquad & \tilde{\mathbb{E}}\big[H^{m,h}_{s\wedge\tau_n}\big] - \tilde{\mathbb{E}}\Big[H^{m,h}_0\Big] \\
&= \tilde{\mathbb{E}}\left[\int_0^{s\wedge\tau_n} H^{m,h}_{r-} \sum_{\alpha\in I^{(m)}_{r-}} \frac{h''(X^\alpha_{r-})}{2h(X^\alpha_{r-})} \mathrm{d}r \right] + \\
&\quad \tilde{\mathbb{E}}\left[\int_0^{s\wedge\tau_n} H^{m,h}_{r-} \sum_{\alpha\in I^{(m)}_{r-}} \frac{b^{(m)}(h(X^\alpha_{r-}))}{h(X^\alpha_{r-})} \mathrm{d}r \right] + \\
&\quad \tilde{\mathbb{E}}\Big[\int_{(0,s\wedge\tau_n]\times\mathcal{R}} \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_{r-}\}} H^{m,h}_{r-} (h(X^\alpha_{r-})^{-1} - 1) \hat{\mathfrak{M}}(\mathrm{d}r, \mathrm{d}(\alpha,\beta)) \Big].
\end{aligned}
$$

Here, recall that $b^{(m)}$ is defined in (4.7).

Step 5. Let us fix and arbitrary $s \geq 0$. Observe that, for each $n \in \mathbb{N}$, the integrands in the first and the second terms on the left hand side of (4.16) are dominated by $1+e^{K^{(m)}_s}$ and 1 repectively; while the integrands in the first and the second terms on the right hand side of (4.16) are dominated by

$$
\frac{1}{2}\|h''\|_\infty \int_0^s e^{K^{(m)}_{r-}} |I^{(m)}_{r-}| \mathrm{d}r \quad \text{and} \quad \|b^{(m)}\|_\infty \int_0^s e^{K^{(m)}_{r-}} |I^{(m)}_{r-}| \mathrm{d}r
$$

respectively. Also observe from (4.15) that the integrand in the third term on the right hand side of (4.16) is dominated by

$$2(1+e^{K_s^{(m)}}) \sum_{\alpha,\beta\in I_{[0,s]}^{(m)}:\alpha\prec\beta} L_s^{\alpha,\beta}.$$

Now, by using Lemmas 3.12, 3.13 and the dominated convergence theorem, after taking $n\to\infty$ in (4.16), we can verify that (4.16) still holds after replacing $s\wedge\tau_n$ by s. That is

(4.17)

$$\begin{aligned}&\tilde{\mathbb{E}}\big[H_s^{m,h}\big]-\tilde{\mathbb{E}}\Big[H_0^{m,h}\Big]\\&=\tilde{\mathbb{E}}\left[\int_0^s H_{r-}^{m,h}\sum_{\alpha\in I_{r-}^{(m)}}\frac{h''(X_{r-}^\alpha)}{2h(X_{r-}^\alpha)}\mathrm{d}r\right]+\tilde{\mathbb{E}}\left[\int_0^s H_{r-}^{m,h}\sum_{\alpha\in I_{r-}^{(m)}}\frac{b^{(m)}(h(X_{r-}^\alpha))}{h(X_{r-}^\alpha)}\mathrm{d}r\right]+\\&\tilde{\mathbb{E}}\left[\int_{(0,s]\times\mathcal{R}}\mathbf{1}_{\{\alpha,\beta\in I_{r-}^{(m)}\}}H_{r-}^{m,h}(h(X_{r-}^\alpha)^{-1}-1)\hat{\mathfrak{M}}(\mathrm{d}r,\mathrm{d}(\alpha,\beta))\right].\end{aligned}$$

Step 6. Fix an arbitrary $t\geq 0$. After replacing the arbitrarily chosen $h\in C_{\mathrm{b}}^2(\mathbb{R},[0,1])$ by $P_\epsilon u_t$ and then taking expectation with respect to $\mathbb{P}$ on both sides of (4.17), we can verify the desired result for this proposition while applying Fubini's theorem. Of course, to use Fubini's theorem, we need to verify the integrability of (4.17) for each term. For instance, for the third term on the right hand side, we can verify, with a similar argument as in (4.15), that

$$\begin{aligned}&\mathbf{E}\left[\int_{(0,s]\times\mathcal{R}}\left|\mathbf{1}_{\{\alpha,\beta\in I_{r-}^{(m)}\}}H_{r-}^{m,P_\epsilon u_t}((P_\epsilon u_t)(X_{r-}^\alpha)^{-1}-1)\right|\hat{\mathfrak{M}}(\mathrm{d}r,\mathrm{d}(\alpha,\beta))\right]\\&\leq 2\mathbf{E}\left[(1+e^{K_s^{(m)}})\sum_{\alpha,\beta\in I_{[0,s]}^{(m)}:\alpha\prec\beta}L_s^{\alpha,\beta}\right]<\infty.\end{aligned}\tag{4.18}$$

Therefore, by Fubini's theorem and Lemma 3.4, we have

$$\begin{aligned}&\int\tilde{\mathbb{E}}\left[\int_{(0,s]\times\mathcal{R}}\mathbf{1}_{\{\alpha,\beta\in I_{r-}^{(m)}\}}H_{r-}^{m,h}(h(X_{r-}^\beta)^{-1}-1)\hat{\mathfrak{M}}(\mathrm{d}r,\mathrm{d}(\alpha,\beta))\right]\Bigg|_{h=P_\epsilon u_t}\mathrm{d}\mathbb{P}\\&=\mathbf{E}\left[\int_{(0,s]\times\mathcal{R}}\mathbf{1}_{\{\alpha,\beta\in I_{r-}^{(m)}\}}H_{r-}^{m,P_\epsilon u_t}((P_\epsilon u_t)(X_{r-}^\beta)^{-1}-1)\hat{\mathfrak{M}}(\mathrm{d}r,\mathrm{d}(\alpha,\beta))\right]\\&=\frac{1}{2}\mathbf{E}\left[\int_0^s\sum_{\alpha,\beta\in I_{r-}^{(m)}:\alpha\prec\beta}(-1)^{J_{r-}^m}e^{K_{r-}^{(m)}}\left(\prod_{\gamma\in I_{r-}^{(m)}\setminus\{\alpha,\beta\}}(P_\epsilon u_t)(X_{r-}^\gamma)\right)\times\right.\end{aligned}$$

$$\left.\Big((P_\epsilon u_t)(X^\alpha_{r-}) - (P_\epsilon u_t)(X^\alpha_{r-})\cdot(P_\epsilon u_t)(X^\beta_{r-})\Big)\mathrm{d}L^{\alpha,\beta}_r\right] = \tilde{\Psi}^{\epsilon,m}_{t,s}.$$

Here, we used the fact that $\mathbf{1}_{\{\alpha,\beta\in I^{(m)}_{r-}\}}\mathrm{d}L^{\alpha,\beta}_r = \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_{r-}, X^\alpha_{r-}=X^\beta_{r-}\}}\mathrm{d}L^{\alpha,\beta}_r$ which is standard for the Brownian local times. Note that (4.18) also implies that

$$\sup_{0\le s,t\le T}\left|\tilde{\Psi}^{\epsilon,m}_{t,s}\right| \le 2\mathbf{E}\left[(1+e^{K^{(m)}_T})\sum_{\alpha,\beta\in I^{(m)}_{[0,T]}:\alpha\prec\beta} L^{\alpha,\beta}_T\right] < \infty.$$

Similar arguments are valid for all the other terms on the right hand side of (4.17), and we are done. □

Proof of Lemma 4.3 (1). The desired result follows from the continuity of u and the dominated convergence theorem, immediately after noticing that the integrand in (4.1) is dominated by $e^{K^{(m)}_s}$, which is integrable with respect to $\mathbf{P}$ by Proposition 2.1. □

Proof of Lemma 4.3 (2). From (4.3), we know that for any $r \ge 0$,

$$\Xi^{0,\infty}_{r,0} = \mathbb{E}\left[\prod_{i=1}^n u_r(x_i)\right] \quad \text{and} \quad \Xi^{0,\infty}_{0,r} = \tilde{\mathbb{E}}\left[(-1)^{|\tilde{J}_r|}e^{K_r}\prod_{\alpha\in I_r} f(X^\alpha_r)\right].$$

From the bounded convergence theorem, and the fact that $r \mapsto u_r(x)$ is a continuous function bounded by 1 for each $x \in \mathbb{R}$, we have that $r \mapsto \Xi^{0,\infty}_{r,0}$ is continuous.

To show that $r \mapsto \Xi^{0,\infty}_{0,r}$ is continuous, we fix an arbitrary (deterministic) $r \ge 0$ and define the event $\tilde{\Omega}_r \subset \tilde{\Omega}$ such that

$$\tilde{\Omega}_r = \big\{\mathfrak{N}(\{r\}\times\mathcal{U}\times\bar{\mathbb{N}}) = 0\big\} \cap \{\mathfrak{M}(\{r\}\times\mathcal{R}) = 0\} \cap \left\{\int_0^t |I_s|\mathrm{d}s < \infty, \forall t \ge 0\right\} \cap \{J_t < \infty, \forall t \ge 0\}.$$

From the property of the Poisson random measure, we have

$$\tilde{\mathbb{E}}[\mathfrak{N}(\{r\}\times\mathcal{U}\times\mathbb{Z}_+)] = \sum_{\alpha\in\mathcal{U}}\sum_{k\in\bar{\mathbb{N}}}\hat{\mathfrak{N}}(\{r\}\times\{\alpha\}\times\{k\}) = 0,$$

and

$$\tilde{\mathbb{E}}[\mathfrak{M}(\{r\}\times\mathcal{R})] = \sum_{(\alpha,\beta)\in\mathcal{R}}\tilde{\mathbb{E}}[\mathfrak{M}(\{r\}\times\{(\alpha,\beta)\})] = \sum_{(\alpha,\beta)\in\mathcal{R}}\tilde{\mathbb{E}}[\hat{\mathfrak{M}}(\{r\}\times\{(\alpha,\beta)\})] = 0.$$

From this and Theorem 1.4, it is clear that $\tilde{\mathbb{P}}(\tilde{\Omega}_r) = 1$.

Firstly note that, on the event $\tilde{\Omega}_r$, since there are only finitely many branching events up to any finite time and there is no branching occurring at time r, there exists a (random) $\varepsilon := \varepsilon(r) > 0$ such that there is no branching event occurring in the time interval

$(r-2\varepsilon, r+2\varepsilon)$. Secondly note that, on the event $\tilde{\Omega}_r$, since $t \mapsto |I_t|$ is non-increasing on $(r-2\varepsilon, r+2\varepsilon)$ and

$$\int_{r-2\varepsilon}^{r+2\varepsilon} |I_t| \mathrm{d}t < \infty,$$

it must hold that $|I_t| < \infty$ for every $t \in (r-2\varepsilon, r+2\varepsilon)$; in particular, $|I_{r-\varepsilon}| < \infty$. Thirdly note that, on the event $\tilde{\Omega}_r$, since there are only finitely many coalescing events occurring in the time interval $(r-\varepsilon, r+\varepsilon)$ and non of them occurs at the time r, there exists a random $\tilde{\varepsilon} := \tilde{\varepsilon}(r) > 0$ such that there is no change of the total number of particles in the time interval $(r-\tilde{\varepsilon}, r+\tilde{\varepsilon})$. Let us now take an arbitrary (deterministic) sequence $(r_n)_{n\in\mathbb{N}} \subset (0, r+1)$ such that $r_n \to r$ as $n \to \infty$. Then, it can be verified that

$$H_n := \mathbf{1}_{\tilde{\Omega}_r} (-1)^{|\tilde{J}_{r_n}|} e^{K_{r_n}} \prod_{\alpha \in I_{r_n}} f(X_{r_n}^\alpha) \xrightarrow[n\to\infty]{\text{a.s.}} H := \mathbf{1}_{\tilde{\Omega}_r} (-1)^{|\tilde{J}_r|} e^{K_r} \prod_{\alpha \in I_r} f(X_r^\alpha).$$

From Proposition 2.1, we know that each element of $(H_n)_{n\in\mathbb{N}}$ is dominated by $1 + e^{K_{r+1}} \in L_1(\tilde{\mathbb{P}})$. Therefore, by the dominated convergence theorem, we have

$$\Xi_{0,r_n}^{0,\infty} = \tilde{\mathbb{E}}[H_n] \xrightarrow[n\to\infty]{} \tilde{\mathbb{E}}[H] = \Xi_{0,r}^{0,\infty}.$$

Finally, since $r \geq 0$ and $(r_n)_{n\in\mathbb{N}}$ are arbitrary, we obtain the continuity of $r \mapsto \Xi_{0,r}^{0,\infty}$. □

Proof of Lemma 4.3 (3). From Lemma 3.12 and the fact that $\left\|\frac{\Delta}{2} P_\epsilon u_t\right\|_\infty \leq \epsilon^{-1}$, we have

$$\mathbf{E}\left[\iint_{s,t\geq 0, s+t\leq T} \left| (-1)^{|\tilde{J}_s^{(m)}|} e^{K_s^{(m)}} \left(\sum_{\alpha \in I_s^{(m)}} \Big(\frac{\Delta}{2} P_\epsilon u_t\Big)(X_s^\alpha) \prod_{\beta \in I_s^{(m)} \setminus \{\alpha\}} (P_\epsilon u_t)\big(X_s^\beta\big) \right) \right| \mathrm{d}t\mathrm{d}s \right]$$

$$\leq \epsilon^{-1} \mathbf{E}\left[\iint_{s,t\geq 0, s+t\leq T} e^{K_s^{(m)}} \big|I_s^{(m)}\big| \mathrm{d}t\mathrm{d}s\right] \leq \epsilon^{-1} T \tilde{\mathbb{E}}\left[\int_0^T e^{K_s^{(m)}} \big|I_s^{(m)}\big| \mathrm{d}s\right] < \infty.$$

