ON THE SUBCRITICAL SELF-CATALYTIC BRANCHING BROWNIAN MOTIONS

HAOJIE HOU AND ZHENYAO SUN

Abstract. The self-catalytic branching Brownian motions (SBBM) are extensions of the classical one-dimensional branching Brownian motions by incorporating pairwise branchings catalyzed by the intersection local times of the particle pairs. These processes naturally arise as the moment duals of certain reaction-diffusion equations perturbed by multiplicative space-time white noise. For the subcritical case of the catalytic branching mechanism, we construct the SBBM allowing an infinite number of initial particles. Additionally, we establish the coming down from infinity (CDI) property for these systems and characterize their CDI rates.

1. Introduction

1.1. **Motivation.** The branching Brownian motion (BBM) is a classical probabilistic model describing a system of particles that move independently according to Brownian motions and branch independently into random number of offspring at random times. This model serves as a cornerstone of modern probability theory, with applications spanning partial differential equations, statistical physics, and mathematical biology. Foundational works such as [McK75] and [Bra78] established its dual connection to the FKPP equation, while a modern overview is provided in [Ber14].

While the classical BBM has been extensively studied, more complex models have emerged to incorporate intraspecific competition. Examples include the branching coalescing Brownian motions [Shi88], where pairs of particles merge randomly based on their pairwise intersection local times; the N-BBM [Mai16], which maintains a fixed population size through a selection mechanism; the L-BBM [Pai16], where particles farther than a distance L from the leading particle are deleted; and BBM with decaying mass [ABBP19], where each particle carries a mass that decays at a rate proportional to its neighboring mass field.

A further extension of the branching coalescing Brownian motion is the self-catalytic branching Brownian motion (SBBM), first introduced in [Ath98] and further developed in [AT00]. In this model, branching events are catalyzed by the intersection local times of particle pairs, allowing not only the competitive but also the cooperative interactions between particles. A defining mathematical feature of SBBM is its dual relationship with

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China

E-mail addresses: houhaojie@bit.edu.cn, zhenyao.sun@gmail.com.

2020 *Mathematics Subject Classification.* 60J80, 60H15, 60F25.

Key words and phrases. branching Brownian motion, coming down from infinity, duality, SPDE.

a family of stochastic reaction-diffusion equations, akin to the connection between BBM and the FKPP equation. This duality provides a comprehensive framework for studying both SBBMs and their dual SPDEs.

For example, the foundational work [Shi88] employed branching coalescing Brownian motion to establish the weak uniqueness of the stochastic FKPP equation. This result was later extended in [AT00], which used SBBM to prove the well-posedness of a family of stochastic reaction-diffusion equations. Another important contribution is [Tri95], where local-time coalescing Brownian motions (LCBM) were utilized to demonstrate the compact interface property of the Wright-Fisher stochastic heat equation (Wright-Fisher SHE). Later, an advanced small-noise asymptotic analysis of the propagation speed of the stochastic FKPP equation was established in [MMQ11], which, via duality, confirmed the Brunet-Derrida conjecture for the branching coalescing Brownian motions. The duality framework was also employed in [DF16] to demonstrate the convergence of the biased voter model to the stochastic FKPP equation. The branching coalescing Brownian motions on metric graphs were utilized in [Fan21] to study the stochastic FKPP equation on metric graphs. A two-type coalescing Brownian motion system was applied in [FT23] to derive the quasi-stationary distributions of the stochastic FKPP equation on the circle. Moreover, the analytical properties of the Wright-Fisher SHE were applied in [BMS24a] to prove the coming down from infinity (CDI) property of LCBM. Building on this result, [BMS24b] showed that a system of branching coalescing Brownian motions, allowing for infinitely many offspring in its branching mechanism, has its total population reflected from infinity. This phenomenon was further used in [BMS24b] to demonstrate a regularization-by-noise effect of the Wright-Fisher space-time white noise.

In this article, we continue the study of the SBBM by addressing the following fundamental question:

- *What happens if there are infinitely many initial particles in an SBBM?*

This question was previously explored in [HT05] in the context of LCBM with a comprehensive resolution provided in [BMS24a]. We will show that, under the assumptions that the catalytic branching is subcritical and not parity-preserving, an SBBM model supporting infinitely many initial particles can be defined as the limit of a sequence of SBBMs with finitely many initial particles (Theorem 1.2). The law of this limiting process is characterized by the initial trace—a key concept introduced in [BMS24a]—and the branching mechanisms. Additionally, we will establish the CDI result: a necessary and sufficient condition for the finiteness of the number of particles in any region at any time (Theorem 1.3 (i) & (ii)); and characterize the corresponding CDI rates in terms of the particle system's mean-field equation (Theorem 1.3 (iii), (iv) & (v)); thereby generalizing the results in [BMS24a]. Our findings reveal an universal behavior: the CDI rates, while depending on the initial trace and the mean-field effect of the pairwise interaction, are independent of the precise form of the branching mechanisms.

1.2. **Main results.** The SBBM model has five parameters:

(1.1) The initial configuration $(x_i)_{i=1}^n$, which is a finite list in $\mathbb{R}$. Without loss of generality, we assume that it is the first n elements of an infinite sequence $(x_i)_{i=1}^\infty$ in $\mathbb{R}$ where $n \in \mathbb{N}$ is arbitrary.
(1.2) The ordinary branching rate $\beta_{\mathrm{o}} \geq 0$;
(1.3) The ordinary offspring law $(p_k)_{k=0}^\infty$, which is a probability measure on $\mathbb{Z}_+$, the space of non-negative integers;
(1.4) The catalytic branching rate $\frac{1}{2}\beta_{\mathrm{c}} > 0$;
(1.5) The catalytic offspring law $(q_k)_{k=0}^\infty$, which is a probability measure on $\mathbb{Z}_+$.

The SBBM with the above parameters is a particle system which evolves according to the following rules (1.6)–(1.9).

(1.6) At time 0, there are n particles located in the real line $\mathbb{R}$ whose locations are given by the initial configuration $(x_i)_{i=1}^n$.
(1.7) Each particle moves as independent Brownian motions unless one of the following ordinary branching or catalytic branching occurs.
(1.8) Each particle induces an ordinary branching according to an independent exponential clock of rate β_{o}. When an ordinary branching occurs, the corresponding particle will be killed and be replaced, at its location of death, by a random number of new particles. This random number will be independently sampled according to the ordinary offspring law $(p_k)_{k=0}^\infty$.
(1.9) Each unordered pair of particles induces a catalytic branching according to an independent exponential clock of rate $\frac{1}{2}\beta_{\mathrm{c}}$ with respect to their intersection local time. When a catalytic branching occurs, the corresponding pair of particles will both be killed and be replaced, at their mutual location of death, by a random number of new particles. This random number will be independently sampled according to the catalytic offspring distribution $(q_k)_{k=0}^\infty$.

Note that, producing one child in an ordinary branching, or two children in a catalytic branching, does not change the configuration of the particle profile at the occurring time of that branching. Therefore, we can assume, without loss of generality, that $p_1 = q_2 = 0$. We say the ordinary branching is subcritical, critical, or supercritical, according to $\sum_{k\in\mathbb{Z}_+} kp_k$ is strictly less than, equals to, or strictly greater than, 1, respectively. We say that the catalytic branching is subcritical, critical, or supercritical, according to $\sum_{k\in\mathbb{Z}_+} kq_k$ is strictly less than, equals to, or strictly greater than, 2, respectively. We will only be considering the case when the catalytic branching is subcritical, that is, we assume that

$$\sum_{k\in\mathbb{Z}_+} kq_k < 2. \tag{1.10}$$

For technical reasons, we also assume the existence of the exponential moments of the two offspring laws:

(1.11) There exists $R > 1$ such that $\sum_{k=0}^\infty R^k p_k < \infty$ and $\sum_{k=0}^\infty R^k q_k < \infty$.

A priori speaking, a particle system following the rules (1.6)–(1.9) can only be defined up to its explosion time. In order to be more precise, let $\tau_0^{(n)} := 0$, and inductively for every $k \in \mathbb{Z}_+$, let $\tau_{k+1}^{(n)}$ be the earliest occurring time of a branching after the time

$\tau_k^{(n)}$ (if $\tau_k^{(n)} = \infty$, or if there is no branching occurring after the time $\tau_k^{(n)} < \infty$, then set $\tau_{k+1}^{(n)} := \infty$ for convention.) Thanks to the strong Markov property of the Brownian motions, one can construct a particle system following the rules (1.6)–(1.9) in the time interval $[0, \tau_\infty^{(n)})$ where $\tau_\infty^{(n)} := \lim_{k\to\infty} \tau_k^{(n)}$ is called the explosion time. (We omit the details of the construction since it is tedious but straightforward.)

Our first result, whose proof is postponed in Section 2, says that this explosion won't really happen.

Proposition 1.1. *Almost surely, $\tau_\infty^{(n)} = \infty$.*

For every $t \geq 0$, we denote by $I_t^{(n)}$ the collection of unique labels of the particles that are alive at time t. (How we label the particles are not crucial for our purpose.) For every $t \geq 0$ and $\alpha \in I_t^{(n)}$, denote by $X_t^{(n),\alpha}$ the spatial location of the particle labeled by α at time t. For every $t \geq 0$ and $U \in \mathcal{B}(\mathbb{R})$, denote by $Z_t^{(n)}(U) := |\{\alpha \in I_t^{(n)} : X_t^{(n),\alpha} \in U\}|$ the number of alive particles at time t whose locations belong to U. Here, $\mathcal{B}(\mathbb{R})$ represents the collection of Borel subsets of $\mathbb{R}$, and $|A|$ represents the cardinality of a given set A.

As a convention, at the occurring time of a branching, we always consider the corresponding children as alive but not their parent. By this convention, $(Z_t^{(n)})_{t\geq 0}$ is a càdlàg process taking values in $\mathcal{N}$, the space of locally finite integer-valued non-negative measures on $\mathbb{R}$. We ask $\mathcal{N}$ to endow the vague topology, i.e. the coarsest topology such that the map $\mu \mapsto \mu(f)$ from $\mathcal{N}$ to $\mathbb{R}$ is continues for every $f \in \mathcal{C}_c(\mathbb{R})$. Here, $\mathcal{C}_c(\mathbb{R})$ represents the collection of compactly supported continuous functions on $\mathbb{R}$; and for any Borel measure ν and Borel measurable function g on $\mathbb{R}$, $\nu(g)$ represents the integral $\int g(x)\nu(\mathrm{d}x)$ whenever it is well-defined. It is known that $\mathcal{N}$ is Polish [Kal17, Theorem 4.2]. For any Borel measure ν and non-negative Borel measurable function g on $\mathbb{R}$, let $g \cdot \nu$ be the unique Borel measure on $\mathbb{R}$ such that $(g \cdot \nu)(A) = \nu(\mathbf{1}_A g)$ for any Borel subset A of $\mathbb{R}$.

Note that the law of the process $(Z_t^{(n)})_{t\geq 0}$ induced on $\mathbb{D}(\mathbb{R}_+, \mathcal{N})$, the space of $\mathcal{N}$-valued càdlàg paths indexed by $\mathbb{R}_+$, is uniquely determined by the parameters listed in (1.1)–(1.5). Any $\mathcal{N}$-valued càdlàg process who shares the same law as $(Z_t^{(n)})_{t\geq 0}$ will be therefore referred to as an SBBM with respect to those parameters.

Notice that $Z_0^n := \sum_{k=1}^n \delta_{x_i}$ where $(x_i)_{i=1}^n$ is the first n elements of the infinite sequence $(x_i)_{i=1}^\infty$. We are interested in the asymptotic behavior of the process $(Z_t^{(n)})_{t>0}$ when $n \to \infty$. If one can show the existence of a limiting process in some sense, then it is reasonable to regard that limit as an SBBM with infinitely many initial particles. With this purpose in mind, we introduce several notations. Let $\mathcal{T}$ be the collection of the pair $(\tilde{\Lambda}, \tilde{\mu})$ where $\tilde{\Lambda}$ is a closed subset of $\mathbb{R}$ and $\tilde{\mu}$ is a Radon measure on $\tilde{\Lambda}^c = \mathbb{R} \setminus \tilde{\Lambda}$. In particular, define $(\Lambda, \mu) \in \mathcal{T}$ so that

$$\Lambda := \left\{ y \in \mathbb{R} : \sum_{i=1}^\infty \mathbf{1}_{(y-r,y+r)}(x_i) = \infty, \forall r > 0 \right\} \tag{1.12}$$

and

$$\mu := \sum_{i=1}^\infty \mathbf{1}_{\Lambda^c}(x_i)\delta_{x_i} \tag{1.13}$$

where the sequence $(x_i)_{i=1}^{\infty}$ is given as in (1.1). This pair (Λ,μ) will be referred to as the initial trace. The support of the initial trace (Λ,μ) is defined by $\mathrm{supp}(\Lambda,\mu) := \Lambda \cup \mathrm{supp}(\mu)$ where $\mathrm{supp}(\mu)$ represents the support of the measure μ. For every $z\in\mathbb{R}$, we define

$$\Phi(z) := \beta_{\mathrm{o}}\left(\sum_{k=0}^{\infty} p_k(1-z)^k - (1-z)\right) \tag{1.14}$$

and

$$\Psi(z) := \beta_{\mathrm{c}}\left(\sum_{k=0}^{\infty} q_k(1-z)^k - (1-z)^2\right) \tag{1.15}$$

whenever the infinite series on the right hand sides are absolutely summable. The functions Φ and Ψ will be referred to as the ordinary branching mechanism and the catalytic branching mechanism, respectively.

Let us also assume the following:

(1.16) The catalytic offspring law is not parity-preserving, that is, there exists an odd number k such that $q_k > 0$.

This assumption is crucial for our establishment of the distributional convergence of $(Z_t^{(n)})_{t>0}$ as $n\to\infty$. We will comment on this assumption in Subsubsection 1.3.1.

Theorem 1.2. *There exists an $\mathcal{N}$-valued càdlàg Markov process $(Z_t)_{t>0}$ such that $(Z_t^{(n)})_{t>0}$ converges to $(Z_t)_{t>0}$ as $n\to\infty$ in finite dimensional distributions. Moreover, the law of the process $(Z_t)_{t>0}$ induced on $\mathbb{D}((0,\infty),\mathcal{N})$, the space of $\mathcal{N}$-valued càdlàg paths indexed by $(0,\infty)$, is uniquely determined by (Λ,μ), Φ and Ψ.*

The proof of Theorem 1.2 is postponed to Section 5.

In light of Theorem 1.2, any $\mathcal{N}$-valued càdlàg process indexed by $(0,\infty)$ who shares the same law as the process $(Z_t)_{t>0}$ in Theorem 1.2 will be referred to as an SBBM with ordinary branching mechanism Φ, catalytic branching mechanism Ψ, and initial trace (Λ,μ).

In the rest of this subsection, let $(Z_t)_{t>0}$ be such a process, whose corresponding probability measure and expectation operator is denoted by $\mathbb{P}_{(\Lambda,\mu)}$ and $\mathbb{E}_{(\Lambda,\mu)}$, respectively. Our next result establishes the CDI property, and the corresponding CDI rates, for this process. Let us denote by $\mathcal{C}^{1,2}((0,\infty)\times\mathbb{R})$ the collection of real-valued functions $(h_t(x))_{t>0,x\in\mathbb{R}}$ which is continuously differentiable in t and twice continuously differentiable in x. For every $(\tilde\Lambda,\tilde\mu)\in\mathcal{T}$, from [LG96, Theorem 4], there exists a unique non-negative $v^{(\tilde\Lambda,\tilde\mu)} = (v_t^{(\tilde\Lambda,\tilde\mu)}(x))_{t>0,x\in\mathbb{R}} \in \mathcal{C}^{1,2}((0,\infty)\times\mathbb{R})$ satisfying the following equation in the classical pointwise sense:

$$\begin{cases} \partial_t v_t^{(\tilde\Lambda,\tilde\mu)}(x) = \frac{1}{2}\partial_x^2 v_t^{(\tilde\Lambda,\tilde\mu)}(x) - \frac{\Psi'(0+)}{2} v_t^{(\tilde\Lambda,\tilde\mu)}(x)^2, \quad t>0, x\in\mathbb{R};\\ \left\{ y\in\mathbb{R} : \forall r>0, \lim_{t\downarrow 0}\int_{y-r}^{y+r} v_t^{(\tilde\Lambda,\tilde\mu)}(x)\mathrm{d}x = \infty \right\} = \tilde\Lambda;\\ \lim_{t\downarrow 0}\int \phi(x) v_t^{(\tilde\Lambda,\tilde\mu)}(x)\mathrm{d}x = \int \phi(x)\tilde\mu(\mathrm{d}x), \quad \phi\in\mathcal{C}_{\mathrm{c}}(\tilde\Lambda^{\mathrm{c}}). \end{cases} \tag{1.17}$$

Here, $\mathcal{C}_c(\tilde{\Lambda}^c)$ is the collection of compactely supported continuous functions on $\tilde{\Lambda}^c$ and

$$\Psi'(0+) := \beta_c\left(2 - \sum_{k=0}^{\infty} kq_k\right) \in (0, \infty). \tag{1.18}$$

Equation (1.17) will be referred to as the mean-field equation of the SBBM $(Z_t)_{t>0}$. We say a set $A \subset \mathbb{R}$ is bounded if $\sup\{|x| : x \in A\} < \infty$. The closure of a set $A \subset \mathbb{R}$ will be denoted by $\bar{A}$.

Theorem 1.3. *For arbitrary open interval $U \subset \mathbb{R}$:*

(i) *If $U \cap \operatorname{supp}(\Lambda, \mu)$ is unbounded, then $\mathbb{P}_{(\Lambda,\mu)}(Z_t(U) = +\infty, \forall t > 0) = 1$.*
(ii) *If $U \cap \operatorname{supp}(\Lambda, \mu)$ is bounded, then $\mathbb{P}_{(\Lambda,\mu)}(Z_t(U) < \infty, \forall t > 0) = 1$.*

Moreover, when $U \cap \operatorname{supp}(\Lambda, \mu)$ is bounded:

(iii) *It holds that $\mathbb{E}_{(\Lambda,\mu)}[Z_t(U)] < \infty$ for every $t > 0$.*
(iv) *If $\bar{U} \cap \Lambda = \emptyset$, then $\limsup_{t\downarrow 0} \mathbb{E}_{(\Lambda,\mu)}[Z_t(U)] < \infty$.*
(v) *If $\bar{U} \cap \Lambda \neq \emptyset$, then $\lim_{t\downarrow 0} \mathbb{E}_{(\Lambda,\mu)}[Z_t(U)] = \infty$ and*

$$\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)^{-1} Z_t(U) \xrightarrow{t\downarrow 0} 1, \quad \text{in } L^1 \text{ w.r.t. } \mathbb{P}_{(\Lambda,\mu)}.$$

The proof of Theorem 1.3 will be given in Section 6. We note here that the CDI rate

$$t \mapsto \int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x, \quad t > 0$$

while dependent on the initial trace (Λ, μ) and the constant $\Psi'(0+)$, which captures the mean-field effect of the pairwise interaction, are independent of the precise form of the branching mechanisms Φ and Ψ.

1.3. **Perspectives.**

1.3.1. *Discussion of assumptions.* The subcritical assumption (1.10) on the catalytic branching is crucial for our results. In the supercritical regime, where $\sum_{k\in\mathbb{Z}_+} kq_k > 2$, a naive physicist's mean-field analysis suggests that, with positive probability, an everywhere explosion occurs in finite time, even when the initial number of particles is finite. This leads to several questions: What is the probability of the explosion? How can the explosion time and the growth of the population before the explosion be characterized? What is the behavior of the system conditioned on non-explosion? We refer our readers to [OP24] where these type of questions are considered for the (non-spatial) branching processes with pairwise interactions.

The critical case, $\sum_{k\in\mathbb{Z}_+} kq_k = 2$, is perhaps more intriguing. We do not expect the CDI property to hold in this case. However, under suitable assumptions and rescaling, we anticipate that the empirical measure of the SBBM with critical catalytic branching is likely to converge to the multiplicative linear SHE, $\partial_t z = \frac{\Delta}{2}z + z\dot{\xi}$, where $\dot{\xi}$ denotes the space-time white noise. Notably, the multiplicative linear SHE is closely related to the KPZ equation, $\partial_t h = \frac{1}{2}\partial_x^2 h - \frac{1}{2}(\partial_x h)^2 + \dot{\xi}$, for which we refer our readers to [Qua12]. This connection invites a further question: Do certain observables of the SBBM with critical catalytic branching belong to the KPZ universality class?

The non-parity-preserving assumption (1.16) is also crucial for our main results. To illustrate this, consider the scenario where $\beta_{\mathrm{o}} = 0$ and $q_0 = 1$, in which (1.16) fails to hold. The SBBM model with these parameters is referred to as the local-time annihilating Brownian motions. Using the methods in this article, it can be shown that for any $t > 0$, the sequence of $\mathcal{N}$-valued random elements $(Z_t^{(n)})_{n\in\mathbb{N}}$ is tight. However, we do not expect the sub-sequential convergence-in-distribution limit to be unique. Instead, we anticipate that, in addition to the initial trace (Λ, μ), one needs to introduce an initial parity field to fully characterize all possible sub-sequential limits. We refer interested readers to [HOV21] for a detailed analysis of such questions in the setting of (instant) annihilating Brownian motions, where particles annihilate immediately upon contact.

The technical assumption (1.11) is included primarily for convenience, facilitating the application of the duality result in [AT00, Theorem 1] (see Proposition 3.3 below). We believe this requirement can be significantly weakened, particularly for ordinary branching mechanisms. Indeed, when the pairwise interaction is given by the coalescing ($q_1 = 1$), [BMS24b] extends the duality result of [AT00, Theorem 1] without imposing any moment conditions on the ordinary branching. Remarkably, their approach even accommodates cases where ordinary branching can produce infinitely many offspring with positive probability. Exploring the optimal moment conditions on the branching mechanisms necessary for our results to hold is an interesting question, but lies beyond the scope of the current paper.

1.3.2. *Coming down from infinity.* The CDI property describes how certain observables in a time-homogeneous dynamical system, starting from an infinite value, become finite immediately after the system starts to evolve. For example, the solution to the ordinary differential equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x(t) = -x(t)^2, & t > 0, \\ x(0) = \infty, \end{cases}$$

which is explicitly given by $x(t) = 1/t$, exhibits the CDI property.

Recently, the CDI phenomenon has also been observed across various stochastic dynamical systems. A well-known example is Kingman's coalescent, where every pair of particles coalesce according to independent exponential clocks [Ald99]. Other examples include the Λ-coalescent, which generalizes Kingman's coalescent by allowing simultaneous mergers of multiple particles [Sch00], [BBL10]; the spatial Λ-coalescent, where particles perform continuous-time independent random walks on a lattice, and particles occupying the same site coalesce according to the usual Λ-coalescent [LS06], [ABL12]; the branching process with pairwise interactions (BPI), which generalizes the Galton-Watson process by incorporating pairwise branchings [OP24]; the logistic continuous-state branching processes (logistic CSBP), which is the continuous-state analog of the BPI where the interaction is competitive [Lam05], [Fou19]; the non-linear CSBP, which generalizes the logistic CSBP by allowing more complex density-dependent interactions [LYZ19], [LZ24]; the time-changed Lévy processes, which is closely related to the non-linear CSBP [FLZ21], [BDS24]; branching random walk with non-local competition [MP24]; and, last but not least, the dynamical Φ^4_3 model, which formally solves the 3-dimensional singular SPDE $\partial_t \rho = \frac{\Delta}{2}\rho - \rho^3 + m\rho + \dot{\xi}$, where $\dot{\xi}$ denotes the space-time white noise [MW17].

1.3.3. *The Mean-field equations.* The general idea of the MFE is to study the behavior of a high-dimensional random model using a simpler model that approximates the original by averaging over degrees of freedom (the number of values that are free to vary). By solving the MFE, some insight into the behavior of the original system can be obtained at a lower computational cost.

In the SBBM model, the values that are free to vary include:

(1.19) the movement of the particles;
(1.20) the occurrence time and the number of children of the ordinary branchings;
(1.21) the occurrence time and the number of children of the catalytic branchings.

We call the following equation

$$\partial_t \tilde{v}_t(x) = \frac{\Delta}{2}\tilde{v}_t(x) + \Phi'(0+)\tilde{v}_t(x) - \frac{\Psi'(0+)}{2}\tilde{v}_t(x)^2, \tag{1.22}$$

subjected to the initial condition similar to that of the equation (1.17), the MFE of the SBBM model, with the idea that $\tilde{v}_t(x)\mathrm{d}x$ is an approximation of the empirical measure, and that the three terms on the right-hand side of (1.22) are the mean-field averages of the three groups of randomness (1.19)–(1.21), respectively. Note that, in Theorem 1.3, the rate of CDI is given by the solution to the equation (1.17) instead of (1.22), where the linear term corresponding to the ordinary branching is absent. This is fine, since one can verify that, uniformly in $x \in \mathbb{R}$, $v_t(x)$ and $\tilde{v}_t(x)$ are asymptotically equivalent as $t \downarrow 0$. By that, we mean $\lim_{t\to 0}\sup_{x\in\mathbb{R}} |v_t(x)/\tilde{v}_t(x) - 1| = 0$.

1.3.4. *Proof Strategy: Duality.* In general, we say a Markov process $(X_t)_{t\geq 0}$, with state space $\mathbb{X}$ and transition kernel $(P_t)_{t\geq 0}$, and a Markov process $(Y_t)_{t\geq 0}$, with state space $\mathbb{Y}$ and transition kernel $(Q_t)_{t\geq 0}$, satisfy a dual relationship w.r.t. a given dual function $H : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$, if

$$\int_{\mathbb{X}} H(x, y_0)P_t(x_0, \mathrm{d}x) = \int_{\mathbb{Y}} H(x_0, y)Q_t(y_0, \mathrm{d}y), \quad x_0 \in \mathbb{X}, y_0 \in \mathbb{Y}.$$

For example, the one-dimensional standard Brownian motion $(B_t)_{t\geq 0}$ and the solution $(h_t)_{t\geq 0}$ to the heat equation

$$\partial_t h_t(x) = \frac{\Delta}{2}h_t(x), \quad t \geq 0, x \in \mathbb{R},$$

which can be regarded as a (deterministic) Markov process with state space $\mathcal{C}_{\mathrm{b}}(\mathbb{R})$, satisfy a dual relationship w.r.t. the dual function $(B, h) \in \mathbb{R} \times \mathcal{C}_{\mathrm{b}}(\mathbb{R}) \mapsto h(B)$. Here, $\mathcal{C}_{\mathrm{b}}(\mathbb{R})$ represents the collection of bounded continuous functions on $\mathbb{R}$.

Our proofs of the main theorems rely on a moment duality between the SBBM $(Z_t)_{t\geq 0}$ and the following stochastic reaction-diffusion equation:

$$\partial_t u_t(x) = \frac{\Delta}{2}u_t(x) - \Phi(u_t(x)) + \sqrt{\Psi(u_t(x))}\dot{W}_t(x), \quad t \geq 0, x \in \mathbb{R}, \tag{1.23}$$

where $\dot{W}$ is a space-time white noise. The corresponding dual function H is given by

$$H(u, Z) := \prod_{x\in\mathbb{R}} (1 - u(x))^{Z(\{x\})}, \quad u \in \mathcal{C}(\mathbb{R}, [0, z^*]), Z \in \mathcal{N}, \tag{1.24}$$

where $z^* \in [1, 2)$ is a constant determined by Ψ. This duality first appeared in [Shi88] for the LCBM and was later generalized for a large family of SBBM in [AT00]. As we will see in this paper, this duality can be further generalized to incorporate infinitely many initial particles. We will be more precise about the solution theory of the SPDE (1.23), the possibly infinite product in (1.24), and the generalized version of this moment dual, in Sections 3, 4, and 5, respectively. We note here that, if there is no catalytic branching, i.e., $\beta_{\mathrm{c}} = 0$, then this duality degenerates to McKean's duality between the BBM and the FKPP equations. Furthermore, if there is no ordinary branching, i.e., $\beta_{\mathrm{o}} = \beta_{\mathrm{c}} = 0$, this duality degenerates to the trivial duality between Brownian motions and the heat equation.

Our proof strategy is similar to that of [BMS24a]. The idea is to compare the above moment dual with a Laplacian dual connecting the MFE (1.22) to a super-Brownian motion, which is a measure-valued Markov process whose density evolves according to the SPDE:

$$\partial_t \tilde{u}_t(x) = \frac{\Delta}{2}\tilde{u}_t(x) - \Phi'(0+)\tilde{u}_t(x) + \sqrt{\Psi'(0+)\tilde{u}_t(x)}\dot{W}_t(x), \quad t \geq 0, x \in \mathbb{R}, \tag{1.25}$$

w.r.t. the dual function $(\tilde{v}, \tilde{u}) \in \mathcal{C}(\mathbb{R})^2 \mapsto \exp\{-\int \tilde{v}(x)\tilde{u}(x)\mathrm{d}x\}$. The super-Brownian motions arise originally as the rescaling limit of the empirical measure of the (near) critical BBM, and its study has expanded into a major area of research over the last few decades. For the precise dual connection between the PDE (1.22) and the SPDE (1.25), see [KS88] and [LG96]. For a modern overview of the super-Brownian motions, we refer our readers to [Li11]. Note that the coefficients $-\Phi'(0+)\tilde{u}_t(x)$ and $\Psi'(0+)\tilde{u}_t(x)$ in the SPDE (1.25) can be considered as the linearizations of $-\Phi(\tilde{u}_t(x))$ and $\Psi(\tilde{u}_t(x))$, respectively. Thus, the behavior of the solutions to the SPDEs (1.23) and (1.25) are similar at small times provided that they share a small initial value. Because of this, many properties of the dual SPDE (1.23), including its compact support property among others, can be obtained via analytical tools such as the weak comparison principle, the Feynman-Kac formula, Itô's formula, and the BDG inequality, etc. It is exactly those analytical results on the dual SPDE (1.23) that can be translated via duality into the CDI property of the SBBM.

We want to mention here that, even though the ideas are the same, the proofs in [BMS24a] also rely heavily on the monotonicity of the LCBM, which is not available for the more general SBBM model. Because of this, the actual proofs are quite different from that of [BMS24a] at several technical levels. For example, the result of Theorem 1.3 (i) for the LCBM is proved in [BMS24a] via a straightforward coupling argument. But here, the proof for the SBBM is much more involved relying on the construction of certain super-martingales. Another technical thing is that one cannot take the logarithm of the dual function H in (1.24) as it is done in [BMS24a] for the sake of a better comparison between the moment dual and the Laplacian dual. This is because the function H might take negative values in our general settings.

Acknowledgement. We want to thank Omer Angle, Clayton Barnes, Louis Fan, Pei-Sen Li, Leonid Mytnik, and Fan Yang for helpful conversations.

Haojie Hou is partially supported by the China Postdoctoral Science Foundation no. 2024M764112. Zhenyao Sun is partially supported by National Key R&D Program of China no. 2023YFA1010102 and National Natural Science Foundation of China no. 12301173.

Zhenyao Sun is the corresponding author.

2. Non-explosion

Let $(x_i)_{i=1}^n$, β_{o}, $(p_k)_{k=0}^\infty$, β_{c} and $(q_k)_{k=0}^\infty$ be given as in (1.1)–(1.5). Assume that (1.10) and (1.11) hold. As it has been mentioned in Subsection 1.2, an SBBM, following the rules (1.6)–(1.9), is a priori defined only up to its explosion time $\tau_\infty^{(n)}$.

In this section, we will prove Proposition 1.1 which says that the explosion won't really happen. Since we are fixing n, the initial number of particles, we will drop the superscript "(n)" to simplify the notations. In particular, $\tau_\infty := \tau_\infty^{(n)}$, and for every $t \in [0, \tau_\infty)$, we denote by $I_t := I_t^{(n)}$ the collection of unique labels of the particles that is alive at time t. For every $t \in [0, \tau_\infty)$ and $\alpha \in I_t$, denote by $X_t^\alpha := X_t^{(n),\alpha}$ the spatial location of the particle labeled by α at time t. For every $t \in [0, \tau_\infty)$ and $U \in \mathcal{B}(\mathbb{R})$, denote by $Z_t(U) := Z_t^{(n)}(U) = |\{\alpha \in I_t : X_t^\alpha \in U\}|$ the number of alive particles at time t whose locations belong to U. Also, define

$$\lambda_{\mathrm{o}} := \beta_{\mathrm{o}} \sum_{k=0}^\infty k p_k \in [0, \infty), \qquad \lambda_{\mathrm{c}} := \beta_{\mathrm{c}} \sum_{k=0}^\infty k q_k \in [0, \infty), \tag{2.1}$$

and

$$\Phi'(0+) := \beta_{\mathrm{o}} \left(1 - \sum_{k=0}^\infty k p_k \right) = \beta_{\mathrm{o}} - \lambda_{\mathrm{o}}. \tag{2.2}$$

Let us warm up with a simpler version of PropotisIon 1.1 by excluding the ordinary branching. That is, we first consider the case that $\beta_{\mathrm{o}} = 0$. For every $t \in [0, \infty]$, define $N_t := |\{k \in \mathbb{N} : \tau_k < t\}|$ to be the total number of branchings in the time interval $[0, t)$. In particular, N_∞ is the total number of branchings that will ever occur in finite time.

Lemma 2.1. *Suppose that there is no ordinary branching, i.e. $\beta_{\mathrm{o}} = 0$, then $N_\infty < \infty$ almost surely. In particular, $\tau_\infty \geq \tau_{N_\infty+1} = \infty$ almost surely.*

Proof. In this case, we know that all the branchings are catalytic branchings. It is not hard to see that the process $(Z_{\tau_k}(\mathbb{R}))_{k=0}^{N_\infty}$ is a Markov chain taking values in $\mathbb{Z}_+$ stopped at the (possibly infinite) random step N_∞. Observe from (1.6) and (1.9), the initial value of this Markov chain is n, and the corresponding transition matrix is

$$P_{i,j} := \mathbf{1}_{\{i \geq 2, j \geq i-2\}} q_{j+2-i} + \mathbf{1}_{\{i=j=1\}} + \mathbf{1}_{\{i=j=0\}}, \quad i, j \in \mathbb{Z}_+.$$

Since we assumed that $q_2 = 0$, the only two absorbing states of this Markov chain are 0 and 1. From (1.11) and the standard theory of the Markov chains, we can verify that a Markov chain with the above transition matrix P will be absorbed by its absorbing states $\{0, 1\}$ in finite steps almost surely. Now, from the fact that the 1-d Brownian motion is recurrent, it is not hard to see that $(Z_{\tau_k}(\mathbb{R}))_{k=0}^{N_\infty}$ is exactly a Markov chain with the transition matrix P stopped upon hitting its absorbing states $\{0, 1\}$. Therefore, $N_\infty < \infty$ almost surely, as desired. □

In the rest of this section, let us consider the more general case $\beta_{\mathrm{o}} \geq 0$. Lemma 2.1 says that the SBBM can be defined up to all finite time if $\beta_{\mathrm{o}} = 0$. This fact is crucial for a coupling argument below, where we couple two SBBMs together with one of them not allowed to have ordinary branching.