Now by Fubini's theorem we know that both

$$\int_0^T \Lambda_{T-s,s}^{\epsilon,m} \mathrm{d}s \quad \text{and} \quad \int_0^T \tilde{\Lambda}_{t,T-t}^{\epsilon,m} \mathrm{d}t$$

are equal to

$$\mathbf{E}\left[\iint_{s,t\geq 0, s+t\leq T} (-1)^{|\tilde{J}_s^{(m)}|} e^{K_s^{(m)}} \left(\sum_{\alpha \in I_s^{(m)}} \Big(\frac{\Delta}{2} P_\epsilon u_t\Big)(X_s^\alpha) \prod_{\beta \in I_s^{(m)} \setminus \{\alpha\}} (P_\epsilon u_t)\big(X_s^\beta\big) \right) \mathrm{d}t\mathrm{d}s \right].$$

The desired result now follows. □

Proof of Lemma 4.3 (4). Step 1. For any $m \in \mathbb{N}$ and $s, t \geq 0$, define $\tilde{\Psi}_{t,s}^{0,m}$ by replacing ϵ by 0 in (4.6). Also, define random variables

$$\mathcal{K}_{t,r}^{\epsilon,m}(\alpha,\beta) := \mathbf{1}_{\{\alpha,\beta \in I_r^{(m)}\}} (-1)^{|\tilde{J}_r^{(m)}|} e^{K_r^{(m)}} (\sigma \circ (P_\epsilon u_t))(X_r^\alpha)^2 \left(\prod_{\gamma \in I_r^{(m)} \setminus \{\alpha,\beta\}} (P_\epsilon u_t)(X_r^\gamma) \right)$$

which clearly satisfies that

$$\left|\mathcal{K}^{\epsilon,m}_{t,r}(\alpha,\beta)\right| \leq \|\sigma\|^2_\infty \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_r\}} e^{K^{(m)}_r}$$

for every $\epsilon \geq 0$, $m \in \mathbb{N}$, $t, r \geq 0$ and $(\alpha,\beta) \in \mathcal{R}$. Therefore, we can verify from (3.13) that

$$\begin{aligned}
&\int_0^T \mathbf{E}\left[\sum_{(\alpha,\beta)\in\mathcal{R}} \int_0^{T-t} \left|\mathcal{K}^{\epsilon,m}_{t,r}(\alpha,\beta)\right| \mathrm{d}L^{\alpha,\beta}_r\right] \mathrm{d}t \\
&\leq \int_0^T \mathbf{E}\left[\sum_{(\alpha,\beta)\in\mathcal{R}} \|\sigma\|^2_\infty \int_0^{T-t} \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_r\}} e^{K^{(m)}_r} \mathrm{d}L^{\alpha,\beta}_r\right] \mathrm{d}t \\
&\leq \|\sigma\|^2_\infty T \mathbf{E}\left[(1+e^{K^{(m)}_T}) \sum_{\alpha,\beta\in I^{(m)}_{[0,T]}:\alpha\prec\beta} L^{\alpha,\beta}_T\right] < \infty.
\end{aligned}$$

Therefore, by the Fubini's theorem and dominated convergence theorem, we can verify

$$\begin{aligned}
&\int_0^T \tilde{\Psi}^{\epsilon,m}_{t,T-t}\mathrm{d}t = \frac{1}{2}\int_0^T \mathbf{E}\left[\sum_{\alpha,\beta\in\mathcal{R}} \int_0^{T-t} \mathcal{K}^{\epsilon,m}_{t,r}(\alpha,\beta)\mathrm{d}L^{\alpha,\beta}_r\right]\mathrm{d}t \\
&\xrightarrow[\epsilon\to 0]{} \frac{1}{2}\int_0^T \mathbf{E}\left[\sum_{\alpha,\beta\in\mathcal{R}} \int_0^{T-r} \mathcal{K}^{0,m}_{t,r}(\alpha,\beta)\mathrm{d}L^{\alpha,\beta}_r\right]\mathrm{d}t = \int_0^T \tilde{\Psi}^{0,m}_{t,T-t}\mathrm{d}t.
\end{aligned}$$

Step 2. It can be verified from Fubini's theorem that for any $s, t \geq 0$, $\epsilon > 0$, and $m \in \mathbb{N}$,

$$\Psi^{\epsilon,m}_{t,s} = \mathbf{E}\left[\sum_{(\alpha,\beta)\in\mathcal{R}} \int p_\epsilon(y - X^\alpha_s) p_\epsilon(y - X^\beta_s) Y^{\epsilon,m}_{t,s}(y;\alpha,\beta)\mathrm{d}y\right]$$

where

$$Y^{\epsilon,m}_{t,s}(y;\alpha,\beta) := (-1)^{|\tilde{J}^{(m)}_s|} e^{K^{(m)}_s} \mathbf{1}_{\{\alpha,\beta\in I^{(m)}_s\}} \int_0^t \sigma(u_r(y))^2 \left(\prod_{\gamma\in I^{(m)}_s\setminus\{\alpha,\beta\}} (P_\epsilon u_r)(X^\gamma_s)\right)\mathrm{d}r.$$

By Fubini's theorem again, and by substituting y with $y + X^\alpha_s$, we have

$$\int_0^T \Psi^{\epsilon,m}_{T-s,s}\mathrm{d}s = \mathbf{E}\left[\sum_{(\alpha,\beta)\in\mathcal{R}} \int \mathrm{d}y \int_0^T p_\epsilon(y) p_\epsilon(y + X^\alpha_s - X^\beta_s) Y^{\epsilon,m}_{T-s,s}(y + X^\alpha_s;\alpha,\beta)\mathrm{d}s\right].$$

Step 3. Recall from (2.4) that $L^{\alpha,\beta}_{\cdot,z}$ is the local time of the process $\tilde{X}^\alpha_\cdot - \tilde{X}^\beta_\cdot$ at the level $z \in \mathbb{R}$. By the theorem of the occupation density, c.f. [24, Theorem 29.5] and [4, Lemma

2], we can verify that for each $(\alpha,\beta)\in\mathcal{R}$, $\epsilon>0$ and $y\in\mathbb{R}$, almost surely

$$\begin{aligned}(4.19)\qquad &2\int_0^T p_\epsilon(y)p_\epsilon(y+X_s^\alpha-X_s^\beta)Y^{\epsilon,m}_{T-s,s}(y+X_s^\alpha;\alpha,\beta)\mathrm{d}s\\ &=\int_0^T p_\epsilon(y)p_\epsilon(y+\tilde X_s^\alpha-\tilde X_s^\beta)Y^{\epsilon,m}_{T-s,s}(y+X_s^\alpha;\alpha,\beta)\mathrm{d}\Big\langle \tilde X_s^\alpha-\tilde X_s^\beta\Big\rangle\\ &=\int \mathrm{d}z\int_0^T p_\epsilon(y)p_\epsilon(y+z)Y^{\epsilon,m}_{T-s,s}(y+X_s^\alpha;\alpha,\beta)\mathrm{d}L^{\alpha,\beta}_{s,z}.\end{aligned}$$

Using the dominated convergence theorem, we can verify that the expression in (4.19) is almost surely continuous in $y\in\mathbb{R}$. Therefore, (4.19) actually holds for every $y\in\mathbb{R}$ and $(\alpha,\beta)\in\mathcal{R}$, almost surely, for every $\epsilon>0$. Therefore, we have

$$2\int_0^T \Psi^{\epsilon,m}_{T-s,s}\mathrm{d}s=\mathbf{E}\left[\sum_{(\alpha,\beta)\in\mathcal{R}}\int\mathrm{d}y\int\mathrm{d}z\int_0^T p_\epsilon(y)p_\epsilon(y+z)Y^{\epsilon,m}_{T-s,s}(y+X_s^\alpha;\alpha,\beta)\mathrm{d}L^{\alpha,\beta}_{s,z}\right].$$

Step 4. By an argument similar to [4, p. 1725], we can verify that almost surely

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\sum_{(\alpha,\beta)\in\mathcal{R}}\int\mathrm{d}y\int\mathrm{d}z\int_0^T p_\epsilon(y)p_\epsilon(y+z)Y^{\epsilon,m}_{T-s,s}(y+X_s^\alpha;\alpha,\beta)\mathrm{d}L^{\alpha,\beta}_{s,z}\\ &=\sum_{(\alpha,\beta)\in\mathcal{R}}\int_0^T Y^{0,m}_{T-s,s}(X_s^\alpha;\alpha,\beta)\mathrm{d}L^{\alpha,\beta}_s.\end{aligned}$$

Notice also that almost surely

$$\int_0^T\left|Y^{\epsilon,m}_{T-s,s}(y+X_s^\alpha;\alpha,\beta)\right|\mathrm{d}L^{\alpha,\beta}_{s,z}\le(1+e^{K_T^{(m)}})T\|\sigma\|_\infty^2\mathbf{1}_{\{\alpha,\beta\in I^{(m)}_{[0,T]}\}}\sup_{z_0\in\mathbb{R}}L^{\alpha,\beta}_{T,z_0},$$

and therefore

$$\begin{aligned}&\sum_{(\alpha,\beta)\in\mathcal{R}}\left|\int\mathrm{d}y\int\mathrm{d}z\int_0^T p_\epsilon(y)p_\epsilon(y+z)Y^{\epsilon,m}_{T-s,s}(y+X_s^\alpha;\alpha,\beta)\mathrm{d}L^{\alpha,\beta}_{s,z}\right|\\ &\le(1+e^{K_T^{(m)}})T\|\sigma\|_\infty^2\sum_{\alpha,\beta\in I^{(m)}_{[0,T]}:\alpha\prec\beta}\sup_{z_0\in\mathbb{R}}L^{\alpha,\beta}_{T,z_0}\end{aligned}$$

which is integrable, thanks to Lemma 3.13. Now, by Steps 2 and 3, the dominated convergence theorem, and Fubini's theorem, we can verify that

$$2\int_0^T\Psi^{\epsilon,m}_{T-s,s}\mathrm{d}s\xrightarrow[\epsilon\downarrow 0]{}\mathbf{E}\left[\sum_{(\alpha,\beta)\in\mathcal{R}}\int_0^T Y^{0,m}_{T-s,s}(X_s^\alpha;\alpha,\beta)\mathrm{d}L^{\alpha,\beta}_s\right]=2\int_0^T\tilde\Psi^{0,m}_{t,T-t}\mathrm{d}t.$$

Combining this with Step 1, we are done. □

Proof of Lemma 4.3 (5). Step 1. For every $t, s \geq 0$ and $m \in \mathbb{N}$, define $\Phi^{0,m}_{t,s}$ and $\tilde{\Phi}^{0,m}_{t,s}$ by taking $\epsilon = 0$ in (4.4) and (4.5) respectively. For every $t, s \geq 0$, define $\Phi^{0,\infty}_{t,s}$ and $\tilde{\Phi}^{0,\infty}_{t,s}$ by

$$\Phi^{0,\infty}_{t,s} := \mathbf{E}\left[(-1)^{|\tilde{J}_s|}e^{K_s}\int_0^t \left(\sum_{\alpha\in I_s}(b\circ u_r)(X^\alpha_s)\prod_{\beta\in I_s\setminus\{\alpha\}}u_r\big(X^\beta_s\big)\right)\mathrm{d}r\right]$$

and

$$\tilde{\Phi}^{0,\infty}_{t,s} := \mathbf{E}\left[\int_0^s (-1)^{|\tilde{J}_r|}e^{K_r}\left(\sum_{\alpha\in I_r}(b\circ u_t)(X^\alpha_r)\prod_{\beta\in I_r\setminus\{\alpha\}}u_t\big(X^\beta_r\big)\right)\mathrm{d}r\right].$$

Step 2. We will show that, for a fixed arbitrary $T > 0$ and $m \in \mathbb{N}$ with $m \geq 2$,

$$\lim_{\epsilon\downarrow 0}\int_0^T \tilde{\Phi}^{\epsilon,m}_{t,T-t}\mathrm{d}t = \int_0^T \tilde{\Phi}^{0,m}_{t,T-t}\mathrm{d}t.$$

Firstly, we note that almost surely with respect to $\mathbf{P}$, for any $t, s \geq 0$ and $\epsilon \geq 0$, the random variable

$$\left|\mathbf{1}_{\{s+t\leq T\}}(-1)^{|\tilde{J}^{(m)}_s|}e^{K^{(m)}_s}\mathbf{1}_{\{\alpha\in I^{(m)}_s\}}\big(b^{(m)}\circ(P_\epsilon u_t)\big)(X^\alpha_s)\prod_{\beta\in I^{(m)}_s\setminus\{\alpha\}}(P_\epsilon u_t)\big(X^\beta_s\big)\right|$$

is bounded by $C_4\mathbf{1}_{\{\alpha\in I_s\}}(1+e^{K_s})$ where $C_4 := \sum_{k\in\bar{\mathbb{N}}}|b_k| + 1 < \infty$. Secondly notice that, by Lemma 3.12,

$$\int_0^T \mathbf{E}\left[\sum_{\alpha\in\mathcal{U}}\int_0^T \mathbf{1}_{\{\alpha\in I_s\}}\big(1+e^{K_s}\big)\mathrm{d}t\right]\mathrm{d}s = T\int_0^T \mathbf{E}\big[|I_s|\big(1+e^{K_s}\big)\big]\mathrm{d}s < \infty.$$

Thirdly notice that, almost surely with respect to $\mathbf{P}$, for any $t, s \geq 0$ and $\alpha \in \mathcal{U}$, the random variables

$$\mathbf{1}_{\{\alpha\in I^{(m)}_s\}}(P_\epsilon u_t)(X^\alpha_s) \text{ and } \mathbf{1}_{\{\alpha\in I^{(m)}_s\}}\big(b^{(m)}\circ(P_\epsilon u_t)\big)(X^\alpha_s)$$

converge, as $\epsilon \downarrow 0$, to

$$\mathbf{1}_{\{\alpha\in I^{(m)}_s\}}u_t(X^\alpha_s) \text{ and } \mathbf{1}_{\{\alpha\in I^{(m)}_s\}}\big(b^{(m)}\circ u_t\big)(X^\alpha_s),$$

respectively. Here, we used the fact that $z \mapsto b^{(m)}(z)$ is a bounded continuous map, and almost surely, $x \mapsto u_t(x)$ is a bounded continuous map. From those, we can verify using Fubini's theorem and dominated convergence theorem that the desired result for this step holds.