Define $\tau_0^{\mathrm{o}} := 0$, and inductively for every $k \in \mathbb{Z}_+$, let τ_{k+1}^{o} be the earliest occurring time of an ordinary branching after the time τ_k^{o} (if $\tau_k^{\mathrm{o}} = \tau_\infty$, or if $\tau_k^{\mathrm{o}} < \tau_\infty$ and there is no ordinary branching occurring in the time interval $(\tau_k^{\mathrm{o}}, \tau_\infty)$, we define $\tau_{k+1}^{\mathrm{o}} := \tau_\infty$ for convention.) Similarly, denote by $(\tau_k^{\mathrm{c}})_{k\in\mathbb{Z}_+}$ the occuring times of the catalytic branchings.

Lemma 2.2. *Almost surely, for every $k \in \mathbb{Z}_+$, there are only finitely many catalytic branching occurring in the time interval $(\tau_k^{\mathrm{o}}, \tau_{k+1}^{\mathrm{o}})$.*

Proof. Let us fix an arbitrary $k \in \mathbb{Z}_+$. On one hand, almost surely on the event $\{\tau_k^{\mathrm{o}} = \tau_\infty\}$, the time interval $(\tau_k^{\mathrm{o}}, \tau_{k+1}^{\mathrm{o}}) = \emptyset$. So obviously, there is no catalytic branching occurring in this empty time interval. On the other hand, by the strong Markov property of the Brownian motions, the process $(Z_t)_{t\in[0,\tau_\infty)}$ can be coupled with a process $(\tilde{Z}_t)_{t\geq 0}$, which is an SBBM without the ordinary branching, such that almost surely on the event $\{\tau_k^{\mathrm{o}} < \tau_\infty\}$, $Z_{t+\tau_k^{\mathrm{o}}} = \tilde{Z}_t$ for every $t \in [0, \tau_{k+1}^{\mathrm{o}} - \tau_k^{\mathrm{o}})$. From Lemma 2.1, we know that $(\tilde{Z}_t(\mathbb{R}))_{t\geq 0}$ jumps only finite many times. So almost surely on the event $\{\tau_k^{\mathrm{o}} < \tau_\infty\}$, $(Z_t(\mathbb{R}))_{t\in(\tau_k^{\mathrm{o}},\tau_{k+1}^{\mathrm{o}})}$ jumps only finite many times. The desired result now follows. □

Lemma 2.3. *For every $m \in \mathbb{N}$, almost surely on the event $\{\tau_m^{\mathrm{o}} = \tau_\infty\}$, $\tau_m^{\mathrm{o}} = \tau_\infty = \infty$.*

Proof. Almost surely on the event $\{\tau_m^{\mathrm{o}} = \tau_\infty\}$, there exists an $N \in \mathbb{N}$ such that $\tau_N^{\mathrm{o}} < \tau_{N+1}^{\mathrm{o}} = \tau_\infty$. Therefore, almost surely on this event, the time interval $[0, \tau_\infty)$ can be decomposed into finitely many disjoint sub-intervals in the following way:

$$[0, \tau_\infty) = \bigcup_{k=0}^{N} [\tau_k^{\mathrm{o}}, \tau_{k+1}^{\mathrm{o}}). \tag{2.3}$$

Note from Lemma 2.2 that, almost surely on the event $\{\tau_m^{\mathrm{o}} = \tau_\infty\}$, there are only finitely many catalytic branchings occurring in each of the sub-intervals on the right hand side of (2.3). Therefore, almost surely on the event $\{\tau_m^{\mathrm{o}} = \tau_\infty\}$, $N_{\tau_\infty} < \infty$, i.e. there are only finitely many branchings occurring before the explosion; from how the explosion time τ_∞ is defined, we must have $\tau_\infty = \infty$, as desired. □

In the rest of this section, let us fix an arbitrary $m \in \mathbb{N}$ and define a new process $(Z_t^{\mathrm{o},m})_{t\geq 0}$ so that almost surely on the event $\{\tau_m^{\mathrm{o}} = \tau_\infty\} = \{\tau_m^{\mathrm{o}} = \tau_\infty = \infty\}$, $Z_t^{\mathrm{o},m} = Z_t$

for every $t \in [0, \infty)$; and almost surely on the event $\{\tau_m^{\mathrm{o}} < \tau_\infty\}$,

$$Z_t^{\mathrm{o},m} := \begin{cases} Z_t, & t \in [0, \tau_m^{\mathrm{o}}), \\ \tilde{Z}_{t-\tau_m^{\mathrm{o}}}^{\mathrm{o},m}, & t \in [\tau_m^{\mathrm{o}}, \infty), \end{cases}$$

where $(\tilde{Z}_t^{\mathrm{o},m})_{t\geq 0}$ is an SBBM without ordinary branching whose initial value is given by $Z_{\tau_m^{\mathrm{o}}}$. Loosely speaking, $(Z_t^{\mathrm{o},m})_{t\geq 0}$ is an SBBM with at most m many ordinary branchings allowed.

Lemma 2.4. *The process* $(e^{\Phi'(0+)(t\wedge\tau_m^{\mathrm{o}})} Z_t^{\mathrm{o},m}(\mathbb{R}))_{t\geq 0}$ *is a local super-martingale where* $\Phi'(0+)$ *is given as in* (2.2).

Proof. Consider the following disjoint decomposition of the time interval $[0, \infty)$:

$$[0, \infty) = \left(\bigcup_{k=0}^{m-1} [\tau_k^{\mathrm{o}}, \tau_{k+1}^{\mathrm{o}}) \right) \bigcup [\tau_m^{\mathrm{o}}, \infty). \tag{2.4}$$

Almost surely, in each of the time intervals on the right hand side of (2.4), we can verify that the integer-valued process $Z_\cdot^{\mathrm{o},m}(\mathbb{R})$ jumps only finite many times. (To see this, we apply Lemma 2.2 to the intervals $[\tau_k^{\mathrm{o}}, \tau_{k+1}^{\mathrm{o}})$ with $k \in \{0, \dots, m-1\}$, and apply Lemma 2.1 for the last interval $[\tau_m^{\mathrm{o}}, \infty)$.) Therefore, $Z_\cdot^{\mathrm{o},m}(\mathbb{R})$ jumps only finite many times in the full time interval $[0, \infty)$.

Now, we have the finite sum decomposition

$$Z_t^{\mathrm{o},m}(\mathbb{R}) - n = \sum_{s\in(0,t]\cap\{\tau_1^{\mathrm{o}},\dots,\tau_m^{\mathrm{o}}\}} \mathbf{\Delta} Z_s^{\mathrm{o},m}(\mathbb{R}) + \sum_{s\in(0,t]\setminus\{\tau_1^{\mathrm{o}},\dots,\tau_m^{\mathrm{o}}\}} \mathbf{\Delta} Z_s^{\mathrm{o},m}(\mathbb{R}), \quad t \geq 0, \text{a.s.} \tag{2.5}$$

where $\mathbf{\Delta}\gamma_t := \gamma_t - \gamma_{t-}$ for every $t \geq 0$ and real-valued càdlàg process $(\gamma_t)_{t\geq 0}$. The first term on the right hand side of (2.5) are the jumps induced by the ordinary branchings, and the second term are the jumps induced by the catalytic branchings. From how those jumps are induced, we can find their compensators. In fact, it is not hard to see that the processes:

$$M_t^{\mathrm{o},m} := \sum_{s\in(0,t]\cap\{\tau_1^{\mathrm{o}},\dots,\tau_m^{\mathrm{o}}\}} \mathbf{\Delta} Z_s^{\mathrm{o},m}(\mathbb{R}) + \Phi'(0+) \int_0^{t\wedge\tau_m^{\mathrm{o}}} Z_s^{\mathrm{o},m}(\mathbb{R}) \mathrm{d}s, \quad t \geq 0,$$

and

$$M_t^{\mathrm{c},m} := \sum_{s\in(0,t]\setminus\{\tau_1^{\mathrm{o}},\dots,\tau_m^{\mathrm{o}}\}} \mathbf{\Delta} Z_s^{\mathrm{o},m}(\mathbb{R}) + \frac{1}{2}\Psi'(0+) L_t^{\mathrm{o},m}, \quad t \geq 0,$$

are local-martingales (c.f. [BMS24b, Lemma 3.3].) Here, $L_t^{\mathrm{o},m}$ is the total amount of intersection local times induced by all the unordered pairs of atoms in the process of atomic measures $Z_\cdot^{\mathrm{o},m}$ up to the time $t \geq 0$; and $\Psi'(0+) > 0$ according to (1.11). Therefore, almost surely,

$$Z_t^{\mathrm{o},m}(\mathbb{R}) - n = M_t^{\mathrm{o},m} + M_t^{\mathrm{c},m} - \Phi'(0+) \int_0^{t\wedge\tau_m^{\mathrm{o}}} Z_s^{\mathrm{o},m}(\mathbb{R}) \mathrm{d}s - \frac{1}{2}\Psi'(0+) L_t^{\mathrm{o},m}, \quad t \geq 0.$$

Now, almost surely for every $t \geq 0$,

$$\begin{aligned} & e^{\Phi'(0+)(t\wedge\tau^{\mathrm{o}}_m)} Z^{\mathrm{o},m}_t(\mathbb{R}) - n \\ &= \int_0^t e^{\Phi'(0+)(s\wedge\tau^{\mathrm{o}}_m)} \mathrm{d}Z^{\mathrm{o},m}_s(\mathbb{R}) + \Phi'(0+) \int_0^{t\wedge\tau^{\mathrm{o}}_m} Z^{\mathrm{o},m}_s(\mathbb{R}) e^{\Phi'(0+)s}\mathrm{d}s \\ &= \int_0^t e^{\Phi'(0+)(s\wedge\tau^{\mathrm{o}}_m)} (\mathrm{d}M^{\mathrm{o},m}_s + \mathrm{d}M^{\mathrm{c},m}_s) - \frac{1}{2}\Psi'(0+) \int_0^t e^{\Phi'(0+)(s\wedge\tau^{\mathrm{o}}_m)} \mathrm{d}L^{\mathrm{o},m}_s(\mathbb{R}). \end{aligned} \tag{2.6}$$

Note that the first term on the right hand side of (2.6) is a local-martingale, while the second term is a non-increasing process with locally finite variation. The desired result follows. □

For every $t \geq 0$, define a random variable $N^{\mathrm{o},m}_t := |\{k \in \mathbb{N} : \tau^{\mathrm{o}}_k < t, k \leq m\}|$ which is the total number of ordinary branchings for the process $(Z^{\mathrm{o},m}_t)_{t\geq 0}$ in the time interval $[0,t)$.

Lemma 2.5. *For every $t \geq 0$, $\mathbb{E}[N^{\mathrm{o},m}_t] \leq n\beta_{\mathrm{o}} \int_0^t e^{-\Phi'(0+)s}\mathrm{d}s$.*

Proof. From Lemma 2.4, the process $(e^{\Phi'(0+)(t\wedge\tau^{\mathrm{o}}_m)} Z^{\mathrm{o},m}_t(\mathbb{R}))_{t\geq 0}$ is a local super-martingale. Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of stopping times increasingly converges to ∞ such that $(e^{\Phi'(0+)(t\wedge\rho_k\wedge\tau^{\mathrm{o}}_m)} Z^{\mathrm{o},m}_{t\wedge\rho_k}(\mathbb{R}))_{t\geq 0}$ is a (true) super-martingale for every $k \in \mathbb{N}$. By Fatou's lemma, for every $t \geq 0$,

$$\begin{aligned} \mathbb{E}\Big[e^{\Phi'(0+)(t\wedge\tau^{\mathrm{o}}_m)} Z^{\mathrm{o},m}_t(\mathbb{R})\Big] &= \mathbb{E}\Big[\lim_{k\uparrow\infty} e^{\Phi'(0+)(t\wedge\rho_k\wedge\tau^{\mathrm{o}}_m)} Z^{\mathrm{o},m}_{t\wedge\rho_k}(\mathbb{R})\Big] \\ &\leq \liminf_{k\to\infty} \mathbb{E}\Big[e^{\Phi'(0+)(t\wedge\rho_k\wedge\tau^{\mathrm{o}}_m)} Z^{\mathrm{o},m}_{t\wedge\rho_k}(\mathbb{R})\Big] \leq n. \end{aligned}$$

In particular, for every $t \geq 0$,

$$\mathbb{E}\Big[\int_0^{t\wedge\tau^{\mathrm{o}}_m} Z^{\mathrm{o},m}_s(\mathbb{R})\mathrm{d}s\Big] \leq \int_0^t e^{-\Phi'(0+)s} \mathbb{E}\Big[e^{\Phi'(0+)(s\wedge\tau^{\mathrm{o}}_m)} Z^{\mathrm{o},m}_s(\mathbb{R})\Big]\mathrm{d}s \leq n \int_0^t e^{-\Phi'(0+)s}\mathrm{d}s.$$

Note that the following holds almost surely for every $t \geq 0$:

$$N^{\mathrm{o},m}_t \leq N^{\mathrm{o},m}_{t+} = \sum_{s\in(0,t]\cap\{\tau^{\mathrm{o}}_1,\dots,\tau^{\mathrm{o}}_m\}} 1.$$

From how the ordinary branching are induced, we can find the compensator for the process $(N^{\mathrm{o},m}_{t+})_{t\geq 0}$. In fact, it is not hard to see that the process

$$\hat{N}^{\mathrm{o},m}_{t+} := N^{\mathrm{o},m}_{t+} - \int_0^{t\wedge\tau^{\mathrm{o}}_m} \beta_{\mathrm{o}} Z^{\mathrm{o},m}_s(\mathbb{R})\mathrm{d}s, \quad t \geq 0,$$

is a (true) martingale (c.f. [BMS24b, Lemma 3.3].) Now we have

$$\mathbb{E}[N^{\mathrm{o},m}_t] \leq \mathbb{E}[N^{\mathrm{o},m}_{t+}] = \mathbb{E}\Big[\int_0^{t\wedge\tau^{\mathrm{o}}_m} \beta_{\mathrm{o}} Z^{\mathrm{o},m}_s(\mathbb{R})\mathrm{d}s\Big] \leq n \int_0^t \beta_{\mathrm{o}} e^{-\Phi'(0+)s}\mathrm{d}s,$$

as desired. □

We are now ready to present the proof of Proposition 1.1.

Proof of Proposition 1.1. Note that m is arbitrarily chosen, and by lemma 2.5,

$$\mathbb{E}[|\{k \in \mathbb{N} : \tau_k^{\mathrm{o}} < t, k \leq m\}|] \leq n \int_0^t \beta_{\mathrm{o}} e^{-\Phi'(0+)s} \mathrm{d}s, \quad t \geq 0.$$

Taking $m \uparrow \infty$, we obtain from the monotone convergence theorem that

$$\mathbb{E}[|\{k \in \mathbb{N} : \tau_k^{\mathrm{o}} < t\}|] \leq n \int_0^t \beta_{\mathrm{o}} e^{-\Phi'(0+)s} \mathrm{d}s, \quad t \geq 0.$$

In particular, we can define the almost surely finite random variable

$$N_t^{\mathrm{o},\infty} := |\{k \in \mathbb{N} : \tau_k^{\mathrm{o}} < t\}| < \infty, \quad t \geq 0.$$

Therefore, almost surely $\tau_\infty \geq \tau_k^{\mathrm{o}}|_{k=N_t^{\mathrm{o},\infty}+1} \geq t$ for every $t \geq 0$. This implies the desired result. □

In the rest of this section, we establish a result which will be used later in Section 5. As it has been explained in Subsection 1.2 that, from Proposition 1.1, $(Z_t)_{t\geq 0}$ is a càdlàg process taking values in $\mathcal{N}$. It is also clear that $Z_t(\mathbb{R}) < \infty$ almost surely for every $t \geq 0$.

Proposition 2.6. *Suppose that g is a smooth function with bounded derivatives of any orders. Then the process*

$$\begin{aligned} M_t^g := {} & e^{\Phi'(0+)t} Z_t(g) - \frac{1}{2} \int_0^t e^{\Phi'(0+)s} Z_s(g'') \mathrm{d}s \\ & + \frac{1}{2} \Psi'(0+) \int_0^t e^{\Phi'(0+)s} \sum_{\{\alpha,\beta\} \subset I_s : \alpha \neq \beta} g(X_s^\alpha) \mathrm{d}L_s^{\{\alpha,\beta\}}, \quad t \geq 0 \end{aligned} \tag{2.7}$$

is a (true) martingale. Here, $(L_t^{\{\alpha,\beta\}})_{t\geq 0}$, the intersection local time between any two particles labelled by α and β, is defined as the unique continuous process such that

$$L_t^{\{\alpha,\beta\}} := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{\{\{\alpha,\beta\} \subset I_s\} \cap \{|X_s^\alpha - X_s^\beta| \leq \epsilon\}} \mathrm{d}s, \quad a.s., \ t \geq 0.$$

Proof. Define $\Theta^{\mathrm{o}} := \{\tau_k^{\mathrm{o}} : k \in \mathbb{N}\} \cap [0, \infty)$, and $\Theta^{\mathrm{c}} := \{\tau_k^{\mathrm{c}} : k \in \mathbb{N}\} \cap [0, \infty)$, the set of the occurring times of the ordinary branchings, and the set of the occurring times of the catalytic branchings, respectively. From Proposition 1.1, we know that almost surely for every $t \geq 0$, $\Theta^{\mathrm{o}} \cap (0, t]$ and $\Theta^{\mathrm{c}} \cap (0, t]$ are finite sets. Noticing that, between each two consecutive branching times, $Z_\cdot$ evolves as the empirical measure of a system of independent Brownian motions. Therefore, we have the decomposition that almost surely for every $t \geq 0$,

$$Z_t(g) - Z_0(g) = \tilde{m}_t^{g'} + \frac{1}{2} \int_0^t Z_s(g'') \mathrm{d}s + \sum_{s \in \Theta^{\mathrm{o}} \cap (0,t]} \mathbf{\Delta} Z_s(g) + \sum_{s \in \Theta^{\mathrm{c}} \cap (0,t]} \mathbf{\Delta} Z_s(g)$$

where $\tilde{m}_\cdot^{g'}$ is a continuous local martingale with quadratic variation

$$\left\langle \tilde{m}_\cdot^{g'} \right\rangle_t = \int_0^t Z_s\big((g')^2\big) \mathrm{d}s, \quad t \geq 0.$$

Here, $\mathbf{\Delta}\gamma_s := \gamma_s - \gamma_{s-}$ for any $s \geq 0$ and real-valued càdlàg process $(\gamma_t)_{t\geq 0}$. Let $\mathcal{U}$ be the (deterministic) countable set of all possible labels of the particles, and let $\mathcal{U}_2 := \{\{\alpha, \beta\} \subset \mathcal{U} : \alpha \neq \beta\}$ be the set of all the possible unordered pairs of labels. It is standard to see that

$$\sum_{s\in\Theta^{\mathrm{o}}\cap(0,t]} \mathbf{\Delta} Z_s(g) = \int_{\mathbb{Z}_+\times\mathcal{U}\times(0,t]} (k-1)g(X^\alpha_{s-})N^{\mathrm{o}}(\mathrm{d}k, \mathrm{d}\alpha, \mathrm{d}s), \quad t \geq 0, \text{a.s.}$$

and

$$\sum_{s\in\Theta^{\mathrm{c}}\cap(0,t]} \mathbf{\Delta} Z_s(g) = \int_{\mathbb{Z}_+\times\mathcal{U}_2\times(0,t]} (k-2)g(X^\alpha_{s-})N^{\mathrm{c}}(\mathrm{d}k, \mathrm{d}\{\alpha,\beta\}, \mathrm{d}s), \quad t \geq 0, \text{a.s.}$$

where N^{o} is a point process on $\mathbb{Z}_+ \times \mathcal{U} \times \mathbb{R}_+$ with compensator

$$\hat{N}^{\mathrm{o}}(\{k\} \times \{\alpha\} \times \mathrm{d}s) := \beta_{\mathrm{o}} p_k \mathbf{1}_{\{\alpha\in I_s\}}\mathrm{d}s, \quad (k, \alpha, s) \in \mathbb{Z}_+ \times \mathcal{U} \times \mathbb{R}_+,$$

and N^{c} is a point process on $\mathbb{Z}_+ \times \mathcal{U}_2 \times \mathbb{R}$ with compensator

$$\hat{N}^{\mathrm{c}}(\{k\} \times \{\{\alpha,\beta\}\} \times \mathrm{d}s) := \frac{1}{2}\beta_{\mathrm{c}} q_k \mathrm{d}L^{\{\alpha,\beta\}}_s, \quad (k, \{\alpha,\beta\}, s) \in \mathbb{Z}_+ \times \mathcal{U}_2 \times \mathbb{R}_+,$$

in the sense of [IW89, Definition 3.1]. Now, by Ito's formula [IW89, Theorem 5.1], we have almost surely for every $t \geq 0$,

$$\begin{aligned}
&e^{\Phi'(0+)t}Z_t(g) - Z_0(g)\\
&= \int_0^t e^{\Phi'(0+)s}\mathrm{d}\tilde{m}^{g'}_s + \frac{1}{2}\int_0^t e^{\Phi'(0+)s}Z_s(g'')\mathrm{d}s + \Phi(0+)\int_0^t Z_s(g)e^{\Phi'(0+)s}\mathrm{d}s\\
&\quad + \int_{\mathbb{Z}_+\times\mathcal{U}\times(0,t]} e^{\Phi'(0+)s}(k-1)g(X^\alpha_{s-})N^{\mathrm{o}}(\mathrm{d}k, \mathrm{d}\alpha, \mathrm{d}s)\\
&\quad + \int_{\mathbb{Z}_+\times\mathcal{U}_2\times(0,t]} e^{\Phi'(0+)s}(k-2)g(X^\alpha_{s-})N^{\mathrm{c}}(\mathrm{d}k, \mathrm{d}\{\alpha,\beta\}, \mathrm{d}s).
\end{aligned}$$

Observe that $(M^g_t)_{t\geq 0}$ is a local martingale, since almost surely for every $t \geq 0$,

$$M^g_t = Z_0(g) + m^{g'}_t + m^{\mathrm{o},g}_t + m^{\mathrm{c},g}_t$$

where

$$m^{g'}_t := \int_0^t e^{\Phi'(0+)s}\mathrm{d}\tilde{m}^{g'}_s,$$

$$m^{\mathrm{o},g}_t := \int_{\mathbb{Z}_+\times\mathcal{U}\times(0,t]} e^{\Phi'(0+)s}(k-1)g(X^\alpha_{s-})\Big(N^{\mathrm{o}}(\mathrm{d}k, \mathrm{d}\alpha, \mathrm{d}s) - \hat{N}^{\mathrm{o}}(\mathrm{d}k, \mathrm{d}\alpha, \mathrm{d}s)\Big),$$

and

$$m^{\mathrm{c},g}_t := \int_{\mathbb{Z}_+\times\mathcal{U}_2\times(0,t]} e^{\Phi'(0+)s}(k-2)g(X^\alpha_{s-})\Big(N^{\mathrm{c}}(\mathrm{d}k, \mathrm{d}\{\alpha,\beta\}, \mathrm{d}s) - \hat{N}^{\mathrm{c}}(\mathrm{d}k, \mathrm{d}\{\alpha,\beta\}, \mathrm{d}s)\Big).$$

Now, replacing g by $\mathbf{1}_\mathbb{R}$ in (2.7), from the fact that $\Psi'(0+) > 0$, it is easy to see that $(e^{\Phi'(0+)t}Z_t(\mathbb{R}))_{t\geq 0}$ is a local super-martingale. Therefore, there exists a sequence of optional times $(\rho_k)_{k\in\mathbb{N}}$ converging increasingly to ∞ such that $(e^{\Phi'(0+)t\wedge\rho_k}Z_{t\wedge\rho_k}(\mathbb{R}))_{t\geq 0}$ is a super-martingale for every $k \in \mathbb{N}$. By Fatou's lemma, for every $t \geq 0$,

$$\mathbb{E}\Big[e^{\Phi'(0+)t}Z_t(\mathbb{R})\Big] \leq \liminf_{k\to\infty} \mathbb{E}\Big[e^{\Phi'(0+)t\wedge\rho_k}Z_{t\wedge\rho_k}(\mathbb{R})\Big] \leq Z_0(\mathbb{R}) = n. \tag{2.8}$$

There also exists a sequence of optional times $(\rho'_k)_{k\in\mathbb{N}}$ converging increasingly to ∞ such that $(M^{\mathbf{1}_\mathbb{R}}_{t\wedge\rho'_k})_{t\geq 0}$ is a martingale for every $k \in \mathbb{N}$. Taking expectation on the both sides of (2.7) while replacing g by $\mathbf{1}_\mathbb{R}$ and t by $t \wedge \rho'_k$, we obtain that for every $k \in \mathbb{N}$ and $t \geq 0$,

$$n = \mathbb{E}\Big[M^{\mathbf{1}_\mathbb{R}}_{t\wedge\rho'_k}\Big] = \mathbb{E}\Big[e^{\Phi'(0+)t\wedge\rho'_k}Z_{t\wedge\rho'_k}(\mathbb{R})\Big] + \frac{1}{2}\Psi'(0+)\mathbb{E}\left[\int_0^{t\wedge\rho'_k} e^{\Phi'(0+)s} \sum_{\{\alpha,\beta\}\subset I_s:\alpha\neq\beta} \mathrm{d}L^{\{\alpha,\beta\}}_s\right].$$

Taking $k \uparrow \infty$, from the monotone convergence theorem, we have for any $t \geq 0$,

$$\mathbb{E}\left[\int_0^t e^{\Phi'(0+)s} \sum_{\{\alpha,\beta\}\subset I_s:\alpha\neq\beta} \mathrm{d}L^{\{\alpha,\beta\}}_s\right] \leq 2n/\Psi'(0+). \tag{2.9}$$

From (2.8), we can verify that

$$\mathbb{E}\left[\int_0^t e^{2\Phi'(0+)s}Z_s\big((g')^2\big)\mathrm{d}s\right] < \infty, \tag{2.10}$$

$$\mathbb{E}\left[\int_{\mathbb{Z}_+\times\mathcal{U}\times(0,t]} \left|e^{\Phi'(0+)s}(k-1)g(X^\alpha_{s-})\right|\hat{N}^{\mathrm{o}}(\mathrm{d}k,\mathrm{d}\alpha,\mathrm{d}s)\right] < \infty, \tag{2.11}$$

and from (2.9) that

$$\mathbb{E}\left[\int_{\mathbb{Z}_+\times\mathcal{U}_2\times(0,t]} \left|e^{\Phi'(0+)s}(k-2)g(X^\alpha_{s-})\right|\hat{N}^{\mathrm{c}}(\mathrm{d}k,\mathrm{d}\{\alpha,\beta\},\mathrm{d}s)\right] < \infty. \tag{2.12}$$

From (2.10), (2.11) and (2.12) we can verify that $(m^{g'}_t)_{t\geq 0}$, $(m^{\mathrm{o},g}_t)_{t\geq 0}$ and $(m^{\mathrm{c},g}_t)_{t\geq 0}$ are (true) martingales, respectively. The desired result of this proposition follows. □

3. The dual SPDEs

Let the parameters $(x_i)_{i=1}^n$, β_o, $(p_k)_{k=0}^\infty$, β_c and $(q_k)_{k=0}^\infty$ be given as in (1.1)–(1.5). Assume that (1.10) and (1.11) hold. Due to Proposition 1.1 (which is proved in the previous section), an SBBM w.r.t. above parameters can be constructed up to all time. Let $(I^{(n)}_t)_{t\geq 0}$, $(X^{(n),\alpha}_t)_{\alpha\in I^{(n)}_t, t\geq 0}$ and $(Z^{(n)}_t)_{t\geq 0}$ be the corresponding notations for this SBBM, given as in Subsection 1.2 (right after Proposition 1.1.)

In this section, we discuss the duality relation between this SBBM and the following 1-d stochastic partial differential equation (SPDE)

$$\begin{cases} \partial_t u_t(x) = \dfrac{\Delta}{2}u_t(x) - \Phi(u_t(x)) + \sqrt{\Psi(u_t(x))}\dot{W}_{t,x}, \quad t > 0, x \in \mathbb{R}, \\ u_0(x) = f(x), \quad x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where Φ and Ψ are defined as in (1.14) and (1.15) respectively, and $\dot{W}$ is a space-time white noise on $[0,\infty)\times\mathbb{R}$. We need to be careful about the solution concept of the SPDE (3.1). In particular, we want the random variable $u_t(x)$, for every $t\geq 0$ and $x\in\mathbb{R}$, to take its values in a subinterval of $\mathbb{R}$ such that $\Psi(u_t(x))$ is non-negative. To this end, let us first analyze the function $\Psi(\cdot)$. Define

$$z^* := \inf\{z\in[1,2] : \Psi(z)=0\}$$

with the convention that $\inf\emptyset=\infty$. The following analytic lemma suggests that the random field $(u_t(x))_{t\geq 0,x\in\mathbb{R}}$ should take its value in $[0,z^*]$. (We include the proof of this analytic lemma in the Supplement Material [HS25].)

Lemma 3.1. *It can be verified that $z^*\in[1,2]$, $\Psi(z^*)=0$, $\Phi(z^*)\geq 0$, $\Phi(0)=\Psi(0)=0$, and $\Psi(z)\geq 0$ for every $z\in[0,z^*]$. Furthermore, if (1.16) holds then $z^*<2$.*

We now give the solution concept of SPDE (3.1). Denote by $\mathcal{C}(\mathbb{R},[0,z^*])$ the collection of $[0,z^*]$-valued continuous functions on $\mathbb{R}$, equipped with the topology of uniform convergence on compact sets. Let f, the initial value of the SPDE (3.1), be an arbitrary element of $\mathcal{C}(\mathbb{R},[0,z^*])$. We say $(\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t)_{t\geq 0},\tilde{\mathbb{P}}_f,W)$ is a stochastic basis, if $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}_f)$ is a complete probability space equipped with an augmented filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ in the sense of [Kal21, Lemma 9.8], and $W:=(W_t(\phi):t\geq 0,\phi\in L^2(\mathbb{R}))$ is an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-adapted cylindrical Wiener process on $L^2(\mathbb{R})$ with covariance structure

$$\tilde{\mathbb{E}}_f[W_t(\phi)W_s(\psi)]=(t\wedge s)\int\phi(x)\psi(x)\mathrm{d}x,\quad t,s\geq 0,\phi,\psi\in L^2(\mathbb{R})$$

in the sense of [DPZ14, Section 4.1.2.]. Given a stochastic basis $(\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t)_{t\geq 0},\tilde{\mathbb{P}}_f,W)$, we say an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-adapted $\mathcal{C}(\mathbb{R},[0,z^*])$-valued continuous process $(u_t)_{t\geq 0}$ solves the SPDE (3.1), if $u_0=f$ and for every $(t,x)\in(0,\infty)\times\mathbb{R}$, almost surely

$$u_t(x)=\int p_t(x-y)f(y)\mathrm{d}y-\iint_0^t p_{t-s}(x-y)\Phi(u_s(y))\mathrm{d}s\mathrm{d}y+ \iint_0^t p_{t-s}(x-y)\sqrt{\Psi(u_s(y))}W(\mathrm{d}s\mathrm{d}y). \tag{3.2}$$

Here,

$$p_t(x):=e^{-x^2/(2t)}/\sqrt{2\pi t},\quad (t,x)\in(0,\infty)\times\mathbb{R}$$

is the heat kernel, and the third term on the right hand side of (3.2) is the stochastic integral driven by the space-time white noise (see [Wal86] or equivalently [DPZ14, Section 4.2.1.]). In particular, for any $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-predictable $L^2(\mathbb{R})$-valued process $(H_t)_{t\geq 0}$ satisfying that almost surely $\int_0^t\|H_s\|^2_{L^2(\mathbb{R})}\mathrm{d}s<\infty$ for every $t\geq 0$, the stochastic integral $(\iint_0^t H_s(y)W(\mathrm{d}s\mathrm{d}y))_{t\geq 0}$ is an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-adapted continuous martingale with quadratic variation $(\int_0^t\|H_s\|^2_{L^2(\mathbb{R})}\mathrm{d}s)_{t\geq 0}$. Equation (3.2) is also known as the mild form of the SPDE (3.1).

Let us be more precise about the existence of the solutions. In this paper, we will be only considering the weak existence. By that, we mean the existence of a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}_f, W)$, and an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-adapted $\mathcal{C}(\mathbb{R}, [0, z^*])$-valued continuous process $(u_t)_{t\geq 0}$ solving the SPDE (3.1).

Let us also be more precise about the uniqueness of the solutions. We will be only considering the uniqueness in law. We say the uniqueness in law holds for the SPDE (3.1) if any two solutions sharing the same initial value, but not necessarily the same stochastic basis, induce the same law on the path space $\mathcal{C}([0, \infty), \mathcal{C}(\mathbb{R}, [0, z^*]))$.

Lemma 3.2. *The weak existence holds for the SPDE* (3.1)*.*

Proof. Thanks to Lemma 3.1, the existence of the SPDE (3.1) is standard. See [Shi94, Theorem 2.6] and [MMR21, Section 2.1] for example. □

The uniqueness in law for the SPDE (3.1) also holds. To show this, let us first give the moment duality relation between the SPDE (3.1) and the SBBM.