Step 3. We will show that, for fixed arbitrary $T > 0$ and $m \in \mathbb{N}$ with $m \geq 2$,

$$\lim_{\epsilon\downarrow 0}\int_0^T \Phi^{\epsilon,m}_{T-s,s}\mathrm{d}s = \int_0^T \Phi^{0,m}_{T-s,s}\mathrm{d}s.$$

Recall that

$$b(z) = \sum_{k=0}^{\infty} b_k z^k + b_\infty z^\infty, \quad z \in [0,1],$$

where $z^\infty := \mathbf{1}_{\{z=1\}}$ for $z \in [0,1]$. If $b_\infty = 0$ then the map $z \mapsto b(z)$ is continuous on $[0,1]$, and the desired result for this step follows from an argument similar to Step 2. However, if $b_\infty > 0$, then $b(z)$ is not continuous at $z = 1$. So we will use a different argument here which depends on a technical result: Proposition A.1 in Appendix A.3.

Notice that for any $\epsilon \geq 0$,

$$\int_0^T \Phi^{\epsilon,m}_{T-s,s} \mathrm{d}s = \int_0^T \mathbf{E}\Big[\int_0^{T-s} \mathfrak{B}^{\epsilon,m}_{r,s} \mathrm{d}r\Big] \mathrm{d}s, \tag{4.20}$$

where

$$\mathfrak{B}^{\epsilon,m}_{r,s} := (-1)^{|\tilde{J}^{(m)}_s|} e^{K^{(m)}_s} \left(\sum_{\alpha \in I^{(m)}_s} (P_\epsilon(b \circ u_r))(X^\alpha_s) \prod_{\beta \in I^{(m)}_s \setminus \{\alpha\}} (P_\epsilon u_r)\big(X^\beta_s\big) \right).$$

Also note that, for any $\epsilon \geq 0$ and $r, s \in [0,T]$,

$$|\mathfrak{B}^{\epsilon,m}_{r,s}| \leq e^{K^{(m)}_s} \|b\|_\infty |I^{(m)}_s| \tag{4.21}$$

and that by Lemma 3.12,

$$\int_0^T \mathbf{E}\Big[\int_0^{T-s} e^{K^{(m)}_s} \|b\|_\infty |I^{(m)}_s| \mathrm{d}r\Big] \mathrm{d}s = (T-s)\mathbf{E}\Big[\int_0^T e^{K^{(m)}_s} \|b\|_\infty |I^{(m)}_s| \mathrm{d}s\Big] < \infty.$$

Therefore, by Fubini's theorem,

(4.22) the orders of the integration and the expectations on the right hand side of (4.20) are interchangeable.

On the other hand, we can verify from Lemma 3.1 that

(4.23) $\tilde{\tau}_n \uparrow \infty$ as $n \uparrow \infty$ where $0 = \tilde{\tau}_0 < \tilde{\tau}_1 < \tilde{\tau}_2 < \cdots$ are the occurring times of the branching/coalescing events for the m-truncated branching-coalescing particle system $\{(X^{(m),\alpha}_t)_{t\geq 0} : \alpha \in \mathcal{U}\}$.

Therefore, from (4.20), (4.22) and (4.23), for any $\epsilon \geq 0$,

$$\int_0^T \Phi^{\epsilon,m}_{T-s,s} \mathrm{d}s = \int_0^T \mathbf{E}\Big[\int_0^{T-r} \mathfrak{B}^{\epsilon,m}_{r,s} \mathrm{d}s\Big] \mathrm{d}r = \int_0^T \mathbf{E}\left[\sum_{k=1}^\infty \int_0^{T-r} \mathbf{1}_{\{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)\}} \mathfrak{B}^{\epsilon,m}_{r,s} \mathrm{d}s\right] \mathrm{d}r.$$

Using Fubini's theorem again, we have for every $\epsilon \geq 0$,

$$\begin{aligned}\int_0^T \Phi^{\epsilon,m}_{T-s,s} \mathrm{d}s &= \int_0^T \mathbb{E}_f\left[\sum_{k=1}^\infty \int_0^{T-r} \tilde{\mathbb{E}}\big[\mathbf{1}_{\{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)\}} \mathfrak{B}^{\epsilon,m}_{r,s}\big] \mathrm{d}s\right] \mathrm{d}r \\ &= \int_0^T \mathbb{E}_f\left[\sum_{k=1}^\infty \int_0^{T-r} \tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\Big[\mathbf{1}_{\{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)\}} \mathfrak{B}^{\epsilon,m}_{r,s} \Big| \mathscr{F}_{\tilde{\tau}_{k-1}}\Big]\Big] \mathrm{d}s\right] \mathrm{d}r. \end{aligned} \tag{4.24}$$

Recall here that $\mathbb{E}_f$ is the expectation corresponding to the random field u, and $\tilde{\mathbb{E}}$ is the expectation corresponding to the m-truncated branching-coalescing Brownian particles system.

Fixing arbitrary $r, s \in [0,T]$ and $k \in \mathbb{N}$, we can write for every $\epsilon \geq 0$,

$$\begin{aligned}
&\tilde{\mathbb{E}}\Big[\mathbf{1}_{\{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)\}}\mathfrak{B}^{\epsilon,m}_{r,s}\Big|\tilde{\mathscr{F}}_{\tilde{\tau}_{k-1}}\Big] = \mathbf{1}_{\{s\geq\tilde{\tau}_{k-1}\}}\tilde{\mathbb{E}}\Big[\mathbf{1}_{\{s<\tilde{\tau}_k\}}\mathfrak{B}^{\epsilon,m}_{r,s}\Big|\tilde{\mathscr{F}}_{\tilde{\tau}_{k-1}}\Big] \\
&= \mathbf{1}_{\{s\geq\tilde{\tau}_{k-1}\}}(-1)^{\left|\tilde{J}^{(m)}_{\tilde{\tau}_{k-1}}\right|}e^{K^{(m)}_{\tilde{\tau}_{k-1}}}\exp\Big\{(\mu+b_1+\frac{1}{m})(s-\tilde{\tau}_{k-1})\Big|I^{(m)}_{\tilde{\tau}_{k-1}}\Big|\Big\}\times \\
&\quad\tilde{\mathbb{E}}\left[\mathbf{1}_{\{s<\tilde{\tau}_k\}}\left(\sum_{\alpha\in I^{(m)}_s}(P_\epsilon(b\circ u_r))(X^\alpha_s)\prod_{\beta\in I^{(m)}_s\setminus\{\alpha\}}(P_\epsilon u_r)\big(X^\beta_s\big)\right)\middle|\tilde{\mathscr{F}}_{\tilde{\tau}_{k-1}}\right].
\end{aligned}$$

Notice that, from the strong Markov property of Brownian motions, after the time $\tilde{\tau}_{k-1}$, the particles in the m-truncated branching-coalescing Brownian particle system will evolve as independent Brownian motions until the next occurring time of its branching/coalescing event. Therefore, we can further write for every $\epsilon \geq 0$ that

$$\begin{aligned}
&\tilde{\mathbb{E}}\Big[\mathbf{1}_{\{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)\}}\mathfrak{B}^{\epsilon,m}_{r,s}\Big|\tilde{\mathscr{F}}_{\tilde{\tau}_{k-1}}\Big] \\
&= \mathbf{1}_{\{s\geq\tilde{\tau}_{k-1}\}}(-1)^{\left|\tilde{J}^{(m)}_{\tilde{\tau}_{k-1}}\right|}e^{K^{(m)}_{\tilde{\tau}_{k-1}}}e^{(\mu+b_1+\frac{1}{m})(s-\tilde{\tau}_{k-1})\left|I^{(m)}_{\tilde{\tau}_{k-1}}\right|}(\mathfrak{P}^\epsilon_{s-\tilde{\tau}_{k-1}}F)(\tilde{X}).
\end{aligned}\tag{4.25}$$

Here,

$$\tilde{X} := (X^{\alpha_1}_{\tilde{\tau}_{k-1}}, \cdots, X^{\alpha_N}_{\tilde{\tau}_{k-1}}) \in \mathbb{R}^N$$

is a (random) finite list of real numbers with $N \in \mathbb{N}$ and $(\alpha_k)_{k=1}^N$ given so that

$$\alpha_1 \prec \cdots \prec \alpha_N \quad \text{and} \quad \{\alpha_1, \cdots, \alpha_N\} = I^{(m)}_{\tilde{\tau}_{k-1}};$$

F is a (random) bounded measurable function on $\mathbb{R}^N$ given so that

$$F(x_1,\cdots,x_N) = \sum_{i=1}^N (b\circ u_r)(x_i)\prod_{j\in\{1,\cdots,N\}\setminus\{i\}}u_r(x_j), \quad (x_1,\ldots,x_n)\in\mathbb{R}^N;$$

and $(\mathfrak{P}^\epsilon_t)_{t\geq0,\epsilon\geq0}$ are the operators given as in (A.6) with n replaced by the random N.

Now, from (4.25), Proposition A.1 and the fact that $\tilde{\mathbb{P}}(\tilde{\tau}_{k-1} = s) = 0$, we have almost surely

$$\tilde{\mathbb{E}}\Big[\mathbf{1}_{\{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)\}}\mathfrak{B}^{\epsilon,m}_{r,s}\Big|\tilde{\mathscr{F}}_{\tilde{\tau}_{k-1}}\Big] \xrightarrow[\epsilon\to0]{} \tilde{\mathbb{E}}\Big[\mathbf{1}_{\{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)\}}\mathfrak{B}^{0,m}_{r,s}\Big|\tilde{\mathscr{F}}_{\tilde{\tau}_{k-1}}\Big].$$

Also observe from (4.21) that

$$\Big|\tilde{\mathbb{E}}\Big[\mathbf{1}_{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)}\mathfrak{B}^{\epsilon,m}_{r,s}\Big|\tilde{\mathscr{F}}_{\tilde{\tau}_{k-1}}\Big]\Big| \leq \tilde{\mathbb{E}}\Big[\mathbf{1}_{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)}e^{K^{(m)}_s}\|b\|_\infty|I^{(m)}_s|\Big|\tilde{\mathscr{F}}_{\tilde{\tau}_{k-1}}\Big]$$

and that

$$\begin{aligned}
&\int_0^T \mathbb{E}_f\left[\sum_{k=1}^\infty\int_0^{T-r}\tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\Big[\mathbf{1}_{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)}e^{K^{(m)}_s}\|b\|_\infty|I^{(m)}_s|\Big|\tilde{\mathscr{F}}_{\tilde{\tau}_{k-1}}\Big]\Big]\mathrm{d}s\right]\mathrm{d}r \\
&= \int_0^T \mathbf{E}\left[\sum_{k=1}^\infty\int_0^{T-r}\mathbf{1}_{s\in[\tilde{\tau}_{k-1},\tilde{\tau}_k)}e^{K^{(m)}_s}\|b\|_\infty|I^{(m)}_s|\mathrm{d}s\right]\mathrm{d}r
\end{aligned}$$

$$= \int_0^T \mathbf{E}\Big[\int_0^{T-r} e^{K_s^{(m)}} \|b\|_\infty |I_s^{(m)}| \mathrm{d}s\Big] \mathrm{d}r < \infty.$$

So by applying the dominated convergence theorem on the right hand side of (4.24), we get the desired result for this step.