Proposition 3.3. *Suppose that $\mathcal{C}(\mathbb{R}, [0, z^*])$-valued continuous process $(u_t)_{t\geq 0}$ is a solution to the SPDE* (3.1) *w.r.t. a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}_f, W)$. Then it holds for every $t \geq 0$ that*

$$\tilde{\mathbb{E}}_f\left[\prod_{i=1}^{n}(1 - u_t(x_i))\right] = \mathbb{E}\left[\prod_{\alpha\in I_t^{(n)}}\left(1 - f(X_t^{(n),\alpha})\right)\right]. \tag{3.3}$$

Proof. In [AT00], Athreya and Tribe considered the moment dual for the SPDEs

$$\partial_t w_t(x) = \frac{1}{2}\Delta w_t(x) + b(w_t(x)) + \sqrt{\sigma(w_t(x))}\dot{\xi}_{t,x}, \quad t \geq 0, x \in \mathbb{R} \tag{3.4}$$

where $\dot{\xi}$ is a space-time white noise on $[0, \infty) \times \mathbb{R}$,

$$b(z) := \sum_{k=0}^{\infty} b_k z^k, \quad z \in (-R_b, R_b), \tag{3.5}$$

and

$$\sigma(z) := \sum_{k=0}^{\infty} \sigma_k z^k, \quad z \in (-R_\sigma, R_\sigma). \tag{3.6}$$

Here, $R_b > 0$ and $R_\sigma > 0$ are the convergence radius for the infinite series on the right hand sides of (3.5) and (3.6) respectively. To utilize the result in [AT00], let us take

$$b_k := \begin{cases} \beta_0 p_k, & k \in \mathbb{Z}_+ \setminus \{1\}, \\ -\beta_0, & k = 1, \end{cases} \tag{3.7}$$

and

$$\sigma_k := \begin{cases} \beta_{\mathrm{c}} q_k, & k \in \mathbb{Z}_+ \setminus \{2\} \\ -\beta_{\mathrm{c}}, & k = 2. \end{cases}$$

In this way, it is clear from (1.11) that $R_b > 1$ and $R_\sigma > 1$. One can also verify that the $[1 - z^*, 1]$-valued continuous random field $(1 - u_t(x))_{t\geq 0, x\in\mathbb{R}}$ satisfies the SPDE (3.4). (Recall that we assumed $p_1 = q_2 = 0$ for convention.)

Let us assume for the moment that the ordinary offspring law is subcritical:

$$\sum_{k=0}^{\infty} k p_k < 1. \tag{3.8}$$

Define

$$\tilde{b}(z) := \sum_{k\in\mathbb{Z}_+ : z\neq 1} |b_k| z^{k-1} = \beta_{\mathrm{o}} \sum_{k=0}^{\infty} p_k z^{k-1}, \quad z \in (-R_\sigma, R_\sigma) \setminus \{0\},$$

and

$$\tilde{\sigma}(z) = \sum_{k\in\mathbb{Z}_+ : z\neq 2} |\sigma_k| z^{k-2} = \beta_{\mathrm{c}} \sum_{k=0}^{\infty} q_k z^{k-2}, \quad z \in (-R_\sigma, R_\sigma) \setminus \{0\}.$$

Note that in this case,

$$\tilde{b}'(1) = \beta_{\mathrm{o}} \sum_{k=0}^{\infty} p_k (k-1) < 0, \tag{3.9}$$

and

$$\tilde{\sigma}'(1) = \beta_{\mathrm{c}} \sum_{k=0}^{\infty} q_k (k-2) < 0.$$

Therefore, the condition (H1) in [AT00, Theorem 1] holds. Moreover, from

$$\tilde{b}(1) = \sum_{k\in\mathbb{Z}_+ : z\neq 1} |b_k| = \beta_0 \sum_{k=0}^{\infty} p_k = \beta_0,$$

and (3.9), there exists a $\gamma > 0$ such that

$$\tilde{b}(e^\gamma) < \beta_0 = -b_1. \tag{3.10}$$

Similarly, there exists a $\gamma' > 0$ such that $\tilde{\sigma}(e^{\gamma'}) < -\sigma_2$. These give the condition (H2) in [AT00, Theorem 1]. So, under the assumption (3.8), the desired relation (3.3) is a corollary of [AT00, Theorem 1].

Note that the condition (3.8) is used to deduce (3.9), and (3.10), which is mainly used in [AT00] to prevent the explosion of the dual particle system, and to ensure the finiteness of the expectation of the following term

$$\sup_{0\leq t\leq T} \exp\left\{ (\mu + b_1) \int_0^t |I_s^{(n)}| \mathrm{d}s \right\}, \tag{3.11}$$

respectively, where $T > 0$ and $\mu := \sum_{k\in\mathbb{Z}_+, k\neq 1} |b_k|$. See Section 2.2, especially Lemma 3, of [AT00].

For our case when $\sum_{k=0}^{\infty} k p_k \geq 1$, since the explosion for the dual particle system won't happen by Proposition 1.1, and that the term (3.11) equals to 1 under our choice of parameters (3.7), one can still verify the desired duality (3.3) following the steps of [AT00]. We omit the details. □

Lemma 3.4. *The uniqueness in law holds for the SPDE* (3.1).

Proof. This follows from Proposition 3.3 and [AT00, Lemma 1]. □

Recall that the initial value $f \in \mathcal{C}(\mathbb{R}, [0, z^*])$ of the SPDE (3.1) is chosen arbitrary. Let $\mathscr{L}_f$ represent the law of the unique in law solution $u = (u_t(x))_{t\geq 0, x\in\mathbb{R}}$ to the SPDE (3.1) on $\mathcal{C}([0,\infty), \mathcal{C}(\mathbb{R}, [0, z^*]))$. The following lemma is standard, and is known as the weak comparison principle. (We include its proof in the Supplement Material [HS25].)

Lemma 3.5. *Suppose that $u^{(1)} = (u_t^{(1)}(x))_{t\geq 0, x\in\mathbb{R}}$ and $u^{(2)} = (u_t^{(2)}(x))_{t\geq 0, x\in\mathbb{R}}$ are two solutions to the SPDE* (3.1) *with initial values $f^{(1)}$ and $f^{(2)}$ in $\mathcal{C}(\mathbb{R}, [0, z^*])$ respectively. Suppose that $f^{(1)} \leq f^{(2)}$ on $\mathbb{R}$. Then $u^{(1)}$ is stochastically dominated by $u^{(2)}$, in the sense that, there exists a probability kernel $\mathscr{K}_{f^{(1)}, f^{(2)}}$ on $\mathcal{C}([0,\infty), \mathcal{C}(\mathbb{R}, [0, z^*]))$ such that, for $\mathscr{L}_{f^{(1)}}$-a.s. $w^{(1)} \in \mathcal{C}([0,\infty), \mathcal{C}(\mathbb{R}, [0, z^*]))$ and $\mathscr{K}_{f^{(1)}, f^{(2)}}(w^{(1)}, \cdot)$-a.s. $w^{(2)}$, we have*

$$w_t^{(1)}(x) \leq w_t^{(2)}(x), \quad t \geq 0, x \in \mathbb{R};$$

and that, for any Borel subset A of $\mathcal{C}([0,\infty), \mathcal{C}(\mathbb{R}, [0, z^*]))$,

$$\mathscr{L}_{f^{(2)}}(A) = \int \mathscr{K}_{f^{(1)}, f^{(2)}}(w^{(1)}, A) \mathscr{L}_{f^{(1)}}(\mathrm{d}w^{(1)}).$$

For our purpose, we sometimes need the initial value of the SPDE (3.1) to be the non-decreasing limit of a sequence of continuous functions on $\mathbb{R}$.

Proposition 3.6. *Let g be a measurable function on $\mathbb{R}$ which can be approximated by the elements of $\mathcal{C}(\mathbb{R}, [0, z^*])$ monotonically from below, i.e. there exists a pointwisely non-decreasing sequence $(f^{(m)})_{m=1}^\infty$ in $\mathcal{C}(\mathbb{R}, [0, z^*])$ such that $f^{(m)}(x) \uparrow g(x)$ for every $x \in \mathbb{R}$ as $m \uparrow \infty$. Then, there exists a $\mathcal{C}(\mathbb{R}, [0, z^*])$-valued continuous process $(u_t)_{t>0}$ with initial value $u_0 := g$, on a probability space whose probability measure will be denoted by $\tilde{\mathbb{P}}_g$, such that the following two statements hold.*

(3.12) *For each $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, almost surely*

$$\int u_t(x)\phi(x)\mathrm{d}x = \int g(x)\phi(x)\mathrm{d}x + \iint_0^t u_s(y)\frac{\phi''(y)}{2}\mathrm{d}s\mathrm{d}y - \iint_0^t \Phi(u_s(y))\phi(y)\mathrm{d}s\mathrm{d}y + M_t^{(\phi)}, \quad t \geq 0,$$

where $(M_t^{(\phi)})_{t\geq 0}$ is a $(\mathcal{G}_t)_{t\geq 0}$-adapted continuous martingale with quadratic variation

$$\langle M_\cdot^{(\phi)} \rangle_t = \iint_0^t \Psi(u_s(y))\phi(y)^2\mathrm{d}s\mathrm{d}y, \quad t \geq 0$$

and $(\mathcal{G}_t)_{t\geq 0}$ is the natural filtration of the process $(u_t)_{t\geq 0}$.

(3.13) *For every $t \geq 0$, the duality formula* (3.3) *holds with f being replaced by g.*

The proof of the above proposition is standard using the weak comparison principle. (We include a proof in the Supplement Material [HS25].)

4. Probabilistic estimates for the dual SPDE

In this section, we give some probabilistic estimates for the dual SPDE $(u_t)_{t\geq 0}$. Especially, we give upper/lower bounds for some finite/infinite moments of the random field $1-u_t$ when the initial value u_0 is close to 0. Those bounds will be crucial for the proof of Theorems 1.2 and 1.3 in the later sections.

Let $(x_i)_{i=1}^\infty$ and n be given as in (1.2). Let $(I_t^{(n)})_{t\geq 0}$, $(X_t^{(n),\alpha})_{\alpha\in I_t^{(n)},t\geq 0}$ and $(Z_t^{(n)})_{t\geq 0}$ be notations given as in Subsection 1.2 (right after Proposition 1.1) for an SBBM with initial configuration $(x_i)_{i=1}^n$, and parameters β_{o}, $(p_k)_{k=0}^\infty$, β_{c} and $(q_k)_{k=0}^\infty$ given as in (1.2)–(1.5). Assume that (1.10), (1.11) and (1.16) hold. Let $(\Lambda,\mu)\in\mathcal{T}$ be given as in (1.12) and (1.13). For every $(\tilde{\Lambda},\tilde{\mu})\in\mathcal{T}$, let $(v_t^{(\tilde{\Lambda},\tilde{\mu})}(x))_{t>0,x\in\mathbb{R}}\in\mathcal{C}^{1,2}((0,\infty)\times\mathbb{R})$ be the unique non-negative solution to the MFE (1.17). Let Φ and Ψ be given as in (1.14) and (1.15) respectively. Let f be a measurable function on $\mathbb{R}$ which can be approximated by the elements of $\mathcal{C}(\mathbb{R},[0,z^*])$ monotonically from below. Let $(u_t)_{t>0}$ be the continuous $\mathcal{C}(\mathbb{R},[0,z^*])$-valued process given as in Proposition 3.6, with initial value $u_0 := f$, on a probability space whose probability measure will be denoted by $\tilde{\mathbb{P}}_f$.

When the initial value u_0 is close to 0, the behavior of the random field $(u_t)_{t\geq 0}$ is largely related to the linearization of the function Φ and Ψ at 0. The following analytical lemma helps us with the linearization technique. (We include its proof in the Supplement Material [HS25].)

Lemma 4.1. *Suppose that $N\in\mathbb{N}$ and $(z_i)_{i=1}^N$ is a finite list in $[0,2]$, then*

$$0\leq 1-\prod_{i=1}^N(1-z_i)\leq\sum_{i=1}^N z_i.$$

Combining Lemmas 3.1 and 4.1, as well as the facts that $(1-z)^k\leq 1$ for every $z\in[0,2]$ and $k\in\mathbb{N}$, we see that for all $z\in[0,z^*]\subset[0,2]$,

$$-\lambda_{\mathrm{o}}z\leq\Phi'(0+)z\leq\Phi(z)\leq\beta_{\mathrm{o}}z\quad\text{and}\quad 0\leq\Psi(z)\leq 2\beta_{\mathrm{c}}z, \tag{4.1}$$

where λ_{o} is defined in (2.1). Moreover, define

$$\kappa(\tilde{\gamma}):=\inf_{w\in[0,\tilde{\gamma}]}\frac{\Psi'(0+)w}{\Psi(w)},\quad\tilde{\gamma}\in[0,1). \tag{4.2}$$

It is clear that $\kappa(\tilde{\gamma})\in[0,1]$ and $\lim_{\tilde{\gamma}\to 0}\kappa(\tilde{\gamma})=1$. With those linear bounds of the functions Φ and Ψ, we have a preliminary upper bound for the first moment of the random field u_t.

Lemma 4.2. *For every $t>0$ and $x\in\mathbb{R}$, it holds that*

$$\tilde{\mathbb{E}}_f[u_t(x)]\leq e^{-\Phi'(0+)t}\mathbf{E}_x[f(B_t)]\leq e^{\lambda_{\mathrm{o}}t}\mathbf{E}_x[f(B_t)],$$

where $(B_t)_{t\geq 0}$ is a 1-d standard Brownian motion with initial value x under the expectation operator $\mathbf{E}_x$.

Proof. Taking expectation in the mild formula (3.2), we see that for every $t > 0$ and $x \in \mathbb{R}$,

$$\tilde{\mathbb{E}}_f[u_t(x)] = \int p_t(x-y)f(y)\mathrm{d}y - \iint_0^t p_{t-s}(x-y)\tilde{\mathbb{E}}_f[\Phi(u_s(y))]\mathrm{d}s\mathrm{d}y.$$

By Feynman-Kac formula and (4.1), we conclude that for every $t > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}\tilde{\mathbb{E}}_f[u_t(x)] &= \mathbf{E}_x\left[\exp\left\{-\int_0^t \frac{\tilde{\mathbb{E}}_f[\Phi(u_{t-s}(z))]}{\tilde{\mathbb{E}}_f[u_{t-s}(z)]}\bigg|_{z=B_s}\mathrm{d}s\right\}f(B_t)\right] \\ &\leq e^{-\Phi'(0+)t}\mathbf{E}_x[f(B_t)] \leq e^{\lambda_0 t}\mathbf{E}_x[f(B_t)]. \end{aligned}$$

□

Rather than the first moment, we are more interested in the infinite moments of the random field $1-u_t$, since they arise naturally when one takes n to ∞ in the duality relation (3.3). To this end, we need to work with the infinite product. For a sequence of real numbers $(z_i)_{i\in\mathbb{N}}$, define

$$\prod_{i=1}^{\infty} z_i = \lim_{m\to\infty}\prod_{i=1}^{m} z_i$$

whenever the limit exists. (This definition is standard, see [Rud87, Definition 15.2].) Recall that $z^* \in [0,2)$. For every $[0,z^*]$-valued measurable function u on $\mathbb{R}$, closed subset $\tilde{\Lambda}$ of $\mathbb{R}$, and integer-valued locally finite measure $\tilde{\mu}$ on $\tilde{\Lambda}^c$, define

$$\prod_{x\in\tilde{\Lambda}^c}(1-u(x))^{\tilde{\mu}(\{x\})} := \prod_{i=1}^{\tilde{\mu}(\tilde{\Lambda}^c)}(1-u(z_i))$$

where $(z_i)_{i=1}^{\tilde{\mu}(\tilde{\Lambda}^c)}$ is a (possibly finite or infinite) sequence in $\tilde{\Lambda}^c$ such that $\tilde{\mu} = \sum_{i=1}^{\tilde{\mu}(\tilde{\Lambda}^c)}\delta_{z_i}$. It is clear that this definition does not depends on the particular choice of the sequence $(z_i)_{i=1}^{\tilde{\mu}(\tilde{\Lambda}^c)}$, since one can verify that

$$\begin{aligned}&\prod_{x\in\tilde{\Lambda}^c}(1-u(x))^{\tilde{\mu}(\{x\})} \qquad (4.3)\\ &= \mathbf{1}_{\left\{\tilde{\mu}\left(\left\{x\in\tilde{\Lambda}^c:u(x)\in(1,z^*]\right\}\right)<\infty\right\}}(-1)^{\tilde{\mu}(\{x\in\tilde{\Lambda}^c:u(x)\in(1,z^*]\})}\exp\left\{\int_{\tilde{\Lambda}^c}\log(|1-u(x)|)\tilde{\mu}(\mathrm{d}x)\right\}.\end{aligned}$$

It is worth to note from the above expression that

(4.4) for every $[0,z^*]$-valued measurable function u on $\mathbb{R}$,

$$\tilde{\mu} \mapsto \prod_{x\in\mathbb{R}}(1-u(x))^{\tilde{\mu}(\{x\})}$$

is a measurable map from $\mathcal{N}$ to $(-1,1]$.

The next lemma shows how the infinite moments of the random field $1-u_t$ is related to the SBBM. It motivates the rest of the results of this section.