Step 4. Notice from Lemma 3.2 that almost surely with respect to $\mathbf{P}$, for any $t, s \geq 0$ and $\alpha \in \mathcal{U}$, the random variables

$$(-1)^{|\tilde{J}_s^{(m)}|}, \quad K_s^{(m)}, \quad \mathbf{1}_{\{\alpha \in I_s^{(m)}\}}, \quad \prod_{\beta \in I_s^{(m)} \setminus \{\alpha\}} u_t\big(X_s^\beta\big), \quad \text{and} \quad \mathbf{1}_{\{\alpha \in I_s^{(m)}\}} b^{(m)} \circ u_t(X_s^\alpha),$$

converge, as $m \uparrow \infty$, to

$$(-1)^{|\tilde{J}_s|}, \quad K_s, \quad \mathbf{1}_{\{\alpha \in I_s\}}, \quad \prod_{\beta \in I_s \setminus \{\alpha\}} u_t\big(X_s^\beta\big), \quad \text{and} \quad \mathbf{1}_{\{\alpha \in I_s\}} b \circ u_t(X_s^\alpha),$$

respectively. From this, we can verify, using Fubini's theorem and dominated convergence theorem, that

$$\lim_{m\uparrow\infty} \int_0^T \Phi_{T-s,s}^{0,m} \mathrm{d}s = \int_0^T \Phi_{T-s,s}^{0,\infty} \mathrm{d}s, \tag{4.26}$$

and that

$$\lim_{m\uparrow\infty} \int_0^T \tilde{\Phi}_{t,T-t}^{0,m} \mathrm{d}t = \int_0^T \tilde{\Phi}_{t,T-t}^{0,\infty} \mathrm{d}t. \tag{4.27}$$

Using Fubini's theorem again, we have

$$\int_0^T \Phi_{T-s,s}^{0,\infty} \mathrm{d}s = \int_0^T \tilde{\Phi}_{t,T-t}^{0,\infty} \mathrm{d}t. \tag{4.28}$$

Finally, from the results in Steps 1 and 2, (4.26), (4.27) and (4.28), we have

$$\lim_{m\uparrow\infty} \lim_{\epsilon\downarrow 0} \int_0^T \Phi_{T-s,s}^{\epsilon,m} \mathrm{d}s = \lim_{m\uparrow\infty} \lim_{\epsilon\downarrow 0} \int_0^T \tilde{\Phi}_{t,T-t}^{\epsilon,m} \mathrm{d}t,$$

as desired. □

5. Proof of the weak existence part of Theorem 1.1

As have been noted in Subsection 1.2, the weak existence of SPDE (1.1) is standard for $b_\infty = 0$. So, we have to verify existence only for the case of $b_\infty \neq 0$. For simplicity, let us also assume that $b_k = 0$ for every $k \in \mathbb{N} \setminus \{1\}$, as the argument for the more general cases is similar. Then, with these parameters, the SPDE (1.1) is given by

$$\begin{cases} \partial_t u_t = \frac{1}{2}\partial_x^2 u_t + b_1 u_t + b_\infty \mathbf{1}_{\{1\}}(u_t) + \sqrt{u_t(1-u_t)}\dot{W}, \\ u_0 = f. \end{cases} \tag{5.1}$$

Due to the condition (1.3), we have $b_1 \leq -|b_\infty|$.

The idea of the existence proof is to construct an approximating sequence of $\mathcal{C}(\mathbb{R}, [0,1])$-valued processes $\{(u_t^{(m)}(x))_{t\geq 0, x\in\mathbb{R}} : m \geq 1\}$ to show that this sequence is tight, and it has a limit point that solves (5.1).

This sequence will be constructed to solve the following SPDEs,

$$\begin{cases} \partial_t u_t^{(m)} = \frac{1}{2}\partial_x^2 u_t^{(m)} + b_1 u_t^{(k)} + b_\infty \left(u_t^{(m)}\right)^m + \sqrt{u_t^{(m)}(1-u_t^{(m)})}\dot{W}^{(m)}, \\ u_0^{(m)} = f, \end{cases} \tag{5.2}$$

where $(\dot{W}^{(m)})_{n\in\mathbb{N}}$ is a sequence of space-times white noises.

The main difficulty in the proof is to show that a sub-sequential weak limit point of $\{u^{(m)} : m \geq 1\}$ indeed solves (5.1). It is non-trivial since, it is not clear why convergence of any subsequence $u^{(m_k)}$ to u, would imply convergence of $\left(u_t^{(m_k)}(x)\right)^{m_k}$ to $\mathbf{1}_{\{1\}}(u_t(x))$. The challenge comes from the discontinuity of the function $\mathbf{1}_{\{1\}}(\cdot)$.

This difficulty will be resolved via duality argument,which is based on the convergence of the dual particle system of $u^{(m)}$ to the dual particle system of u. We will give the details below, but first let us introduce the settings for the rest of this section and state some useful lemmas. Let $x \in \mathbb{R}$ be arbitrary. Let $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ be a coalescing-branching Brownian particle system with branching rate $\mu := |b_\infty|$, offspring distribution $p_\infty = 1$, and initial configuration $(x_i)_{i=1}^\infty$ such that $x_i = x$ for every $i \in \mathbb{N}$. This particle system is defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. If we want to emphasize that all the particles start at x, we write $\tilde{\mathbb{P}}^x$ for the probability measure, and $\tilde{\mathbb{E}}^x$ for the corresponding expectation. For each $l, m \in \mathbb{N} \cup \{\infty\}$, recall that the (l, m)-truncated version of this particle system $\{(X_t^{(l,m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$, given as in Subsection 3.1, is a coalescing-branching Brownian particle system with branching rate μ, offspring distribution $p_m = 1$, and initial configuration $(x_i)_{i=1}^l$. Also recall the sets of labels $I_t^{(l,m)}$ and $J_t^{(l,m)}$ for $t \geq 0$ are given as in (3.7) and (3.8) respectively. The following lemma is a variant of Lemma 3.2. It allows us to approximate $|I_t^{(l,\infty)}|$ and $|J_t^{(l,\infty)}|$ from below.

Lemma 5.1. *Almost surely, for each $l, m \in \mathbb{N}$ with $l \leq m$ and $t \geq 0$, we have*

$$I_t^{(l,m)} = \{\alpha \in \mathcal{U} : \|\alpha\|_\infty \leq m, \alpha \in I_t^{(l,\infty)}\}$$

and

$$J_t^{(l,m)} = \{\alpha \in \mathcal{U} : \|\alpha\|_\infty \leq m, \alpha \in J_t^{(l,\infty)}\}.$$

We omit the proof of the above lemma, because it is very similar to the proof of Lemma 3.2 in Appendix A.1. Recall that the sets of indices $(I_t)_{t\geq 0}, (J_t)_{t\geq 0}$ for the non-truncated system were defined in (2.10), (2.11). Let us consider the event $\tilde{\Omega}' := \cup_{n=1}^\infty \{|J_{1/n}| = 0\}$. From Theorem 1.4, we have for every $\epsilon > 0$,

$$\tilde{\mathbb{P}}(|J_\epsilon| = 0) = 1 - \tilde{\mathbb{P}}(|J_\epsilon| \geq 1) \geq 1 - \tilde{\mathbb{E}}[|J_\epsilon|] = 1 - \mu \int_0^\epsilon \tilde{\mathbb{E}}[|I_s|] \mathrm{d}s$$

which implies that

$$\tilde{\mathbb{P}}(\tilde{\Omega}') = \lim_{n\to\infty} \tilde{\mathbb{P}}(|J_{1/n}| = 0) = 1. \tag{5.3}$$

Define the random integer $L_\epsilon = \sup\{\alpha_1 : \alpha \in I_\epsilon\} \vee 0$ for each $\epsilon > 0$. Since for $\epsilon > 0$, almost surely I_ϵ is a finite set (Theorem 1.4), we have almost surely $L_\epsilon < \infty$. The following lemma allows us to approximate $|I_t|$ and $|J_t|$ by $|I_t^{(l,\infty)}|$ and $|J_t^{(l,\infty)}|$ when $l \uparrow \infty$.

Lemma 5.2. *For any $\epsilon > 0$, almost surely on the event $\{|J_\epsilon| = 0\}$, for every finite integer $l \geq L_\epsilon$ and $t \geq 0$, we have*

$$I_t^{(l,\infty)} = \{\alpha \in \mathcal{U} : \alpha_1 \leq l, \alpha \in I_t\}$$

and

$$J_t^{(l,\infty)} = \{\alpha \in \mathcal{U} : \alpha_1 \leq l, \alpha \in J_t\}.$$

We omit the proof of the above lemma, because it is also similar to the proof of Lemma 3.2 in Appendix A.1.

Lemma 5.3. *Suppose that $(Z_k)_{k\in\mathbb{N}}$ is a sequence of $[0,1]$-valued random variables converging almost surely to a $[0,1]$-valued random variable Z. Assume that $\mathbb{E}[Z_k^k]$ converges to $\mathbb{E}[\mathbf{1}_{\{1\}}(Z)]$ when $k \uparrow \infty$. Then Z_k^k converges to $\mathbf{1}_{\{1\}}(Z)$ in L^p for every $p \geq 1$ when $k \uparrow \infty$.*

Proof. Let us first prove the L^1 convergence. Note that for every $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}\big[\big|Z_k^k - \mathbf{1}_{\{1\}}(Z)\big|\big] &= \mathbb{E}\big[\big|Z_k^k - \mathbf{1}_{\{1\}}(Z)\big|\mathbf{1}_{\{Z=1\}}\big] + \mathbb{E}\big[\big|Z_k^k - \mathbf{1}_{\{1\}}(Z)\big|\mathbf{1}_{\{Z<1\}}\big] \\ &= \mathbb{E}\big[\big(1 - Z_k^k\big)\mathbf{1}_{\{Z=1\}}\big] + \mathbb{E}\big[Z_k^k\mathbf{1}_{\{Z<1\}}\big] \\ &= \mathbb{E}\big[\mathbf{1}_{\{1\}}(Z)\big] - \mathbb{E}\big[Z_k^k\big] + 2\mathbb{E}\big[Z_k^k\mathbf{1}_{\{Z<1\}}\big]. \end{aligned} \tag{5.4}$$

It is clear that Z_k^k converges to 0 on the event $\{Z < 1\}$, and therefore the third term on the right hand side of (5.4) converges to 0, when $k \uparrow \infty$. Now (5.4) implies that Z_k^k converges to $\mathbf{1}_{\{1\}}(Z)$ in L^1 when $k \uparrow \infty$.

It is then clear that Z_k^k converges in probability to $\mathbf{1}_{\{1\}}(Z)$. Also, it can be verified form bounded convergence theorem that for any $p \geq 1$,

$$\lim_{k\to\infty} \mathbb{E}\big[\big(Z_k^k\big)^p\big] = \mathbb{E}\big[\big(\mathbf{1}_{\{1\}}(Z)\big)^p\big].$$

Now from [24, Theorem 5.12] we also obtain the L^p convergence for every $p \geq 1$ as desired. □

Now we are ready to present the main steps of the existence proof. The construction of the approximating sequence $u^{(m)}$ and one of its weak limit points will be carried out in the Steps 1, 2. The non-trivial part, as we have mentioned above, is to show that the limit point indeed solves the SPDE (5.1); this will be done in the Steps 3–7.

Proof of the weak existence part of Theorem 1.1. Step 1. Let $\mathcal{C}(\mathbb{R})$ be the space of real-valued continuous functions on $\mathbb{R}$. Define $\|f\|_{(p)} := \sup_{x\in\mathbb{R}} |e^{p|x|}f(x)|$ for every $p \in \mathbb{R}$ and $f \in \mathcal{C}(\mathbb{R})$. Define the complete separable metric space $\mathcal{C}_{\text{tem}}(\mathbb{R}) := \{f \in \mathcal{C}(\mathbb{R}) : \forall p > 0, \|f\|_{(-p)} < \infty\}$ equipped with the metric

$$d_{\text{tem}}(f,g) := \sum_{k=1}^{\infty} 2^{-k}(\|f-g\|_{(-k^{-1})} \wedge 1), \quad f, g \in \mathcal{C}_{\text{tem}}(\mathbb{R}).$$

Denote by $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{\text{tem}}(\mathbb{R}))$ the space of $\mathcal{C}_{\text{tem}}(\mathbb{R})$-valued continuous paths on $\mathbb{R}_+$, equipped with the topology of uniform convergence on compact sets. For each $m \in \mathbb{N}$, we have the existence of a $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{\text{tem}}(\mathbb{R}))$-valued random element $u^{(m)} = (u_t^{(m)})_{t\geq 0} = (u_t^{(m)}(x))_{t\geq 0, x\in\mathbb{R}}$

satisfying the SPDE (5.2) on some probability space. As we have mentioned, existence of solution to (5.2) is standard (see e.g. [39, Theorem 2.6] and [32, Section 2.1]). Moreover, $u_t^{(m)}(x)$ takes values in $[0,1]$ for each $m \in \mathbb{N}$, $t \geq 0$ and $x \in \mathbb{R}$. Note in particular, the random elements $u^{(m)}$, for different $m \in \mathbb{N}$, are not necessarily driven by the same noise, nor necessarily defined in the same probability space.

Step 2. One can also verify, by the standard theory (c.g. [39]), that the sequence of $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{\text{tem}}(\mathbb{R}))$-valued random elements $\{u^{(m)} : m \in \mathbb{N}\}$ is tight. In particular, by using Prokhorov's theorem and Skorokhod's representation theorem, there exists a (deterministic) strictly increasing $\mathbb{N}$-valued sequence $(m_k)_{k\in\mathbb{N}}$, and a sequence of $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{\text{tem}}(\mathbb{R}))$-valued random elements $(\tilde{u}^{(m_k)})_{k\in\mathbb{N}}$ defined in a common probability space, such that

(5.5) for each $k \in \mathbb{N}$, the law of $\tilde{u}^{(m_k)}$ equals to the law of $u^{(m_k)}$;

(5.6) the limit $\tilde{u} = \lim_{k\to\infty} \tilde{u}^{(m_k)}$ exists almost surely with respect to the topology of $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{\text{tem}}(\mathbb{R}))$.

It is also clear that $(\tilde{u}_t(x))_{t\geq 0, x\in\mathbb{R}}$ is a $[0,1]$-valued continuous random field.