Lemma 4.3. *For any $t > 0$,*

$$\lim_{n\to\infty} \mathbb{E}\left[\prod_{\alpha\in I_t^{(n)}} \left(1 - f\left(X_t^{(n),\alpha}\right)\right)\right] = \tilde{\mathbb{E}}_f\left[\prod_{i=1}^{\infty}(1-u_t(x_i))\right]$$
$$= \tilde{\mathbb{E}}_f\left[\mathbf{1}_{\{u_t(x)=0,\forall x\in\Lambda\}} \prod_{x\in\Lambda^c}(1-u_t(x))^{\mu(\{x\})}\right].$$

Proof. Thanks to Proposition 3.3, it suffices to consider the limit

$$\lim_{n\to\infty} \mathbb{E}\left[\prod_{\alpha\in I_t^{(n)}} \left(1 - f\left(X_t^{(n),\alpha}\right)\right)\right] = \lim_{n\to\infty} \tilde{\mathbb{E}}_f\left[\prod_{i=1}^{n}(1-u_t(x_i))\right]. \tag{4.5}$$

Let us first explain that the infinite product

$$\prod_{i=1}^{\infty}(1-u_t(x_i)) := \lim_{n\to\infty}\prod_{i=1}^{n}(1-u_t(x_i))$$

is a well-defined random variable. Note that $u_t(x_i)$ takes their values in $[0, z^*]$. From Lemma 3.1 and (1.16), we have $z^* < 2$. On one hand, if there are infinitely many i such that $u_t(x_i) \in [1, z^*] \subset [1, 2)$, then $\prod_{i=1}^{\infty}(1-u_t)(x_i) = 0$. On the other hand, if there are only finitely many i such that $u_t(x_i) \in [1, z^*]$, then the infinite product is also well-defined since the sequence

$$\prod_{i=1}^{n}(1-u_t)(x_i), \quad n \in \mathbb{N}$$

is eventually monotone. Since $u_t(x)$ is continuous in $x \in \mathbb{R}$, by (4.3), almost surely,

$$\prod_{i=1}^{\infty}(1-u_t(x_i)) = \mathbf{1}_{\{u_t(x)=0,\forall x\in\Lambda\}} \prod_{x\in\Lambda^c}(1-u_t(x))^{\mu(\{x\})}.$$

Now, by bounded convergence theorem, we can exchange the limit and the expectation on the right hand side of (4.5), and get the desired result. □

The next lemma says that the infinite moments of the random field $1 - u_t$ can be estimated by the Laplace transform of u_t.

Lemma 4.4. *Let $\varepsilon \in (0, \frac{1}{2})$ and $U \subset \mathbb{R}$ be an open interval. Suppose that F is a closed interval containing $\{x_i : i \in \mathbb{N}\}$. Define*

$$\theta(\gamma) := \begin{cases} -\log(1-\gamma)/\gamma, & \gamma \in (0,1), \\ 1, & \gamma = 0. \end{cases} \tag{4.6}$$

Then for any $t > 0$ and $\gamma \in (\varepsilon, 1)$,

$$\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1-u_t)(x_i)\right] \geq \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\left\{-\theta(\gamma)\sum_{i=1}^{\infty}u_t(x_i)\right\}\right] - 2\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\left(\sup_{s\le t, y\in F} u_s(y) > \gamma\right),$$

and

$$\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1-u_t)(x_i)\right] \leq \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\left\{-\sum_{i=1}^{\infty}u_t(x_i)\right\}\right] + \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\left(\sup_{s\leq t, y\in F} u_s(y) > \frac{1}{2}\right).$$

We omit the proof of the above lemma because it is similar to that of [BMS24a, Lemma 2.2] except some minor variations to incorporate the fact that one can not take the logarithm of the random field $1-u_t$. (We include a detailed proof in the Supplementary Material [HS25].)

Now, we want to give upper/lower bounds for the Laplace transform of the random field u_t in terms of the solution $(v_t)_{t\geq 0}$ to the MFE (1.17). The idea is to investigate Doob's decomposition of the semimartingale

$$s \mapsto \exp\left\{-\int u_s(y)v_{t-s}(y)\mathrm{d}y\right\}$$

on $[0,t]$. To do this, we need several analytical results for $(v_t)_{t\geq 0}$. For every $(\tilde{\Lambda},\tilde{\mu}) \in \mathcal{T}$, we use $(\hat{v}_t^{(\tilde{\Lambda},\tilde{\mu})}(x))_{t>0,x\in\mathbb{R}} \in \mathcal{C}^{1,2}((0,\infty)\times\mathbb{R})$ to denote the unique non-negative solution of (1.17) with $\Psi'(0+)$ being replaced by 1. It is easy to check that for all $(\tilde{\Lambda},\tilde{\mu}) \in \mathcal{T}$ and $t>0, x\in\mathbb{R}$,

$$v_t^{(\tilde{\Lambda},\tilde{\mu})}(x) = \Psi'(0+)^{-1}\hat{v}_t^{(\tilde{\Lambda},\Psi'(0+)\tilde{\mu})}(x). \tag{4.7}$$

From this and [BMS24a, (2.4)–(2.10)], we can verify that for every $(\tilde{\Lambda},\tilde{\mu}) \in \mathcal{T}$ and $t>0, x\in\mathbb{R}$,

$$v_t^{(\tilde{\Lambda},\tilde{\mu})}(x) \leq v_t^{(\mathbb{R},\mathbf{0})}(x) = \frac{2}{\Psi'(0+)t}, \tag{4.8}$$

$$v_t^{(\tilde{\Lambda},\mathbf{0})}(x) = v_t^{(\tilde{\Lambda}+\{z\},\mathbf{0})}(x+z), \quad \forall z\in\mathbb{R}, \tag{4.9}$$

$$v_t^{(\tilde{\Lambda},\mathbf{0})}(x) = v_t^{(-\tilde{\Lambda},\mathbf{0})}(-x). \tag{4.10}$$

Here, we used Minkowski's notation $aA+bB := \{ax+by : x\in A, y\in B\}$ for $a,b\in\mathbb{R}$ and $A,B\subset\mathbb{R}$; and $\mathbf{0}$ represents the null measure. We also have that

$$v_t^{((-\infty,0],\mathbf{0})}(|x|) \lesssim \frac{1}{t}\left(1+\frac{|x|}{\sqrt{t}}\right)e^{-\frac{x^2}{t}}, \tag{4.11}$$

and

$$v_t^{([-k,k],\mathbf{0})}(x) \lesssim \frac{1}{t}\left(1+\frac{\mathrm{dist}(\{x\},[-k,k])}{\sqrt{t}}\right)e^{-\frac{\mathrm{dist}(\{x\},[-k,k])^2}{2t}} \tag{4.12}$$

uniformly for $t>0, x\in\mathbb{R}$ and $k\geq 0$. Here, $\mathrm{dist}(A,B) := \inf\{|a-b| : a\in A, b\in B\}$ represents the distance between two given sets $A,B\subset\mathbb{R}$. For every $t>0$, $(\tilde{\Lambda},\tilde{\mu})\in\mathcal{T}$ and closed interval $F\subset\mathbb{R}$, define

$$\mathcal{V}_t^{(\tilde{\Lambda},\tilde{\mu},F)} := \int_0^t\int_{F^c} v_r^{(\tilde{\Lambda},\tilde{\mu})}(z)^2\mathrm{d}z\mathrm{d}r. \tag{4.13}$$

Lemma 4.5. *The following statements hold.*

(i) *If U is an open interval such that $U \cap \{x_i : i \in \mathbb{N}\}$ is bounded, then for every $b \geq a > 0$,*

$$\sup_{t\in[a,b]} \int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y < \infty. \tag{4.14}$$

(ii) *If F is a closed interval containing $\cup_{i\in\mathbb{N}}(x_i - 1, x_i + 1)$, then $\mathcal{V}_t^{(\Lambda,\mu,F)} < \infty$ for every $t > 0$. In particular, it holds in this case that $\lim_{t\downarrow 0} \mathcal{V}_t^{(\Lambda,\mu,F)} = 0$.*

(iii) *Let*

$$\Lambda_K := [a_K, \infty) \quad \text{and} \quad F_K := [b_K, \infty), \quad K \in \mathbb{N}$$

be unbounded intervals where $(a_K)_{K\in\mathbb{N}}, (b_K)_{K\in\mathbb{N}}$ are sequences in $\mathbb{R}$ such that

$$\lim_{K\to\infty} \mathrm{dist}(\Lambda_K, F_K^c) = \lim_{K\to\infty} (a_K - b_K)^+ = \infty.$$

Then for every $t > 0$, $\lim_{K\to\infty} \mathcal{V}_t^{(\Lambda_K,\mathbf{0},F_K)} = 0$.

Proof. Let us prove (i). Let us fix an arbitrary open interval U such that $U \cap \{x_i : i \in \mathbb{N}\}$ is bounded. Let $\tilde{F}$ be the smallest closed interval which contains $\{x_i : i \in \mathbb{N}\}$. There are four different cases to consider.

- If $\tilde{F} = \mathbb{R}$, then it must be the case that U is bounded. In this case, (4.14) follows from (4.8).
- If $\tilde{F} = (-\infty, \beta]$ for some $\beta \in \mathbb{R}$, then it must be the case that U is the subset of (α, ∞) for some $\alpha \in \mathbb{R}$. In this case, it was argued in the proof of [BMS24a, Lemma 2.3] that

$$\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y \leq \int_\alpha^\beta v_t^{(\mathbb{R},\mathbf{0})}(y)\mathrm{d}y + \int_0^\infty v_t^{((-\infty,0],\mathbf{0})}(y)\mathrm{d}y.$$

  The desired (4.14) now follows from (4.8) and (4.11).
- If $\tilde{F} = [\alpha, \infty)$ for some $\alpha \in \mathbb{R}$ then, similarly to the previous case, (4.14) holds.
- If $\tilde{F} = [\alpha, \beta]$ for some $-\infty < \alpha \leq \beta < \infty$, then it was argued in the proof of [BMS24a, Lemma 2.3] that

$$\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y \leq \int v_t^{([\frac{\alpha-\beta}{2},\frac{\beta-\alpha}{2}],\mathbf{0})}(y)\mathrm{d}y.$$

  The desired (4.14) now follows from (4.12).

Noticing from (4.7), (ii) of this lemma are essentially given by [BMS24a, Lemma 2.3 and Lemma 3.1 (2)].

We now prove (iii). Without loss of generality, we assume that $c_K := a_K - b_K > 1$ for every $K \in \mathbb{N}$. By (4.9), (4.10) and (4.11), uniformly for every $K \in \mathbb{N}$ and $t > 0$,

$$\mathcal{V}_t^{(\Lambda_K,\mathbf{0},F_K)} = \int_0^t \int_{-\infty}^{b_K} v_r^{([a_K,\infty),\mathbf{0})}(z)^2 \mathrm{d}z\mathrm{d}r = \int_0^t \int_{-\infty}^{-c_K} v_r^{([0,\infty),\mathbf{0})}(z)^2 \mathrm{d}z\mathrm{d}r$$
$$= \int_0^t \int_{c_K}^\infty v_r^{((-\infty,0],\mathbf{0})}(z)^2 \mathrm{d}z\mathrm{d}r \lesssim \int_0^t \int_{c_K}^\infty \frac{z^2}{r}\left(1 + \frac{z}{\sqrt{r}}\right) e^{-\frac{z^2}{r}} \mathrm{d}z\mathrm{d}r.$$

Therefore, noticing

$$\sup\Big\{a^2(1+a)e^{-\frac{a^2}{2}} : a>0\Big\}<\infty,$$

we have uniformly for every $K\in\mathbb{N}$ and $t>0$,

$$\mathcal{V}_t^{(\Lambda_K,\mathbf{0},F_K)} \lesssim \int_0^t \int_{c_K}^{\infty} e^{-\frac{z^2}{2r}}\mathrm{d}z\mathrm{d}r \le t\int_{c_K}^{\infty} e^{-\frac{z^2}{2t}}\mathrm{d}z.$$

The desired result of (iii) now follows. □

We now present our upper/lower bounds for certain Laplace transform of u_t.

Lemma 4.6. *Let F be a closed interval containing $\cup_{i\in\mathbb{N}}(x_i-1,x_i+1)$ and U be an open interval. Let $0\le\varepsilon\le\gamma<1$. Suppose that the initial value of $(u_t)_{t\ge0}$ is $f=\varepsilon\mathbf{1}_U$. Let $\theta(\gamma)$ be given as in* (4.6)*. Then for any $t>0$,*

$$\begin{aligned}&\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\Big\{-\theta(\gamma)\sum_{i=1}^{\infty}u_t(x_i)\Big\}\right] \qquad (4.15)\\ &\ge \exp\Big\{-\frac{\varepsilon e^{\lambda_{\mathrm{o}}t}}{1-\gamma}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\Big\} - \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\Big(\sup_{s\le t,y\in F}u_s(y)>\frac{\gamma}{2\beta_{\mathrm{c}}}\Psi'(0+)\Big) -\\ &\quad \frac{\varepsilon\Psi'(0+)e^{2\lambda_{\mathrm{o}}t}}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)},\end{aligned}$$

and

$$\begin{aligned}&\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\Big\{-\sum_{i=1}^{\infty}u_t(x_i)\Big\}\right] \qquad (4.16)\\ &\le \exp\Big\{-\varepsilon\kappa(\gamma)e^{-\beta_{\mathrm{o}}t}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\Big\} + \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\Big(\sup_{s\le t,y\in F}u_s(y)>\gamma\Big) + \varepsilon\beta_{\mathrm{c}}e^{\lambda_{\mathrm{o}}t}\mathcal{V}_t^{(\Lambda,\mu,F)}.\end{aligned}$$

Here, $\Psi'(0+)$, λ_{o}, $\Phi'(0+)$ and κ are given as in (1.18), (2.1), (2.2) *and* (4.2) *respectively.*

We omit the tedious proof of Lemma 4.6 because it is similar to that of [BMS24a, Lemma 2.4]. (We refer the diligent readers to the Supplementary Material [HS25] for a detailed proof.)

In the upper/lower bounds for the Laplace transform of the random field u_t in Lemma 4.6 above, except the terms involving the solution $(v_t)_{t\ge0}$ to the MFE (1.17), there are still terms related to the probability of the random field u_t itself. In the next lemma, we explain how these terms can be controlled by the initial data $\varepsilon\mathbf{1}_U$.

Lemma 4.7. *Let U be an open interval and F be a closed interval. Let $t>0$ and $\gamma\in(0,1)$. Suppose that $U\cap F$ is bounded. Then*

$$C_1(U,F,t,\gamma) := \sup_{\varepsilon\in(0,\gamma/2)}\frac{1}{\varepsilon}\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\Big(\sup_{s\le t,y\in F}u_s(y)>\gamma\Big)<\infty. \qquad (4.17)$$

In particular, $\limsup_{t\downarrow0}C_1(U,F,t,\gamma)<\infty$. Moreover, if $\tilde{U}$ is an open interval such that its intersection with $F_K:=[K,\infty)$ is bounded for every $K\in\mathbb{R}$, then we also have that $\lim_{K\to\infty}C_1(\tilde{U},F_K,t,\gamma)=0$.

We omit the tedious proof of the above Lemma, because it is standard, and is similar to that of [Tri95, Lemma 3.1]. (We refer the diligent readers to the Supplementary Material [HS25] for a detailed proof.)

As an application of the results in this section, we establish a compact support property for the random field u_t.