In the rest of the proof, we will show that $\tilde{u}$ solves the martingale problem corresponding to the SPDE (5.1), that is, for any compactly supported smooth (testing) function ϕ on $\mathbb{R}$, almost surely for every $t \geq 0$,

$$\begin{aligned}&\int \phi(x)\tilde{u}_t(x)\mathrm{d}x - \int \phi(x)f(x)\mathrm{d}x \\ &= \frac{1}{2}\iint_0^t \phi''(x)\tilde{u}_s(x)\mathrm{d}s\mathrm{d}x + b_1 \iint_0^t \phi(x)\tilde{u}_s(x)\mathrm{d}s\mathrm{d}x + \\ &\quad b_\infty \iint_0^t \phi(x)\mathbf{1}_{\{1\}}(\tilde{u}_s(x))\mathrm{d}s\mathrm{d}x + M_t(\phi)\end{aligned} \tag{5.7}$$

where $(M_t(\phi))_{t\geq 0}$ is an L^2-martingale with quadratic variation

$$\langle M_\cdot(\phi)\rangle_t = \iint_0^t \phi(x)^2 \tilde{u}_s(x)(1-\tilde{u}_s(x))\mathrm{d}s\mathrm{d}x, \quad t \geq 0.$$

Step 3. Fix an arbitrary $x \in \mathbb{R}$. From Proposition 2.2, we have that

$$\begin{aligned}&\mathbb{E}\Big[\Big(u_t^{(m)}(x)\Big)^l\Big] \\ &= \tilde{\mathbb{E}}^x\Bigg[(-1)^{|J_t^{(l,m)}|\mathbf{1}_{\{b_\infty<0\}}} \exp\Big\{(b_1 + |b_\infty|)\int_0^t \big|I_s^{(l,m)}\big|\mathrm{d}s\Big\} \prod_{\alpha\in I_t^{(l,m)}} f(X_t^\alpha)\Bigg]\end{aligned} \tag{5.8}$$

holds for each finite $l, m \in \mathbb{N}$, and $t \geq 0$. While fixing l, replacing m by m_k, and then taking $k \uparrow \infty$ in (5.8), we obtain

$$\begin{aligned}&\mathbb{E}\big[\tilde{u}_t(x)^l\big] \\ &= \tilde{\mathbb{E}}^x\Bigg[(-1)^{|J_t^{(l,\infty)}|\mathbf{1}_{\{b_\infty<0\}}} \exp\Big\{(b_1 + |b_\infty|)\int_0^t \big|I_s^{(l,\infty)}\big|\mathrm{d}s\Big\} \prod_{\alpha\in I_t^{(l,\infty)}} f(X_t^\alpha)\Bigg]\end{aligned}$$

for every $l \in \mathbb{N}$ and $t \geq 0$. Here, we used the bounded convergence theorem (recall that $b_1 + |b_\infty| \leq 0$), Step 2, Lemma 5.1, and the fact (from Theorem 1.4) that $|J_t^{(l,\infty)}| < \infty$ almost surely.

Step 4. In this step, we will show that for every $t \geq 0$,

$$\mathbb{E}\big[\mathbf{1}_{\{1\}}(\tilde{u}_t(x))\big] = \tilde{\mathbb{E}}^x\left[(-1)^{|J_t|\mathbf{1}_{\{b_\infty<0\}}} \exp\left\{(b_1+|b_\infty|)\int_0^t |I_s|\mathrm{d}s\right\} \prod_{\alpha\in I_t} f(X_t^\alpha)\right]. \tag{5.9}$$

Note that, from Step 3, for arbitrary $t \geq 0$ and $\epsilon > 0$,

$$\begin{aligned}&\left|\mathbb{E}\big[\tilde{u}_t(x)^l\big] - \tilde{\mathbb{E}}^x\left[\mathbf{1}_{\{|J_\epsilon|=0\}}(-1)^{|J_t^{(l,\infty)}|\mathbf{1}_{\{b_\infty<0\}}} e^{(b_1+|b_\infty|)\int_0^t |I_s^{(l,\infty)}|\mathrm{d}s} \prod_{\alpha\in I_t^{(l,\infty)}} f(X_t^\alpha)\right]\right|\\ &\leq \tilde{\mathbb{P}}^x(|J_\epsilon|>0);\end{aligned}$$

which, by taking $l \uparrow \infty$ and then $\epsilon \downarrow 0$, implies the desired result for this step. Here, we used the bounded convergence theorem, Lemma 5.2, (5.3) and the fact (from Theorem 1.4) that almost surely $|J_t| < \infty$.

Step 5. While taking $l = m = m_k$ and then $k \uparrow \infty$ in (5.8), we obtain

$$\lim_{k\to\infty} \mathbb{E}\left[\left(\tilde{u}_t^{(m_k)}(x)\right)^{m_k}\right] = \tilde{\mathbb{E}}^x\left[(-1)^{|J_t|\mathbf{1}_{\{b_\infty<0\}}} \exp\left\{(b_1+|b_\infty|)\int_0^t |I_s|\mathrm{d}s\right\} \prod_{\alpha\in I_t} f(X_t^\alpha)\right].$$

Here, we used the bounded convergence theorem, Lemma 3.2, and again the fact that almost surely $|J_t| < \infty$. Combine this with Step 4, we obtain that

$$\lim_{k\to\infty} \mathbb{E}\left[\left(\tilde{u}_t^{(m_k)}(x)\right)^{m_k}\right] = \mathbb{E}\big[\mathbf{1}_{\{1\}}(\tilde{u}_t(x))\big].$$

Combine this further with (5.6) and Lemma 5.3, we obtain that

$$\left(\tilde{u}_t^{(m_k)}(x)\right)^{m_k} \to \mathbf{1}_{\{1\}}(\tilde{u}_t(x))$$

in L^p when $k \to \infty$ for every $p \geq 1$.

Step 6. Fix arbitrary $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, where $\mathcal{C}_c^\infty(\mathbb{R})$ is the space of compactly supported infinitely differentiable functions on $\mathbb{R}$. In this step, we want to show that, for every $t \geq 0$, when $k \uparrow \infty$,

$$\iint_0^t \phi(x)\big(\tilde{u}_s^{(m_k)}(x)\big)^{m_k}\mathrm{d}s\mathrm{d}x \to \iint_0^t \phi(x)\mathbf{1}_{\{1\}}(\tilde{u}_s(x))\mathrm{d}s\mathrm{d}x$$

in L^2. In fact, we can calculate by Hölder's inequality that, for each $t \geq 0$ and $k \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{E}\left[\left(\iint_0^t \phi(x)\big(\tilde{u}_s^{(m_k)}(x)\big)^{m_k}\mathrm{d}s\mathrm{d}x - \iint_0^t \phi(x)\mathbf{1}_{\{1\}}(\tilde{u}_s(x))\mathrm{d}s\mathrm{d}x\right)^2\right]\\ &\leq \mathbb{E}\left[\left(\iint_0^t \phi(x)^2\Big(\big(\tilde{u}_s^{(m_k)}(x)\big)^{m_k} - \mathbf{1}_{\{1\}}(\tilde{u}_s(x))\Big)^2\mathrm{d}s\mathrm{d}x\right)\cdot\left(\iint_0^t \mathbf{1}_{\{\phi(x)\neq 0\}}\mathrm{d}s\mathrm{d}x\right)\right]\end{aligned}$$

$$\leq tC_5(\phi)\iint_0^t \phi(x)^2\mathbb{E}\Big[\Big(\big(\tilde{u}_s^{(m_k)}(x)\big)^{m_k} - \mathbf{1}_{\{1\}}(\tilde{u}_s(x))\Big)^2\Big]\mathrm{d}s \tag{5.10}$$

where $0 < C_5(\phi) < \infty$ is a constant only depending on the support of ϕ. From Step 5, while taking $k \uparrow \infty$, we have for every $s \geq 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}\Big[\Big(\big(\tilde{u}_s^{(m_k)}(x)\big)^{m_k} - \mathbf{1}_{\{1\}}(\tilde{u}_s(x))\Big)^2\Big] \to 0.$$

From this and (5.10), we obtain the desired result for this step using the bounded convergence theorem.

Step 7. Fix arbitrary $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. From (5.5), we know that almost surely for every $t \geq 0$,

$$\begin{aligned}
&\int \phi(x)\tilde{u}_t^{(m_k)}(x)\mathrm{d}x - \int \phi(x)f(x)\mathrm{d}x \\
&= \frac{1}{2}\iint_0^t \phi''(x)\tilde{u}_s^{(m_k)}(x)\mathrm{d}s\mathrm{d}x + b_1\iint_0^t \phi(x)\tilde{u}_s^{(m_k)}(x)\mathrm{d}s\mathrm{d}x + \\
&\quad b_\infty\iint_0^t \phi(x)\big(\tilde{u}_s^{(m_k)}(x)\big)^{m_k}\mathrm{d}s\mathrm{d}x + M_t^{(m_k)}(\phi)
\end{aligned} \tag{5.11}$$

where $(M_t^{(m_k)}(\phi))_{t\geq 0}$ is an L^2-martingale with quadratic variation

$$\big\langle M_\cdot^{(m_k)}(\phi)\big\rangle_t = \iint_0^t \phi(x)^2\tilde{u}_s^{(m_k)}(x)(1-\tilde{u}_s^{(m_k)}(x))\mathrm{d}s\mathrm{d}x.$$

From (5.6) and the bounded convergence, we can verify that, for any $t \geq 0$, the left hand side and the first two terms on the right hand side of (5.11) all converge in L^2 while $k \uparrow \infty$. Combine these with the result in Step 6, we know that for any $t \geq 0$, $M_t^{(m_k)}(\phi)$ converges in L^2 to a random variable $M_t(\phi)$ while $k \uparrow \infty$. Now, by the standard theory for continuous L^2-martingales, see [11, Proposition 1.3 & Theorem 4.6] for example, the limit $(M_t(\phi))_{t\geq 0}$ is an L^2-martingale with quadratic variation

$$\langle M_\cdot(\phi)\rangle_t = \iint_0^t \phi(x)^2\tilde{u}_s(x)(1-\tilde{u}_s(x))\mathrm{d}s\mathrm{d}x.$$

Now, by taking $k \uparrow \infty$ in (5.11), we can verify the desired result (5.7).

Final Step. By extending the probability space if necessary, it is standard to show (c.f. [25, Proof of Lemma 2.4]) that the martingale problem solution $(\tilde{u}_t(x))_{t\geq 0, x\in\mathbb{R}}$ is also a (mild) solution to the SPDE (5.1) with respect to some space-time white noise in some probability space. Therefore, we are done. □

6. Proofs of Lemmas 1.2 and 1.3

Proof of Lemma 1.2. We will prove the lemma by contradiction. Fix arbitrary $b_\infty^{(1)}, b_\infty^{(2)} \in [-1,1]$ with $b_\infty^{(1)} \neq b_\infty^{(2)}$. For $i = 1, 2$, let $w^{(i)}$ be the unique in law solution to (1.8) with

$b_\infty = b_\infty^{(i)}$, $w_0^{(i)} = f \in \mathcal{C}(\mathbb{R},[0,1])$. Assume that $w^{(1)}$ and $w^{(2)}$ have the same laws. Fix arbitrary non-negative and not identically zero $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. Then for $i=1,2$, we have

$$\begin{aligned} &\int \phi(x) w_t^{(i)}(x)\mathrm{d}x - \int \phi(x) f(x)\mathrm{d}x \\ &= \frac{1}{2}\iint_0^t \phi''(x) w_s^{(i)}(x)\mathrm{d}s\mathrm{d}x + \iint_0^t \phi(x)(1-w_s^{(i)}(x))\mathrm{d}s\mathrm{d}x \\ &\quad - b_\infty^{(i)} \iint_0^t \phi(x)\mathbf{1}_{\{0\}}(w_s^{(i)}(x))\mathrm{d}s\mathrm{d}x + M_t^{(i)}(\phi) \end{aligned} \tag{6.1}$$

where $(M_t^{(i)}(\phi))_{t\geq 0}$ is an L^2-martingale with quadratic variation

$$\langle M_\cdot^{(i)}(\phi)\rangle_t = \iint_0^t \phi(x)^2 w_s^{(i)}(x)(1-w_s^{(i)}(x))\mathrm{d}s\mathrm{d}x, \quad t\geq 0.$$

By taking expectation on both sides of (6.1), and recalling that $w^{(1)}$ and $w^{(2)}$ have the same law, we immediately get that

$$b_\infty^{(1)}\mathbb{E}\Big[\iint_0^t \phi(x)\mathbf{1}_{\{0\}}(w_s^{(1)}(x))\mathrm{d}s\mathrm{d}x\Big] = b_\infty^{(2)}\mathbb{E}\Big[\iint_0^t \phi(x)\mathbf{1}_{\{0\}}(w_s^{(2)}(x))\mathrm{d}s\mathrm{d}x\Big], \quad t\geq 0. \tag{6.2}$$

Let us check that the expectations in (6.2) are not zero. To this end, it is enough to check that

$$\mathbb{E}\Big[\mathbf{1}_{\{0\}}(w_t^{(i)}(x))\Big] = \mathbb{P}\left(w_t^{(i)}(x)=0\right) > 0, \quad \forall t>0, x\in\mathbb{R}, i=1,2.$$

We will derive this by comparison argument. Let w be any solution to

$$\begin{cases} \partial_t w_t = \frac{1}{2}\Delta w_t + 2(1-w_t) + \sqrt{w_t(1-w_t)}\dot{W}, \\ w_0 = f. \end{cases} \tag{6.4}$$

Clearly $u_t = 1-w_t, t\geq 0$, satisfies (1.1) with $b(u) = -2u$ and initial conditions $u_0 = 1-f$. Thus, such u is a unique in law solution of (1.1). Thefere, w is a unique in law solution to (6.4). By our assumptions on $b_\infty^{(i)}$, we immediately have $2(1-z) \geq (1-z) - b_\infty^{(i)}\mathbf{1}_{\{0\}}(z)$, for $z\in[0,1]$, and $i=1,2$. Therefore, the drift in (6.4) dominates from above the drifts in equations for $w^{(i)}, i=1,2$. This, by weak uniqueness, implies that w scholastically dominates $w^{(1)}$ and $w^{(2)}$ from above. Therefore,

$$\mathbb{E}\Big[\mathbf{1}_{\{0\}}(w_t^{(i)}(x))\Big] \geq \mathbb{E}\big[\mathbf{1}_{\{0\}}(w_t(x))\big], \quad \forall t\geq 0, x\in\mathbb{R}, i=1,2. \tag{6.5}$$

However, by duality formula (5.9) we obtain

$$\mathbb{E}\big[\mathbf{1}_{\{0\}}(w_t(x))\big] = \mathbb{E}\big[\mathbf{1}_{\{1\}}(u_t(x))\big] = \tilde{\mathbb{E}}^x\left[\exp\left\{-2\int_0^t |I_s|\mathrm{d}s\right\}\prod_{\alpha\in I_t}(1-f)(X_t^\alpha)\right]$$

where $\{(X_t^\alpha)_{t\geq 0} : \alpha\in\mathcal{U}\}$ is a coalescing Brownian particle system with initial configuration $(x_i)_{i=1}^\infty$ such that $x_i = x$ for every $i\in\mathbb{N}$. Since $f\not\equiv 1$, and $\int_0^t |I_s|\mathrm{d}s$, $|I_t|$ are almost surely finite for $t > 0$ by Theprem 1.4, we immediately get from the properties of coalescent Brownian motions that

$$\mathbb{E}\big[\mathbf{1}_{\{0\}}(w_t(x))\big] > 0, \quad t > 0, x \in \mathbb{R}.$$

Then, by (6.5) we have

$$\mathbb{E}\Big[\mathbf{1}_{\{0\}}(w_t^{(i)}(x))\Big] > 0, \;\; \forall t > 0, x \in \mathbb{R}.$$

This and our assumptions of ϕ imply that the expectations in (6.2) do not equal to zero.

Now, from our assumption (by contradiction) that $w^{(1)}$ and $w^{(2)}$ have the same laws, we obtain that $b_\infty^{(1)} = b_\infty^{(2)}$ which contradicts the condition of this lemma. This implies that $w^{(1)}$ and $w^{(2)}$ should have different laws. □

Proof of Lemma 1.3 (i). Let us show the non-uniqueness result when the initial value $X_0 = 0$. It is easy to see that one solution to (1.9) with $b_\infty = 1$ is $X_t \equiv 0, t \geq 0$, while the other one can be the solution to (1.10). Let us show that solution to (1.10) indeed also solves (1.9) (with $b_\infty = 1$). First let us check that if X solves (1.10), then

$$\int_0^t \mathbf{1}_{\{0\}}(X_s)\mathrm{d}s = 0, \; \forall t \geq 0. \tag{6.6}$$

In fact, one can easily get (6.6), by following the steps in the proof of Proposition XI.1.5 in [38]. First, by using Theorem VI.1.7 in [38] one shows that

$$L_t^0(X) = 2\int_0^t \mathbf{1}_{\{0\}}(X_s)(1 - X_s)\mathrm{d}s = 2\int_0^t \mathbf{1}_{\{0\}}(X_s)\mathrm{d}s, \tag{6.7}$$

where $L_t^0(X), t \geq 0$, is the local time of X at zero. Then, again following the proof of of Proposition XI.1.5 in [38] one derives that $L_t^0(X) = 0, \forall t \geq 0$, and then (6.6) follows by (6.7). With (6.6) at hand the result is immediate. □

Proof of Lemma 1.3 (ii). The pathwise (and thus week) uniqueness for (1.10) follows from Theorem IX.3.5 in [38]. Fix arbitrary $b_\infty \in [-1, 1)$. To prove the claim we need to show that any solution X to (1.9) also solves (1.10). This will follow if we show that for any X solving (1.9), the following holds:

$$b_\infty \int_0^t \mathbf{1}_{\{0\}}(X_s)\mathrm{d}s = 0, \; \forall t \geq 0. \tag{6.8}$$

To obtain (6.8), we follow the same strategy as in the proof of (i) of this lemma. By Theorem VI.1.7 in [38] we get that

$$L_t^0(X) = 2\int_0^t \mathbf{1}_{\{0\}}(X_s)\mathrm{d}V_s, \tag{6.9}$$

where $V_t = \int_0^t \big(\mathbf{1}_{(0,1]}(X_s)(1 - X_s) + (1 - b_\infty)\mathbf{1}_{\{0\}}(X_s)\big)\,\mathrm{d}s, t \geq 0$, is the drift in (1.9). Substituting the definition of V into (6.9) we get

$$L_t^0(X) = 2(1 - b_\infty)\int_0^t \mathbf{1}_{\{0\}}(X_s)\mathrm{d}s, \tag{6.10}$$

Then, again following the proof of of Proposition XI.1.5 in [38] we derive that $L_t^0(X) = 0, \forall t \geq 0$, and hence from (6.10) (recall that $b_\infty < 1$) we get that

$$\int_0^t \mathbf{1}_{\{0\}}(X_s)\mathrm{d}s = 0.$$

Thus, (6.8) follows and we are done. □

Appendix A.

A.1. **Proof of Lemma 3.2.**

Proof of Lemma 3.2. We claim that,

(A.1) for each $\alpha \in \mathcal{U}$ and $m \in \mathbb{N}$, $\zeta_\alpha^{(m)} = \mathbf{1}_{\{m<\|\alpha\|_\infty\}}\xi_\alpha + \mathbf{1}_{\{m\geq\|\alpha\|_\infty\}}\zeta_\alpha$ almost surely.

Fixing $m \in \mathbb{N}$, we will prove this claim by induction over $\alpha \in \mathcal{U}$. If $\alpha = 1$, then it is easy to see that $\zeta_\alpha^{(m)} = \zeta_{\alpha,\alpha} = \zeta_\alpha$. Let us fix an arbitrary $\beta \in \mathcal{U}$, and for the sake of induction, assume that the desired claim (A.1) holds for every $\alpha \prec \beta$.

Firstly, we show that almost surely $\zeta_\beta^{(m)} = \xi_\beta$ provided $m < \|\beta\|_\infty$. To do this, we discuss in two different cases.

(i) $m < \|\beta\|_\infty$ and $|\beta| = 1$. In this case, we have $m < \beta$. So by (iii) of (3.5), we have $\zeta_\beta^{(m)} = \xi_\beta$ as desired.

(ii) $m < \|\beta\|_\infty$ and $|\beta| > 1$. In this case, we can show that the event

$$\left\{\zeta_{\overleftarrow{\beta}}^{(m)} = \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}\right\} \cap \left\{\beta_{|\beta|} \leq Z_{\overleftarrow{\beta}} \wedge m\right\}$$

almost surely won't happen. In fact, if the above event happens, we have $\beta_{|\beta|} \leq m$. This, and the condition $m < \|\beta\|_\infty$, implies that $m < \|\overleftarrow{\beta}\|_\infty$. From what we assumed for the sake of induction, we must have $\zeta_{\overleftarrow{\beta}}^{(m)} = \xi_{\overleftarrow{\beta}}$. This further implies that $\xi_{\overleftarrow{\beta}} = \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}$ which has 0 probability. Now, by (iii) of (3.5), we have $\zeta_\beta^{(m)} = \xi_\beta$ as desired.

Secondly, we show that almost surely $\zeta_\beta^{(m)} = \zeta_\beta$ provided $m \geq \|\beta\|_\infty$. To do this, we discuss in four different cases.

(i) $m \geq \|\beta\|_\infty$, $|\beta| = 1$ and $\beta > n$. In this case, by (iii) of (2.7) and (iii) of (3.5) we have $\zeta_\beta^{(m)} = \xi_\beta = \zeta_\beta$ as desired.

(ii) $m \geq \|\beta\|_\infty$, $|\beta| = 1$ and $\beta \leq n$. In this case, since $\beta \leq m$, by (i) of (3.5) we have

$$\zeta_\beta^{(m)} = \inf\bigl(\{\zeta_{\beta,\beta}\} \cup \bigl\{\zeta_{\alpha,\beta} : \alpha \in \mathcal{U}, \alpha \prec \beta, \zeta_{\alpha,\beta} \leq \zeta_\alpha^{(m)}\bigr\}\bigr).$$

Note that, $\alpha \prec \beta$ actually implies that $\|\alpha\|_\infty \leq m$. So by what we assumed for the sake of induction, we have $\zeta_\alpha^{(m)} = \zeta_\alpha$ for every $\alpha \prec \beta$. Now the above equation can be rewritten as

$$\zeta_\beta^{(m)} = \inf(\{\zeta_{\beta,\beta}\} \cup \{\zeta_{\alpha,\beta} : \alpha \in \mathcal{U}, \alpha \prec \beta, \zeta_{\alpha,\beta} \leq \zeta_\alpha\}).$$

This and (i) of (2.7) imply that $\zeta_\beta^{(m)} = \zeta_\beta$ as desired.

(iii) $m \geq \|\beta\|_\infty$, $|\beta| > 1$, $\zeta^{(m)}_{\overleftarrow{\beta}} = \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}$ and $\beta_{|\beta|} \leq Z^{(m)}_{\overleftarrow{\beta}} = Z_{\overleftarrow{\beta}} \wedge m$. In this case, by (ii) of (3.5), we have

$$\zeta^{(m)}_\beta = \inf\big(\{\zeta_{\beta,\beta}\} \cup \big\{\zeta_{\alpha,\beta} : \alpha \in \mathcal{U}, \alpha \prec \beta, \zeta_{\alpha,\beta} \leq \zeta^{(m)}_\alpha\big\}\big).$$

Similar to the previous case, we can rewrite the above as

$$\zeta^{(m)}_\beta = \inf(\{\zeta_{\beta,\beta}\} \cup \{\zeta_{\alpha,\beta} : \alpha \in \mathcal{U}, \alpha \prec \beta, \zeta_{\alpha,\beta} \leq \zeta_\alpha\}).$$

Also note that $\|\overleftarrow{\beta}\|_\infty \leq \|\beta\|_\infty \leq m$. So from what we assumed for the sake of induction, we have $\zeta^{(m)}_{\overleftarrow{\beta}} = \zeta_{\overleftarrow{\beta}}$. This implies that $\zeta_{\overleftarrow{\beta}} = \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}$ and $\beta_{|\beta|} \leq Z_{\overleftarrow{\beta}}$. Now from (ii) of (2.7), we have $\zeta^{(m)}_\beta = \zeta_\beta$ as desired.

(iv) $m \geq \|\beta\|_\infty$, $|\beta| > 1$, and the condition in (iii) does not hold. In this case, by (ii) of (3.5), we have $\zeta^{(m)}_\beta = \xi_\beta$. We can verify by contradiction that the event

$$\Big\{\zeta_{\overleftarrow{\beta}} = \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}\Big\} \cap \Big\{\beta_{|\beta|} \leq Z_{\overleftarrow{\beta}}\Big\}$$

almost surely won't happen. In fact, if otherwise, then from the condition $\|\beta\|_\infty \leq m$, we have $\beta_{|\beta|} \leq Z_{\overleftarrow{\beta}} \wedge m$. Note that we also have $\|\overleftarrow{\beta}\|_\infty \leq \|\beta\|_\infty \leq m$. So from what we have assumed for the sake of induction, $\zeta^{(m)}_{\overleftarrow{\beta}} = \zeta_{\overleftarrow{\beta}}$. Then we arrived at a contradiction that the condition in (iii) holds. Now, by (iii) of (2.7), we have $\zeta^{(m)}_\beta = \zeta_\beta$ as desired.

To sum up, we proved claim (A.1). The desired result for this lemma follows immediately. □

A.2. **Proofs of Lemmas 3.12 and 3.13.**

Proof of Lemma 3.12. Note that by Proposition 3.11,

$$\sup_{0 \leq t \leq T} \tilde{\mathbb{E}}[|I_t|] < \infty.$$

We can also verify that

$$\sup_{0 \leq t \leq T} \tilde{\mathbb{E}}\left[\left|I^{(m)}_t\right|^2\right] < \infty.$$

In fact, by Lemma 3.1, $|I^{(m)}_t|$ is dominated by the total population at time t of a continuous-time Galton-Watson process with a bounded offspring distribution; and therefore, have all finite moments, thank to the standard theory of branching processes [17, p. 103].

It is then clear that the desired result for this lemma is trivial if $\mu + b_1 \leq 0$, because in this case, the term $1 + e^{K_t}$ is almost surely bounded by 2. In particular, the result is trivial if $R = 1$, because $R = 1$ actually implies that $\mu + b_1 \leq 0$ by (1.3). Also note that the result is trivial if $b_\infty \neq 0$, since in order that the condition (1.3) to hold, we must have $R = 1$ in this case. So for the rest of this proof, we only have to show

$$\sup_{0\le t\le T} \tilde{\mathbb{E}}\left[e^{K_t}\left(1+|I_t|+\left|I_t^{(m)}\right|^2\right)\right] < \infty \tag{A.2}$$

under the assumption that $\mu + b_1 > 0$, $b_\infty = 0$ and $R > 1$.