Lemma 4.8. *Let $\varepsilon > 0$ be small enough so that $4\varepsilon < 2 - \sum_{k=1}^{\infty} kq_k = \Psi'(0+)/\beta_c$. Let U be a bounded open subset of $\mathbb{R}$. Let f, the initial value of the SPDE* (3.1)*, be bounded by $\varepsilon \mathbf{1}_U$. Let $t > 0$ be arbitrary. Then, it holds that*

$$\lim_{K\to+\infty} \tilde{\mathbb{P}}_f(u_t(x) = 0, \forall x > K) = 1$$

and

$$\lim_{K\to+\infty} \tilde{\mathbb{P}}_f(u_t(x) = 0, \forall x < -K) = 1.$$

Proof. We only prove the first limit, since the second one follows by symmetry. By the weak comparison principle, i.e. Lemma 3.5, we can assume $f = \varepsilon \mathbf{1}_U$ without loss of generality. Let $K \geq 2$ be arbitrary. We can construct a sequence $(\hat{x}_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that $\{\hat{x}_i : i \in \mathbb{N}\} = (K, +\infty) \cap \mathbb{Q}$. Note that (1.12) and (1.13) holds with $(x_i)_{i=1}^{\infty}$ and (Λ, μ) being replaced by $(\hat{x}_i)_{i=1}^{\infty}$ and $(\Lambda_K, \mathbf{0})$ where $\Lambda_K := [K, +\infty)$. Let $\gamma = \gamma(\varepsilon) < 1$ be close enough to 1 such that $4\varepsilon < \frac{\Psi'(0+)}{\beta_c}\gamma$. In particular, it holds that $0 \leq \varepsilon \leq \gamma < 1$. Fix $\theta = \theta(\gamma)$ according to (4.6). By Lemma 4.6, with $F_K := [K/2, +\infty)$, we have

$$\begin{aligned} &\tilde{\mathbb{P}}_{\varepsilon \mathbf{1}_U}(u_t(x) = 0, \forall x > K) = \tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\left[\exp\left\{-\theta(\gamma)\sum_{i=1}^{\infty} u_t(\hat{x}_i)\right\}\right] \qquad (4.18)\\ &\geq \exp\left\{-\frac{\varepsilon e^{\lambda_o t}}{1-\gamma}\int_U v_t^{(\Lambda_K,\mathbf{0})}(y)\mathrm{d}y\right\} - \tilde{\mathbb{P}}_{\varepsilon \mathbf{1}_U}\left(\sup_{s\leq t, y\in F_K} u_s(y) > \frac{\gamma}{2\beta_c}\Psi'(0+)\right) - \\ &\quad \frac{\varepsilon \Psi'(0+) e^{2\lambda_o t}}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda_K,\mathbf{0},F_K)}. \end{aligned}$$

By (4.9) and (4.11), since U is bounded, we have that

$$\lim_{K\to+\infty} \exp\left\{-\frac{\varepsilon e^{\lambda_o t}}{1-\gamma}\int_U v_t^{(\Lambda_K,\mathbf{0})}(y)\mathrm{d}y\right\} = 1.$$

Also, by Lemma 4.5 (iii), we see that

$$\lim_{K\to+\infty} \mathcal{V}_t^{(\Lambda_K,\mathbf{0},F_K)} = 0.$$

Therefore, it remains to prove that the second term on the right-hand side of (4.18) converges to 0 as $K \to \infty$. From $\varepsilon \in (0, \frac{\gamma}{4\beta_c}\Psi'(0+))$ and Lemma 4.7, the absolute value of this term is bounded by $\varepsilon C_1(U, F_K, t, \frac{\gamma}{2\beta_c}\Psi'(0+)) \overset{K\to\infty}{\longrightarrow} 0$. We arrive at the desired result. □

5. Existence: Proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

Let $(x_i)_{i=1}^\infty$ and n be given as in (1.2). Let $(I_t^{(n)})_{t\geq 0}$, $(X_t^{(n),\alpha})_{\alpha\in I_t^{(n)},t\geq 0}$ and $(Z_t^{(n)})_{t\geq 0}$ be notations given as in Subsection 1.2 (right after Proposition 1.1) for an SBBM w.r.t. initial configuration $(x_i)_{i=1}^n$, ordinary branching rate β_{o}, ordinary branching law $(p_k)_{k=0}^\infty$, catalytic branching rate β_{c}, and catalytic branching law $(q_k)_{k=0}^\infty$ given as in (1.2)–(1.5). Assume that (1.10), (1.11), and (1.16) hold. Let $(\Lambda,\mu)\in\mathcal{T}$ be given as in (1.12) and (1.13). For every $(\tilde\Lambda,\tilde\mu)\in\mathcal{T}$, let $(v_t^{(\tilde\Lambda,\tilde\mu)}(x))_{t>0,x\in\mathbb{R}}\in\mathcal{C}^{1,2}((0,\infty)\times\mathbb{R})$ be the unique non-negative solution to the equation (1.17). Let Φ and Ψ be given as in (1.14) and (1.15) respectively. Let f be a measurable function on $\mathbb{R}$ which can be approximated by the elements of $\mathcal{C}(\mathbb{R},[0,z^*])$ monotonically from below. Let $(u_t)_{t>0}$ be the continuous $\mathcal{C}(\mathbb{R},[0,z^*])$-valued process given as in Proposition 3.6, with initial value $u_0=f$, on a probability space whose probability measure will be denoted by $\tilde{\mathbb{P}}_f$.

The proof of Theorem 1.2 will be divided into three parts. In Subsection 5.1, we will show the convergence in finite-dimensional distributions of $(Z_t^{(n)})_{t>0}$ as $n\to\infty$ to a Markov process $(\tilde Z_t)_{t>0}$. In Subsection 5.2, we will give several preliminary results on the process $(\tilde Z_t)_{t>0}$, in particular, showing that it is stochastically right-continuous. In Subsection 5.3, we will show that $(\tilde Z_t)_{t>0}$ has a càdlàg modification $(Z_t)_{t>0}$.

5.1. **Convergence in finite dimensional distributions.** In this subsection, we will show the convergence in finite-dimensional distributions of $(Z_t^{(n)})$ as $n\to\infty$. Let us start with an analytic lemma which gives the continuity of certain functions on $\mathcal{N}$. (We include its proof in the Supplement Material [HS25].)

Lemma 5.1. *Suppose that ν_m converges to ν in $\mathcal{N}$ when $m\to\infty$. Let $g\in\mathcal{C}(\mathbb{R},[-1,1])$ satisfy that $1-g$ has compact support. Then*

$$\lim_{m\to\infty}\prod_{z\in\mathbb{R}} g(z)^{\nu_m(\{z\})}=\prod_{z\in\mathbb{R}} g(z)^{\nu(\{z\})}. \tag{5.1}$$

Corollary 5.2. *If f has compact support and is bounded by $\Psi'(0+)/(4\beta_{\mathrm{c}})$, then for each $t>0$,*

$$\lim_{m\to\infty}\tilde{\mathbb{E}}_f\left[\prod_{z\in\mathbb{R}}(1-u_t(z))^{\nu_m(\{z\})}\right]=\tilde{\mathbb{E}}_f\left[\prod_{z\in\mathbb{R}}(1-u_t(z))^{\nu(\{z\})}\right]. \tag{5.2}$$

Proof. For each $K>0$ and $t>0$, it holds that

$$\begin{aligned}&\left|\tilde{\mathbb{E}}_f\left[\prod_{z\in\mathbb{R}}(1-u_t(z))^{\nu_m(\{z\})}\right]-\tilde{\mathbb{E}}_f\left[\prod_{z\in\mathbb{R}}(1-u_t(z))^{\nu(\{z\})}\right]\right|\\&\leq\tilde{\mathbb{E}}_f\left[\left|\prod_{z\in\mathbb{R}}(1-u_t(z))^{\nu_m(\{z\})}-\prod_{z\in\mathbb{R}}(1-u_t(z))^{\nu(\{z\})}\right|\mathbf{1}_{\{u_t(x)=0,\forall|x|>K\}}\right]\\&\quad+2\tilde{\mathbb{P}}_f(\exists x\text{ s.t. }|x|>K,u_t(x)>0).\end{aligned}$$

From Lemma 5.1 and the bounded convergence theorem, we know that the first term on the right hand side converges to 0 when $m \to \infty$. Now from Lemma 4.8, taking $m \to \infty$ first and then $K \to \infty$, we get (5.2) as desired. □

In the next proposition, we show that $(Z_t^{(n)})_{t>0}$ converges to some Markov process in finite dimensional distributions.

Proposition 5.3. *There exists an $\mathcal{N}$-valued time-homogeneous Markov process $(\tilde{Z}_t)_{t>0}$ such that $(Z_t^{(n)})_{t>0}$ converges to $(\tilde{Z}_t)_{t>0}$ as $n \to \infty$ in finite dimensional distributions. The entrance law $(\mathscr{P}_t^{(\Lambda,\mu)})_{t>0}$ of $(\tilde{Z}_t)_{t>0}$ are characterized so that, for any non-negative continuous function g on $\mathbb{R}$ with compact support and $t > 0$,*

$$\begin{aligned}\int e^{-\tilde{\nu}(g)} \mathscr{P}_t^{(\Lambda,\mu)}(\mathrm{d}\tilde{\nu}) &= \tilde{\mathbb{E}}_{1-e^{-g}}\left[\prod_{i=1}^{\infty}(1-u_t(x_i))\right] \\ &= \tilde{\mathbb{E}}_{1-e^{-g}}\left[\mathbf{1}_{\{u_t(x)=0,\forall x\in\Lambda\}} \prod_{x\in\Lambda^c}(1-u_t(x))^{\mu(\{x\})}\right].\end{aligned} \tag{5.3}$$

The transition kernels $(\mathscr{Q}_t)_{t\geq 0}$ of $(\tilde{Z}_t)_{t>0}$ are characterized so that, for any non-negative continuous function g on $\mathbb{R}$ with compact support, $t \geq 0$, and $\nu \in \mathcal{N}$,

$$\int e^{-\tilde{\nu}(g)} \mathscr{Q}_t(\nu, \mathrm{d}\tilde{\nu}) = \tilde{\mathbb{E}}_{1-e^{-g}}\left[\prod_{z\in\mathbb{R}}(1-u_t(z))^{\nu(\{z\})}\right]. \tag{5.4}$$

In particular, the finite dimensional distributions of $(\tilde{Z}_t)_{t>0}$ is determined by (Λ, μ), Φ, and Ψ.

Remark 5.4. Comparing (5.3) and (5.4), we have $\mathscr{P}_t^{(\emptyset,\nu)} = \mathscr{Q}_t(\nu, \cdot)$ for any $\nu \in \mathcal{N}$ and $t > 0$.

Proof. Step 1. Let us fix an arbitrary $t > 0$ and show that the $\mathcal{N}$-valued random element $Z_t^{(n)}$ converges in distribution to some $\mathcal{N}$-valued random element $\hat{Z}_t$ as $n \to \infty$. Fix an arbitrary non-negative continuous function g on $\mathbb{R}$ with compact support. From [Kal17, Corollary 4.14], it suffices to show the convergence in distribution of the random variable $Z_t^{(n)}(g)$ as $n \to \infty$. By Lemma 4.3 with $f := 1 - e^{-\theta g}$, we see that the following limit exists for each $\theta \geq 0$:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[e^{-\theta Z_t^{(n)}(g)}\right] &= \lim_{n\to\infty} \mathbb{E}\left[\prod_{\alpha\in I_t^{(n)}} e^{-\theta g(X_t^{(n),\alpha})}\right] \\ = \tilde{\mathbb{E}}_{1-e^{-\theta g}}\left[\prod_{i=1}^{\infty}(1-u_t(x_i))\right] &= \tilde{\mathbb{E}}_{1-e^{-\theta g}}\left[\mathbf{1}_{\{u_t(x)=0,\forall x\in\Lambda\}} \prod_{x\in\Lambda^c}(1-u_t(x))^{\mu(\{x\})}\right].\end{aligned} \tag{5.5}$$

To show the weak convergence of $Z_t^{(n)}(g)$, by Lévy's continuity theorem, it suffices to prove that $\lim_{\theta\to 0}\lim_{n\to\infty}\mathbb{E}[e^{-\theta Z_t^{(n)}(g)}] = 1$. Recall from (1.11) and (1.18) that $\Psi'(0+)/\beta_c =$

$2-\sum_{k=0}^{\infty} kq_k \in (0,2)$. Let $\theta>0$ be small enough so that

$$0<\theta' := 1-e^{-\theta\|g\|_\infty} < \frac{\Psi'(0+)}{8\beta_{\mathrm{c}}} \leq \frac{1}{4}.$$

Let U be a bounded open interval containing the support of g, and F be a closed interval containing $\cup_{i\in\mathbb{N}}(x_i-1,x+1)$. By Lemma 4.3, we have that

$$\lim_{n\to\infty} \mathbb{E}\Big[e^{-\theta Z_t^{(n)}(g)}\Big] \geq \lim_{n\to\infty} \mathbb{E}\Big[(1-\theta')^{Z_t^{(n)}(U)}\Big] = \tilde{\mathbb{E}}_{\theta'\mathbf{1}_U}\Bigg[\prod_{i=1}^{\infty}(1-u_t)(x_i)\Bigg].$$

From Lemmas 4.4 and 4.6, for $\gamma=1/2$,

$$\begin{aligned}
&\tilde{\mathbb{E}}_{\theta'\mathbf{1}_U}\Bigg[\prod_{i=1}^{\infty}(1-u_t)(x_i)\Bigg] \qquad (5.6)\\
&\geq \exp\Big\{-\frac{\theta' e^{\lambda_{\mathrm{o}}t}}{1-\gamma}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\Big\} - \tilde{\mathbb{P}}_{\theta'\mathbf{1}_U}\Big(\sup_{s\leq t, y\in F} u_s(y) > \frac{\gamma}{2\beta_{\mathrm{c}}}\Psi'(0+)\Big) - \\
&\qquad \frac{\theta'\Psi'(0+)e^{2\lambda_{\mathrm{o}}t}}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)} - 2\tilde{\mathbb{P}}_{\theta'\mathbf{1}_U}\Big(\sup_{s\leq t, y\in F} u_s(y) > \gamma\Big).
\end{aligned}$$

Note that $\theta'\to 0$ as $\theta\to 0$. By Lemma 4.7, we have

$$\tilde{\mathbb{P}}_{\theta'\mathbf{1}_U}\Big(\sup_{s\leq t, y\in F} u_s(y) > \frac{\gamma}{2\beta_{\mathrm{c}}}\Psi'(0+)\Big) \leq \theta' C_1\Big(U,F,t,\frac{\gamma}{2\beta_{\mathrm{c}}}\Psi'(0+)\Big) \xrightarrow{\theta\to 0} 0,$$

and

$$\tilde{\mathbb{P}}_{\theta'\mathbf{1}_U}\Big(\sup_{s\leq t, y\in F} u_s(y) > \gamma\Big) \leq \theta' C_1(U,F,t,\gamma) \xrightarrow{\theta\to 0} 0.$$

Here, $C_1(\cdot,\cdot,\cdot,\cdot)$ is the constant given as in Lemma 4.7. From Lemma 4.5, we have $\mathcal{V}_t^{(\Lambda,\mu,F)}<\infty$. Therefore, the third term on the right hand side of (5.6) converges to 0 as $\theta\to 0$. Now, we have

$$\begin{aligned}
\liminf_{\theta\to 0}\lim_{n\to\infty} \mathbb{E}\Big[e^{-\theta Z_t^{(n)}(g)}\Big] &\geq \liminf_{\theta\to 0}\tilde{\mathbb{E}}_{\theta'\mathbf{1}_U}\Bigg[\prod_{i=1}^{\infty}(1-u_t)(x_i)\Bigg] \\
&\geq \lim_{\theta\to 0}\exp\Big\{-\frac{\theta' e^{\lambda_{\mathrm{o}}t}}{1-\gamma}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\Big\} = 1,
\end{aligned}$$

as desired for this step. Moreover, we can verify from (5.5) that the distribution of $\tilde{Z}_t$, denoted by $\mathscr{P}_t^{(\Lambda,\mu)}$, satisfies (5.3). In fact, we know that $\mathscr{P}_t^{(\Lambda,\mu)}$ is the unique probability measure on $\mathcal{N}$ satisfying (5.3), thanks to [Kal17, Theorem 4.11 (iii)].

Step 2. Fixing integer $m>1$ and real numbers $0<t_1<\cdots<t_m$, we will show in this step the convergence in distribution of the $\mathcal{N}^m$-valued random element $(Z_{t_1}^{(n)},\dots,Z_{t_m}^{(n)})$ as $n\to\infty$. From Step 1, for each $k\in\{1,\dots,m\}$, the family of $\mathcal{N}$-valued random elements $\{Z_{t_k}^{(n)}: n\in\mathbb{N}\}$ is tight. From this, it is easy to see that the family of $\mathcal{N}^m$-valued random elements $\{(Z_{t_1}^{(n)},\dots,Z_{t_m}^{(n)}): n\in\mathbb{N}\}$ is also tight, and therefore, by [Kal21, Theorem 23.2], is relatively compact in distribution. This implies the existence of a strictly increasing

sequence $(n_k)_{k=1}^{\infty}$ in $\mathbb{N}$ satisfying that the $\mathcal{N}^m$-valued random element $(Z_{t_1}^{(n_k)}, \dots, Z_{t_m}^{(n_k)})$ converges in distribution as $k \to \infty$. Let the $\mathcal{N}^m$-valued random element $(\hat{Z}_{t_1}, \dots, \hat{Z}_{t_m})$ be the corresponding subsequential convergence in distribution limit. We will show that $(\hat{Z}_{t_1}, \dots, \hat{Z}_{t_m})$ is also the convergence in distribution limit of $(Z_{t_1}^{(n)}, \dots, Z_{t_m}^{(n)})$ as $n \to \infty$. To do this, by [Kal17, Theorem 4.11 (iii)], it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}\Big[e^{-\sum_{i=1}^m Z_{t_i}^{(n)}(g_i)}\Big] \quad \text{exists in } \mathbb{R}, \tag{5.7}$$

where, for each $i \in \{1, \dots, m\}$, g_i is an arbitrarily chosen non-negative continuous function on $\mathbb{R}$ with compact support satisfying that $\|1 - e^{-g_i}\|_\infty < \Psi'(0+)/(4\beta_{\mathrm{c}})$. In fact, if (5.7) holds, it must be the case that

$$\mathbb{E}\Big[e^{-\sum_{i=1}^m \hat{Z}_{t_i}(g_i)}\Big] = \lim_{n\to\infty} \mathbb{E}\Big[e^{-\sum_{i=1}^m Z_{t_i}^{(n)}(g_i)}\Big]. \tag{5.8}$$

By the principle of induction, we can assume without loss of generality that (5.7) holds with m being replaced by $m-1$. In particular, we can assume that $(\hat{Z}_{t_1}, \dots, \hat{Z}_{t_{m-1}})$ is the convergence in distribution limit of $(Z_{t_1}^{(n)}, \dots, Z_{t_{m-1}}^{(n)})$ as $n \to \infty$. By Skorokhod's representation theorem, we can further assume (in this step) without loss of generality that $\{(Z_{t_1}^{(n)}, \dots, Z_{t_{m-1}}^{(n)}) : n \in \mathbb{N}\}$ and $(\hat{Z}_{t_1}, \dots, \hat{Z}_{t_{m-1}})$ are coupled in one probability space so that $(Z_{t_1}^{(n)}, \dots, Z_{t_{m-1}}^{(n)})$ converges almost surely to $(\hat{Z}_{t_1}, \dots, \hat{Z}_{t_{m-1}})$ when $n \to \infty$.