From Theorem 1.4 and that $p_\infty = \mu^{-1}|b_\infty| = 0$, we can verify that the process $(|I_t|)_{t\ge0}$ has finite jumps up to any finite time. This allows us to write down the decomposition

$$R^{|I_t|} - R^{|I_0|} = \int_{\mathcal{U}\times\mathbb{Z}_+}\int_0^t (R^{|I_{s-}|+k-1} - R^{|I_{s-}|})\mathbf{1}_{\{X^\alpha_{s-}\in\mathbb{R}\}}\mathfrak{N}(\mathrm{d}s,\mathrm{d}\alpha,\mathrm{d}k) + \int_{\mathcal{R}}\int_0^t (R^{|I_{s-}|-1} - R^{|I_{s-}|})\mathbf{1}_{\{X^\alpha_{s-},X^\beta_{s-}\in\mathbb{R}\}}\mathfrak{M}(\mathrm{d}s,\mathrm{d}(\alpha,\beta))$$

for every $t \ge 0$ where the integrals are simply finite sums. Consider the process

$$Z_t := e^{K_t}R^{|I_t|}, \quad t\ge 0.$$

From the integration by parts formula, see [24, p. 444] for example, we have for $t \ge 0$,

$$\begin{aligned}
Z_t - Z_0 &= \int_0^t R^{|I_{s-}|}\mathrm{d}e^{K_s} + \int_0^t e^{K_s}\mathrm{d}R^{|I_s|} \\
&= (\mu+b_1)\int_0^t e^{K_s}R^{|I_{s-}|}|I_{s-}|\mathrm{d}s + \\
&\quad \int_{\mathcal{U}\times\mathbb{Z}_+}\int_0^t e^{K_s}(R^{|I_{s-}|+k-1} - R^{|I_{s-}|})\mathbf{1}_{\{X^\alpha_{s-}\in\mathbb{R}\}}\mathfrak{N}(\mathrm{d}s,\mathrm{d}\alpha,\mathrm{d}k) + \\
&\quad \int_{\mathcal{R}}\int_0^t e^{K_s}(R^{|I_{s-}|-1} - R^{|I_{s-}|})\mathbf{1}_{\{X^\alpha_{s-},X^\beta_{s-}\in\mathbb{R}\}}\mathfrak{M}(\mathrm{d}s,\mathrm{d}(\alpha,\beta)).
\end{aligned}$$

From Lemma 3.4, we know that $\mathfrak{N}$ and $\mathfrak{M}$ are QL point processes with compensators $\hat{\mathfrak{N}}$ and $\hat{\mathfrak{M}}$ respectively. (Recall that $\hat{\mathfrak{N}}$ and $\hat{\mathfrak{M}}$ are given in (2.1) and (2.5) respectively.) Now from Lemma 3.3, there exists a local martingale $(m_t)_{t\ge0}$ such that

$$\begin{aligned}
Z_t - Z_0 &= m_t + (\mu+b_1)\int_0^t e^{K_s}R^{|I_{s-}|}|I_{s-}|\mathrm{d}s + \\
&\quad \int_{\mathcal{U}\times\mathbb{Z}_+}\int_0^t e^{K_s}(R^{|I_{s-}|+k-1} - R^{|I_{s-}|})\mathbf{1}_{\{X^\alpha_{s-}\in\mathbb{R}\}}\hat{\mathfrak{N}}(\mathrm{d}s,\mathrm{d}\alpha,\mathrm{d}k) + \\
&\quad \int_{\mathcal{R}}\int_0^t e^{K_s}(R^{|I_{s-}|-1} - R^{|I_{s-}|})\mathbf{1}_{\{X^\alpha_{s-},X^\beta_{s-}\in\mathbb{R}\}}\hat{\mathfrak{M}}(\mathrm{d}s,\mathrm{d}(\alpha,\beta)) \\
&= m_t + \int_0^t e^{K_s}R^{|I_{s-}|}|I_{s-}|\left(b_1 + \sum_{k\in\mathbb{Z}_+\setminus\{1\}}R^{k-1}|b_k|\right)\mathrm{d}s + \\
&\quad \int_{\mathcal{R}}\int_0^t e^{K_s}(R^{|I_{s-}|-1} - R^{|I_{s-}|})\mathbf{1}_{\{X^\alpha_{s-},X^\beta_{s-}\in\mathbb{R}\}}\hat{\mathfrak{M}}(\mathrm{d}s,\mathrm{d}(\alpha,\beta)).
\end{aligned}$$

By (1.3), we know that $(Z_t)_{t\geq 0}$ is a local supermartingale. From the fact that $(Z_t)_{t\geq 0}$ is non-negative, we can verify that it has finite mean for any $t \geq 0$. In fact, since there exists a sequence of stopping time τ_k such that $\tau_k \uparrow \infty$ almost surely as $k \uparrow \infty$ and $(Z_{t\wedge\tau_k})_{t\geq 0}$ is a supermartingale for each $k \in \mathbb{N}$, we have by Fatou's lemma

$$\tilde{\mathbb{E}}[Z_t] = \tilde{\mathbb{E}}\Big[\lim_{k\to\infty} Z_{t\wedge\tau_k}\Big] \leq \liminf_{k\to\infty} \tilde{\mathbb{E}}[Z_{t\wedge\tau_k}] \leq \tilde{\mathbb{E}}[Z_0] = R^n < \infty.$$

It is also clear from Lemma 3.2 that $|I_t^{(m)}| \leq |I_t|$ almost surely for every $t \geq 0$. The desired result (A.2) now follows since

$$\begin{aligned}\tilde{\mathbb{E}}\Big[e^{K_t}(1 + |I_t| + |I_t^{(m)}|^2)\Big] &\leq \tilde{\mathbb{E}}\big[e^{K_t}(1 + |I_t| + |I_t|^2)\big]\\ &\leq C_6(R)\tilde{\mathbb{E}}\big[e^{K_t}R^{|I_t|}\big] \leq C_6(R)R^n\end{aligned}$$

where $C_6(R) = \sup_{x\in\mathbb{R}}(1 + |x| + |x|^2)/R^{|x|} \in (0, \infty)$. □

Proof of Lemma 3.13. Step 1. Let us first mention a result about the all-level supremum of the local time of the Brownian motion. Suppose that $L_{t,z}$ is the local time of a standard 1-dimensional Brownian motion at level $z \in \mathbb{R}$ up to time $t \geq 0$. Without loss of generality, we can assume that $L_{t,z}$ is jointly continuous in $t \geq 0$ and $z \in \mathbb{R}$ [38, Corollary 1.8 Chapter VI]. It is known that for any finite time $t \geq 0$, the all-level supremum of this local time

$$\sup_{z\in\mathbb{R}} L_{t,z} \tag{A.3}$$

up to time t has all finite moments. Indeed, this result has already been used in [4, p. 1725]; and two different characterizations of (A.3) appeared in [38, Excises 1.22] and [8], respectively.

Step 2. Fix $m \in \mathbb{N}$ with $m \geq n$ and $T \geq 0$. Let us construct yet another particle system, denoted by $\{(\bar{X}_t^{(m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$, in the probability space where both the original branching-coalescing Brownian particle system $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ and its m-truncated version $\{(X_t^{(m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ are constructed. This new particle system $\{(\bar{X}_t^{(m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ will be constructed as a branching Brownian particle system which produces exactly m-many children at each of its branching event, and does not induce coalescing event. And it will be sharing the same initial configuration $(x_i)_{i=1}^n$.

More precisely, we construct $\{(\bar{X}_t^{(m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ through (A.4) and (A.5) below. (Recall that $\{\zeta_{\alpha,\alpha} : \alpha \in \mathcal{U}\}$ and $\{(\tilde{X}_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$ are already constructed in Section 2 along with $\{(X_t^\alpha)_{t\geq 0} : \alpha \in \mathcal{U}\}$.)

(A.4) For each $\beta \in \mathcal{U}$, define $\mathbb{R}_+$-valued random variable $\bar{\zeta}_\beta^{(m)}$ inductively so that
- (i) if $|\beta| = 1$ and $\beta \leq n$, then $\bar{\zeta}_\beta^{(m)} := \zeta_{\beta,\beta}$.
- (ii) if $|\beta| > 1$, $\bar{\zeta}_{\overleftarrow{\beta}}^{(m)} = \zeta_{\overleftarrow{\beta},\overleftarrow{\beta}}$ and $\beta_{|\beta|} \leq m$, then $\bar{\zeta}_\beta^{(m)} := \zeta_{\beta,\beta}$;
- (iii) if neither of the conditions in (i) nor (ii) holds, then $\bar{\zeta}_\beta^{(m)} := \xi_\beta$.

(A.5) For each $\beta \in \mathcal{U}$, define $\mathbb{R} \cup \{\dagger\}$-valued process

$$\bar{X}_t^{(m),\beta} := \begin{cases} \dagger, & t \in [0, \xi_\beta), \\ \tilde{X}_t^\beta, & t \in [\xi_\beta, \bar{\zeta}_\beta^{(m)}), \\ \dagger, & t \in [\bar{\zeta}_\beta^{(m)}, \infty). \end{cases}$$

It is also clear, c.f. Lemma 3.1, that the m-truncated branching-coalescing Brownian particle system is dominated by $\{(\bar{X}_t^{(m),\alpha})_{t\geq 0} : \alpha \in \mathcal{U}\}$ in the sense that almost surely,

$$I_t^{(m)} \subset \bar{I}_t^{(m)} := \{\alpha \in \mathcal{U} : \bar{X}_t^{(m),\alpha} \in \mathbb{R}\},$$

and

$$J_t^{(m)} \subset \bar{J}_t^{(m)} := \{\alpha \in \mathcal{U} : \zeta_{\alpha,\alpha} = \bar{\zeta}_\alpha^{(m)} \leq t\}.$$

Step 3. Fix arbitrary $(\alpha, \beta) \in \mathcal{R}$. From the construction of the Brownian motions $(\tilde{X}_t^\alpha)_{t\geq 0}$ and $(\tilde{X}_t^\beta)_{t\geq 0}$, and the strong Markov property of the Brownian motions, we know that there exists a stopping time $\tau_{\alpha,\beta}$ satisfying $\tilde{X}_t^\alpha = \tilde{X}_t^\beta$ on $[0, \tau_{\alpha,\beta}]$; and that

$$\hat{X}_t^\alpha := \tilde{X}^\alpha_{\tau_{\alpha,\beta}+t} - \tilde{X}^\alpha_{\tau_{\alpha,\beta}}, \quad t \geq 0$$

and

$$\hat{X}_t^\beta := \tilde{X}^\beta_{\tau_{\alpha,\beta}+t} - \tilde{X}^\beta_{\tau_{\alpha,\beta}}, \quad t \geq 0$$

are two independent Brownian motions with zero initial values. Denote by $\hat{L}_{t,z}^{\alpha,\beta}$ the local time of the process $\hat{X}_\cdot^\alpha - \hat{X}_\cdot^\beta$ up to time $t \geq 0$ at level $z \in \mathbb{R}$. We can assume without loss of generality that $\hat{L}_{t,z}^{\alpha,\beta}$ is continuous in both $t \geq 0$ at $z \in \mathbb{R}$, c.f. [24, Theorem 29.4]. It is clear that

$$L_{T,z}^{\alpha,\beta} = \mathbf{1}_{\{\tau_{\alpha,\beta} \leq T\}} \hat{L}^{\alpha,\beta}_{T-\tau_{\alpha,\beta},z} \leq \hat{L}_{T,z}^{\alpha,\beta}, \quad T \geq 0, z \in \mathbb{R}, \text{a.s.}$$

Step 4. Denote by $\mathscr{G}$ the minimal σ-field containing all the information about the genealogical structure of the branching Brownian motions $\{(\bar{X}_t^{(m),\gamma})_{t\geq 0} : \gamma \in \mathcal{U}\}$, i.e. the σ-field generated by the death-times $\{\bar{\zeta}_\gamma^{(m)} : \gamma \in \mathcal{U}\}$. It is clear that the Brownian motions $\hat{X}_\cdot^\alpha$ and $\hat{X}_\cdot^\beta$ are independent of the σ-field $\mathscr{G}$. Therefore, using the result in Step 1, it can be shown that, for any $T \geq 0$ and $\varrho \geq 0$, there exists a constant $C_7(T, \varrho) < \infty$, which is independent of the choice of the arbitrary $(\alpha, \beta) \in \mathcal{R}$, such that

$$\begin{aligned} &\tilde{\mathbb{E}}\left[\left(1 + \sup_{z\in\mathbb{R}} L_{T,z}^{\alpha,\beta}\right)^{1+\varrho} \middle| \mathscr{G}\right] \leq \tilde{\mathbb{E}}\left[\left(1 + \sup_{z\in\mathbb{R}} \hat{L}_{T,z}^{\alpha,\beta}\right)^{1+\varrho} \middle| \mathscr{G}\right] \\ &= \tilde{\mathbb{E}}\left[\left(1 + \sup_{z\in\mathbb{R}} \hat{L}_{T,z}^{\alpha,\beta}\right)^{1+\varrho}\right] = C_7(T, \varrho). \end{aligned}$$

Step 5. Using Jensen's inequality, we can verify that for any $\varrho \geq 0$,

$$\tilde{\mathbb{E}}\left[\left(\sum_{\alpha,\beta\in I_{[0,T]}^{(m)}:\alpha\prec\beta} \left(1 + \sup_{z\in\mathbb{R}} L_{T,z}^{\alpha,\beta}\right)\right)^{1+\varrho}\right] \leq \tilde{\mathbb{E}}\left[\left(\sum_{\alpha,\beta\in \bar{I}_{[0,T]}^{(m)}:\alpha\prec\beta} \left(1 + \sup_{z\in\mathbb{R}} L_{T,z}^{\alpha,\beta}\right)\right)^{1+\varrho}\right]$$

$$\leq \tilde{\mathbb{E}}\left[\left|\mathcal{R}\cap\left(\bar{I}^{(m)}_{[0,T]}\times\bar{I}^{(m)}_{[0,T]}\right)\right|^{\varrho}\sum_{\alpha,\beta\in\bar{I}^{(m)}_{[0,T]}:\alpha\prec\beta}\left(1+\sup_{z\in\mathbb{R}}L^{\alpha,\beta}_{T,z}\right)^{1+\varrho}\right]$$
$$=\tilde{\mathbb{E}}\left[\left|\mathcal{R}\cap\left(\bar{I}^{(m)}_{[0,T]}\times\bar{I}^{(m)}_{[0,T]}\right)\right|^{\varrho}\sum_{\alpha,\beta\in\bar{I}^{(m)}_{[0,T]}:\alpha\prec\beta}\tilde{\mathbb{E}}\left[\left(1+\sup_{z\in\mathbb{R}}L^{\alpha,\beta}_{T,z}\right)^{1+\varrho}\middle|\mathscr{G}\right]\right]$$
$$\leq C_7(T,\varrho)\tilde{\mathbb{E}}\left[\left|\bar{I}^{(m)}_{[0,T]}\right|^{2+2\varrho}\right]<\infty.$$

The last inequality is due to the standard theory of branching processes [17, p. 103], and our assumption that the number of offspring at each branching event of the branching Brownian particle system $\{(\bar{X}^\alpha_t)_{t\geq 0}:\alpha\in\mathcal{U}\}$ is exactly m.

Step 6. Thanks to Step 5, it is clear that the desired result for this lemma is trivial if $\mu+b_1-\frac{1}{m}\leq 0$, because in this case, the term $1+e^{K^{(m)}_t}$ is almost surely bounded by 2. In particular, the result is trivial if $R=1$, because $R=1$ actually implies that $\mu+b_1-\frac{1}{m}\leq\mu+b_1\leq 0$ by (1.3). So, for the rest of this proof, we assume that $\mu+b_1-\frac{1}{m}>0$ and $R>1$ holds. And we only have to prove

$$\tilde{\mathbb{E}}\left[e^{K^{(m)}_T}\sum_{\alpha,\beta\in I^{(m)}_{[0,T]}:\alpha\prec\beta}\left(1+\sup_{z\in\mathbb{R}}L^{\alpha,\beta}_{T,z}\right)\right]<\infty.$$

Step 7. Let us show that there exists a deterministic $\vartheta_m>0$ such that

$$\tilde{\mathbb{E}}\left[e^{(1+\vartheta_m)K^{(m)}_T}\right]<\infty.$$

In fact, by Lemmas 3.2 and 3.12, we have

$$\tilde{\mathbb{E}}\left[\exp\left\{(\mu+b_1)\int_0^t\left|I^{(m)}_s\right|\mathrm{d}s\right\}\right]<\infty.$$

Therefore, by taking a deterministic $\vartheta_m>0$ such that $(1+\vartheta_m)(\mu+b_1-\frac{1}{m})=\mu+b_1$, we have the desired result for this step.

Step 8. Let $\vartheta_m>0$ be given as in Step 7. Define $\varrho_m>0$ so that $(1+\vartheta_m)^{-1}+(1+\varrho_m)^{-1}=1$. Now by Hölder's inequality, we have

$$\tilde{\mathbb{E}}\left[e^{K^{(m)}_T}\sum_{\alpha,\beta\in I^{(m)}_{[0,T]}:\alpha\prec\beta}\left(1+\sup_{z\in\mathbb{R}}L^{\alpha,\beta}_{T,z}\right)\right]$$
$$\leq\tilde{\mathbb{E}}\left[e^{(1+\vartheta_m)K^{(m)}_T}\right]^{\frac{1}{1+\vartheta_m}}\tilde{\mathbb{E}}\left[\left(\sum_{\alpha,\beta\in I^{(m)}_{[0,T]}:\alpha\prec\beta}\left(1+\sup_{z\in\mathbb{R}}L^{\alpha,\beta}_{T,z}\right)\right)^{1+\varrho_m}\right]^{\frac{1}{1+\varrho_m}},$$

which is finite, thanks to Steps 5 and 7. We are done. $\square$

A.3. **A technical result.** In this subsection, let us fix an arbitrary $n \in \mathbb{N}$, and let $\{X_t = (X_t^1, \ldots, X_t^n) : t \geq 0\}$ be an n-dimensional Brownian motion, with initial values denoted by $(x_1, \ldots, x_n)$, living in a probability space with its probability measure denoted by $\Pi_{(x_1,\ldots,x_n)}$. Define stopping time

$$\tau = \inf\left\{ t \geq 0 : \frac{1}{4} \sum_{i,j=1}^{n} L_t^{i,j} + \mu n t \geq \mathbf{e} \right\}$$

where $\mathbf{e}$ is an standard exponential random variable, independent of the Brownian motion $(X_t)_{t\geq 0}$, and $L_t^{i,j}$ is the local time of $(X_t^j - X_t^i)_{t\geq 0}$ up to time $t \geq 0$ at the level 0. Define a family of operators $(\mathfrak{P}_t^\epsilon)_{t\geq 0, \epsilon \geq 0}$ on $\mathrm{b}\mathcal{B}(\mathbb{R}^n)$, the space of bounded measurable functions on $\mathbb{R}^n$, such that for any $F \in \mathrm{b}\mathcal{B}(\mathbb{R}^n)$, $(x_i)_{i=1}^n \in \mathbb{R}^n$, and $\epsilon > 0$,

(A.6)
$$\begin{aligned}&(\mathfrak{P}_t^\epsilon F)(x_1, \ldots, x_n) \\ &= \Pi_{(x_1,\ldots,x_n)} \left[\mathbf{1}_{\{t \leq \tau\}} \int_{\mathbb{R}^n} p_\epsilon(X_t^1 - y_1) \cdots p_\epsilon(X_t^n - y_n) F(y_1, \ldots, y_n) \mathrm{d}(y_1, \cdots, y_n) \right]\end{aligned}$$

and

$$(\mathfrak{P}_t^0 F)(x_1, \ldots, x_n) = \Pi_{(x_1,\ldots,x_n)} \left[F(X_t^1, \ldots, X_t^n) \right]$$

where $(p_\epsilon)_{\epsilon > 0}$ are the one-dimensional heat kernels given as in (1.6).

The main result of this subsection is the following proposition which will be later used in the proof of Proposition 4.3 (5).

Proposition A.1. *For any $t > 0$, $F \in \mathrm{b}\mathcal{B}(\mathbb{R}^n)$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$, it holds that*

$$(\mathfrak{P}_t^\epsilon F)(x_1, \ldots, x_n) \underset{\epsilon \downarrow 0}{\longrightarrow} (\mathfrak{P}_t^0 F)(x_1, \ldots, x_n).$$

Remark A.2. If F is continuous on $\mathbb{R}^n$, then the result of Lemma A.1 follows from the bounded convergence theorem immediately. So the point here is that F can be discontinuous. Also, it is crucial that t is strictly larger than 0.

Before we give the proof of Proposition A.1, we mention an analytical fact. Its proof is elementary, and therefore omitted.

Lemma A.3. *Suppose that h is a bounded measurable function on $\mathbb{R}^n$ and q is a non-negative continuous function on $\mathbb{R}^n$ such that $\bar{q}$ is integrable (w.r.t. the Lebesgue measure on $\mathbb{R}^n$) where for any $(y_1, \ldots, y_n)$ in $\mathbb{R}^n$,*

$$\bar{q}(y_1, \ldots, y_n) := \sup_{|z_i| \leq 1, i=1,\ldots,n} q(y_1 + z_1, \ldots, y_n + z_n).$$

Then

$$\int_{\mathbb{R}^n} h(y_1, \ldots, y_n) q^\epsilon(y_1, \ldots, y_n) \mathrm{d}(y_1, \ldots, y_n) \underset{\epsilon \downarrow 0}{\longrightarrow} \int_{\mathbb{R}^n} h(y_1, \ldots, y_n) q(y_1, \ldots, y_n) \mathrm{d}(y_1, \ldots, y_n)$$

where

$$q^\epsilon(y_1,\dots,y_n) := \int_{\mathbb{R}^n} p_\epsilon(z_1-y_1)\dots p_\epsilon(z_n-y_n)q(z_1,\dots,z_n)\mathrm{d}(z_1,\cdots,z_n).$$

Proof of Lemma A.1. Fix arbitrary $t>0$, $F\in \mathrm{b}\mathcal{B}(\mathbb{R}^n)$ and $(x_1,\dots,x_n)\in\mathbb{R}^n$. We note that

$$\begin{aligned}&\Pi_{(x_1,\dots,x_n)}\left[\mathbf{1}_{\{t\le\tau\}}F(X_t^1,\dots,X_t^n)\right]\\&=\Pi_{(x_1,\dots,x_n)}\left[\exp\left\{-\frac14\sum_{i,j=1}^n L_t^{i,j}-\mu nt\right\}F(X_t^1,\dots,X_t^n)\right].\end{aligned}$$

By using Tanaka's formula and Gilsanov transformation, we can get that

$$\begin{aligned}&\Pi_{(x_1,\dots,x_n)}\left[\mathbf{1}_{\{t\le\tau\}}F(X_t^1,\dots,X_t^n)\right]\\&=e^{n(n^2-1)\frac{t}{24}-\mu nt}\tilde{\Pi}_{(x_1,\dots,x_n)}\left[\frac{e^{-\frac14\sum_{i,j=1}^n|X_t^j-X_t^i|}}{e^{-\frac14\sum_{i,j=1}^n|x_j-x_i|}}F(X_t^1,\dots,X_t^n)\right]\end{aligned}$$

where $\tilde{\Pi}_{(x_1,\dots,x_n)}$ is a new probability measure under which $\{(X_s^i)_{s\ge0}: i=1,\dots,n\}$ is a family of stochastic processes satisfying the SDEs

$$\begin{cases}\mathrm{d}X_s^i=\frac12\sum_{j=1}^n\operatorname{sgn}(X_s^i-X_s^j)\mathrm{d}t+\mathrm{d}B_s^i, & i=1,\dots,n;\\ X_0^i=x_i, & i=1,\dots,n.\end{cases}$$

Here, $\operatorname{sgn}(x):=x\cdot|x|^{-1}$ for $x\in\mathbb{R}\setminus\{0\}$ and $\operatorname{sgn}(0):=0$; $\{(B_s^i)_{s\ge0}: i=1,\dots,n\}$ is a family of standard independent Brownian motions.

It is known from [31, Theorem 1.2] that, under the probability $\tilde{\Pi}_{(x_1,\dots,x_n)}$, the random vector $(X_t^1,\dots,X_t^n)$ has a continuous density, denoted by $\tilde{q}_t$, with respect to the Lebesgue measure on $\mathbb{R}^n$. Therefore, we have

$$\begin{aligned}&\Pi_{(x_1,\dots,x_n)}\left[\mathbf{1}_{\{t\le\tau\}}F(X_t^1,\dots,X_t^n)\right]\\&=e^{n(n^2-1)\frac{t}{24}-\mu nt+\frac14\sum_{i,j=1}^n|x_i-x_j|}\times\\&\qquad\int_{\mathbb{R}^n}e^{-\frac14\sum_{i,j=1}^n|z_i-z_j|}\tilde{q}_t(z_1,\dots,z_n)F(z_1,\dots,z_n)\mathrm{d}(z_1,\dots,z_n)\\&=\int_{\mathbb{R}^n}q_t(z_1,\dots,z_n)F(z_1,\dots,z_n)\mathrm{d}(z_1,\dots,z_n)\end{aligned}\tag{A.7}$$

where

$$q_t(z_1,\dots,z_n):=e^{n(n^2-1)\frac{t}{24}-\mu nt+\frac14\sum_{i,j=1}^n|x_i-x_j|-\frac14\sum_{i,j=1}^n|z_i-z_j|}\tilde{q}_t(z_1,\dots,z_n).$$

It is also known from [31, Theorem 1.2] that $\tilde{q}_t$ is dominated by the n-dimensional heat kernel, up to certain centering and scaling. In particular, we can verify that $\bar{q}_t$ is integrable w.r.t. the Lebesgue measure on $\mathbb{R}^n$ where for any $(y_1,\dots,y_n)\in\mathbb{R}^n$,

$$\bar{q}_t(y_1,\dots,y_n):=\sup_{|z_i|\le1,i=1,\dots,n}q_t(y_1+z_1,\dots,y_n+z_n).$$

Since $F \in b\mathcal{B}(\mathbb{R}^n)$ in (A.7) is arbitrary, by Fubini's theorem, we have

$$(\mathfrak{P}_t^\epsilon F)(x_1, \ldots, x_n) = \int_{\mathbb{R}^n} q_t^\epsilon(y_1, \ldots, y_n) F(y_1, \ldots, y_n) \mathrm{d}(y_1, \cdots, y_n)$$

where

$$q_t^\epsilon(y_1, \ldots, y_n) = \int_{\mathbb{R}^n} p_\epsilon(z_1 - y_1) \ldots p_\epsilon(z_n - y_n) q_t(z_1, \ldots, z_n) \mathrm{d}(z_1, \cdots, z_n).$$

Now the desired result follows from Lemma A.3. □

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