For every $t > 0$ and $h \in \mathcal{C}(\mathbb{R}, [0, \Psi'(0+)/(4\beta_{\mathrm{c}})])$ with compact support, define

$$H_t^h(\nu) := \tilde{\mathbb{E}}_h\Bigg[\prod_{z\in\mathbb{R}} (1 - u_t(z))^{\nu(\{z\})}\Bigg], \quad \nu \in \mathcal{N}, \tag{5.9}$$

which is a bounded continuous function on $\mathcal{N}$, according to Corollary 5.2. From Proposition 3.3 and the Markov property of the process $(Z_t^{(n)})_{t\geq 0}$, almost surely

$$\mathbb{E}\Big[e^{-\sum_{i=1}^m Z_{t_i}^{(n)}(g_i)}\Big|(Z_s^{(n)})_{s\leq t_{m-1}}\Big] = e^{-\sum_{i=1}^{m-1} Z_{t_i}^{(n)}(g_i)} H_{t_m - t_{m-1}}^{1-e^{-g_m}}(Z_{t_{m-1}}^{(n)}).$$

Therefore, by the dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}\Big[e^{-\sum_{i=1}^m Z_{t_i}^{(n)}(g_i)}\Big] &= \lim_{n\to\infty} \mathbb{E}\Big[e^{-\sum_{i=1}^{m-1} Z_{t_i}^{(n)}(g_i)} H_{t_m - t_{m-1}}^{1-e^{-g_m}}(Z_{t_{m-1}}^{(n)})\Big] \\ &= \mathbb{E}\Big[e^{-\sum_{i=1}^{m-1} \hat{Z}_{t_i}(g_i)} H_{t_m - t_{m-1}}^{1-e^{-g_m}}(\hat{Z}_{t_{m-1}})\Big]. \end{aligned} \tag{5.10}$$

This implies (5.7), and therefore, the desired result of this step.

Step 3. For every integer $m \in \mathbb{N}$ and real numbers $0 < t_1 < \cdots < t_m$, denote by $\mathscr{P}_{t_1,\dots,t_m}^{(\Lambda,\mu)}$ the distribution of the $\mathcal{N}^m$-valued random elements $(\hat{Z}_{t_1}, \dots, \hat{Z}_{t_m})$ given as in the previous steps. It is straightforward to verify that the family of distributions $(\mathscr{P}_{t_1,\dots,t_m}^{(\Lambda,\mu)})_{m\in\mathbb{N}, 0<t_1<\cdots<t_m}$ is projective in the sense of [Kal21, p. 179]. Recall that $\mathcal{N}$ is Polish. Therefore, by Kolmogorov's extension theorem (see [Kal21, Theorem 8.23] for example), there exists an $\mathcal{N}$-valued process $(\tilde{Z}_t)_{t>0}$ such that, for every integer $m \in \mathbb{N}$

and real numbers $0 < t_1 < \cdots < t_m$, the distribution of $(\tilde{Z}_{t_1}, \dots, \tilde{Z}_{t_m})$ is given by $\mathscr{P}^{(\Lambda,\mu)}_{t_1,\dots,t_m}$. Moreover, from (5.8) and (5.10), we have

$$\mathbb{E}\Big[e^{-\sum_{i=1}^m \tilde{Z}_{t_i}(g_i)}\Big] = \mathbb{E}\Big[e^{-\sum_{i=1}^{m-1} \tilde{Z}_{t_i}(g_i)} H^{1-e^{-g_m}}_{t_m - t_{m-1}}(\tilde{Z}_{t_{m-1}})\Big] \tag{5.11}$$

for every $m \in \{2, 3, \dots\}$, $0 < t_1 < \cdots < t_m$ and testing functions $(g_i)_{i=1}^m$ given as in Step 2.

Step 4. Note that, the result in Step 1 essentially implies that, for any $t > 0$, closed subset $\tilde{\Lambda}$ of $\mathbb{R}$, and locally finite integer-valued measure $\tilde{\mu}$ on $\tilde{\Lambda}^c$, there exists a unique probability measure $\tilde{\mathscr{P}}$ on $\mathcal{N}$, such that, for any non-negative continuous function g on $\mathbb{R}$ with compact support,

$$\int e^{-\tilde{\nu}(g)} \tilde{\mathscr{P}}(\mathrm{d}\tilde{\nu}) = \tilde{\mathbb{E}}_{1-e^{-g}}\Bigg[\mathbf{1}_{\{u_t(x)=0, \forall x \in \tilde{\Lambda}\}} \prod_{x \in \tilde{\Lambda}^c} (1 - u_t(x))^{\tilde{\mu}(\{x\})}\Bigg].$$

For every $t > 0$ and $\nu \in \mathcal{N}$, by taking $\tilde{\Lambda} = \emptyset$ and $\tilde{\mu} = \nu$ in the above statement, we know that there exists a unique probability measure $\mathscr{Q}_t(\nu, \cdot)$ on $\mathcal{N}$ such that (5.4) holds. (When $t = 0$, we set $\mathscr{Q}_t(\nu, \cdot) = \delta_\nu$.)

It can be verified that $(\mathscr{Q}_t)_{t \geq 0}$ is a family of kernels on $\mathcal{N}$. In fact, fixing $t > 0$, denote by $\mathcal{H}$ the collection of bounded measurable function F on $\mathcal{N}$ such that $\nu \mapsto \int F(\tilde{\nu}) \mathscr{Q}_t(\nu, \mathrm{d}\tilde{\nu})$ is a measurable map from $\mathcal{N}$ to $\mathbb{R}$. It is clear that $\mathcal{H}$ is a monotone vector space (MVS) in the sense of [Sha88, Appendix A0]. Denote by $\mathcal{K}$ the collection of bounded measurable map $\tilde{\nu} \mapsto e^{-\tilde{\nu}(g)}$ from $\mathcal{N}$ to $\mathbb{R}$ where g is a non-negative continuous function on $\mathbb{R}$ with compact support. Now, from (4.4), (5.4) and [Kal21, Lemma 1.28], It can been argued that, for every $F \in \mathcal{K}$, $\nu \mapsto \int F(\tilde{\nu}) \mathscr{Q}_t(\nu, d\tilde{\nu})$ is a measurable map from $\mathcal{N}$ to $\mathbb{R}$. In other word, $\mathcal{K} \subset \mathcal{H}$. Also, note that $\mathcal{K}$ is a multiplicative class of bounded real functions on $\mathcal{N}$ in the sense of [Sha88, Appendix A0]. It is also clear that $\sigma(\mathcal{K})$ is the Borel σ-field $\mathcal{B}_{\mathcal{N}}$ of $\mathcal{N}$ generated by the vague topology. So from [Sha88, Theorem A0.6], we have $\mathcal{B}_{\mathrm{b}}(\mathcal{N}) \subset \mathcal{H}$. Here, $\mathcal{B}_{\mathrm{b}}(\mathcal{N})$ represents the collection of bounded Borel measurable functions on $\mathcal{N}$. This proves that $\mathscr{Q}_t$ is a kernel from $\mathcal{N}$ to itself.

Step 5. Let $(\mathscr{Q}_t)_{t \geq 0}$ be the family of probability kernels given as in Step 4. For every $m \in \{2, 3, \dots\}$, $0 < t_1 < \cdots < t_m$ and testing functions $(g_i)_{i=1}^m$ chosen as in Step 2, from (5.11) we have

$$\mathbb{E}\Big[e^{-\sum_{i=1}^m \tilde{Z}_{t_i}(g_i)}\Big] = \mathbb{E}\Big[e^{-\sum_{i=1}^{m-1} \tilde{Z}_{t_i}(g_i)} \int e^{-\tilde{\nu}(g_m)} \mathscr{Q}_{t_m - t_{m-1}}(\tilde{Z}_{t_{m-1}}, \mathrm{d}\tilde{\nu})\Big].$$

From this, it is clear that $(\tilde{Z}_t)_{t>0}$ is a time-homogeneous Markov process with transition kernels $(\mathscr{Q}_t)_{t \geq 0}$. The entrance laws of this Markov process, i.e. its one-dimensional distributions $(\mathscr{P}^{(\Lambda,\mu)}_t)_{t>0}$, was characterized already in Step 1 through (5.3). We are done. □

5.2. **Stochastic right continuity.** In this subsection, we are going to give several preliminary results on the Markov process $(\tilde{Z}_t)_{t>0}$ given as in Proposition 5.3. Without loss of generality, we assume that $(\tilde{Z}_t)_{t \geq 0}$ is the canonical process defined on the path space $\Omega := \mathcal{N}^{(0,\infty)}$, which is the collection of maps from $(0,\infty)$ to $\mathcal{N}$. More precisely, $\tilde{Z}_t(\omega) = \omega(t)$ for every $\omega \in \Omega$ and $t > 0$. Let $\mathcal{F}^{\tilde{Z}}$ and $(\mathcal{F}_t^{\tilde{Z}})_{t>0}$ be the natural σ-field, and the natural filtration, generated by the process $(\tilde{Z}_t)_{t>0}$. The corresponding probability measure, and the expectation operator, will be denoted by $\mathbb{P}_{(\Lambda,\mu)}$, and $\mathbb{E}_{(\Lambda,\mu)}$, respectively.

To show the existence of a càdlàg modification of a process, one typically needs information about its finite dimensional distributions. Equations (5.3) and (5.4) can be regarded as the duality formulas between the process $(\tilde{Z}_t)_{t>0}$ and the SPDE (3.1), which essentially characterizes the finite dimensional distributions of $(\tilde{Z}_t)_{t>0}$. Note that they hold under the condition that the initial value of the dual SPDE $(u_t)_{t\geq 0}$ is a non-negative, compactly supported, continuous function bounded by 1. This condition will be relaxed in Proposition 5.6 and Corollary 5.7 below where the following analytical lemma will play a role. (The proof of this analytic lemma is included in the Supplement Material [HS25].)

Lemma 5.5. *For each $i \in \mathbb{N}$, let $(z_i^{(m)})_{m\in\mathbb{N}}$ be an increasing sequence in $[0, z^*]$ whose limit is denoted by $z_i \in [0, z^*]$. Then*

$$\lim_{m\to\infty} \prod_{i=1}^{\infty} \left(1 - z_i^{(m)}\right) = \prod_{i=1}^{\infty} (1 - z_i). \tag{5.12}$$

Proposition 5.6. *Let $t > 0$. It holds that*

$$\int \left(\prod_{x\in\mathbb{R}} (1 - f(x))^{\nu(\{x\})} \right) \mathscr{P}_t^{(\Lambda,\mu)}(\mathrm{d}\nu) = \tilde{\mathbb{E}}_f \left[\prod_{i=1}^{\infty} (1 - u_t(x_i)) \right].$$

Proof. Step 1. Let us first assume that f is a continuous $[0, z^*]$-valued function on $\mathbb{R}$ with compact support. From Lemma 5.1, we known that the map $\nu \mapsto \prod_{x\in\mathbb{R}}(1 - f(x))^{\nu(\{x\})}$ from $\mathcal{N}$ to $[-1, 1]$ is continuous. Now, from Proposition 5.3 and Lemma 4.3, we have

$$\mathbb{E}_{(\Lambda,\mu)} \left[\prod_{x\in\mathbb{R}} (1 - f(x))^{\tilde{Z}_t(\{x\})} \right] = \lim_{n\to\infty} \mathbb{E} \left[\prod_{x\in\mathbb{R}} (1 - f(x))^{Z_t^{(n)}(\{x\})} \right] = \tilde{\mathbb{E}}_f \left[\prod_{i=1}^{\infty} (1 - u_t(x_i)) \right]$$

as desired.

Step 2. Let us now assume that f is a $[0, z^*]$-valued measurable function on $\mathbb{R}$ which can be approximated by a sequence $(\tilde{f}_m)_{m\in\mathbb{N}}$ in $\mathcal{C}(\mathbb{R}, [0, z^*])$ monotonically from below. Let $(\tilde{g}_m)_{m\in\mathbb{N}}$ be a sequence of continuous functions on $\mathbb{R}$ with compact support which approximates $\mathbf{1}_{\mathbb{R}}$ from below. Then it is clear that f can be approximated from below by $(f_m)_{i\in\mathbb{N}} := (\tilde{f}_m \tilde{g}_m)_{m\in\mathbb{N}}$, a sequence of $[0, z^*]$-valued continuous functions on $\mathbb{R}$ with compact support. Without loss of generality, it is standard (see the proof of Proposition 3.6 in the Supplementary Material [HS25]) to assume the existence of a sequence of $\mathcal{C}([0,\infty), \mathcal{C}(\mathbb{R}, [0, z^*]))$-valued random element $(u^{(m)})_{m\in\mathbb{N}}$ such that

- for each $m \in \mathbb{N}$, $u^{(m)}$ has the law $\mathscr{L}_{f_m}$;
- almost surely, $u^{(m)} \leq u^{(m+1)}$ on $[0, \infty) \times \mathbb{R}$ for each $m \in \mathbb{N}$; and that
- almost surely, for almost every $(s, y) \in (0, \infty) \times \mathbb{R}$ w.r.t. the Lebesgue measure, $u_s^{(m)}(y) \uparrow u_s(y)$.

From what we have proved in Step 1, we have

$$\mathbb{E}_{(\Lambda,\mu)}\left[\prod_{x\in\mathbb{R}}(1-f_m(x))^{\tilde{Z}_t(\{x\})}\right]=\tilde{\mathbb{E}}_f\left[\prod_{i=1}^{\infty}\Big(1-u_t^{(m)}(x_i)\Big)\right].$$

Taking $m\uparrow\infty$, from Lemma 5.5 and bounded convergence theorem, we obtain the desired result. □

From Proposition 5.6 and Remark 5.4, one can verify the following result which generalizes (5.4).

Corollary 5.7. *Let $t>0$ and $\nu\in\mathcal{N}$. It holds that*

$$\int\left(\prod_{x\in\mathbb{R}}(1-f(x))^{\tilde{\nu}(\{x\})}\right)\mathscr{Q}_t(\nu,\mathrm{d}\tilde{\nu})=\tilde{\mathbb{E}}_f\left[\prod_{x\in\mathbb{R}}(1-u_t(x))^{\nu(\{x\})}\right].$$

As an application of Proposition 5.6, we can control the expected number of particles in any given interval at any given time in the following proposition. This result will be needed later when we show that $(\tilde{Z}_t)_{t>0}$ has a càdlàg version.

Proposition 5.8. *Suppose that F is a closed interval containing $\cup_{i\in\mathbb{N}}(x_i-1,x_i+1)$ and that U is an open interval. Suppose that $U\cap F$ is bounded. Let $\gamma\in(0,1)$ and $\gamma_0=\gamma\Psi'(0+)/(2\beta_c)$. Then for every $t>0$,*

$$\mathbb{E}_{(\Lambda,\mu)}\left[\tilde{Z}_t(U)\right]\leq\frac{e^{\lambda_\circ t}}{1-\gamma}\left(\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y+\frac{\Psi'(0+)e^{\lambda_\circ t}}{2}\mathcal{V}_t^{(\Lambda,\mu,F)}\right)+3C_1(U,F,t,\gamma_0)<\infty.$$

Here, $C_1(\cdot,\cdot,\cdot,\cdot)$ is the constant given as in Lemma 4.7.

We omit the proof of the above proposition because it is similar to that of [BMS24a, Proposition 2.6]. (A detailed proof can be found in the Supplementary Material [HS25].)

Corollary 5.9. *Suppose that g is a bounded non-negative continuous function whose support is contained in an open interval U. Suppose that $U\cap F$ is bounded where F is a closed interval containing $\cup_{i=1}^{\infty}(x_i-1,x_i+1)$. Then for any $b\geq a>0$,*

$$\sup_{t\in[a,b]}\mathbb{E}_{(\Lambda,\mu)}\left[\tilde{Z}_t(g)\right]<\infty.$$

Proof. Fix arbitrary $b\geq a>0$. Let $\gamma\in(0,1)$ and $\gamma_0=\gamma\Psi'(0+)/(2\beta_c)$. From Proposition 5.8, for every $t>0$,

$$\begin{aligned}\mathbb{E}_{(\Lambda,\mu)}\left[\tilde{Z}_t(g)\right]&\leq\|g\|_\infty\mathbb{E}_{(\Lambda,\mu)}\left[\tilde{Z}_t(U)\right]\\&\leq\|g\|_\infty\frac{e^{\lambda_\circ t}}{1-\gamma}\left(\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y+\frac{\Psi'(0+)e^{\lambda_\circ t}}{2}\mathcal{V}_t^{(\Lambda,\mu,\mathbb{R})}\right)+3\|g\|_\infty C_1(U,\mathbb{R},t,\gamma_0).\end{aligned}$$

From Lemma 4.5 (i),

$$\sup_{t\in[a,b]}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y<\infty;$$

from Lemma 4.5 (ii), $t \mapsto \mathcal{V}_t^{(\Lambda,\mu,\mathbb{R})}$ is a finite increasing function on $(0,\infty)$; from Lemma 4.7, $t \mapsto C_1(U,\mathbb{R},t,\gamma_0)$ is a finite increasing function on $(0,\infty)$. Therefore, for every $b \geq t \geq a > 0$, we have

$$\begin{aligned}&\sup_{t\in[a,b]} \mathbb{E}_{(\Lambda,\mu)}\left[\tilde{Z}_t(g)\right] \\ &\leq \|g\|_\infty \frac{e^{\lambda_\circ b}}{1-\gamma}\left(\sup_{t\in[a,b]}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y + \frac{\Psi'(0+)e^{\lambda_\circ b}}{2}\mathcal{V}_b^{(\Lambda,\mu,\mathbb{R})}\right) + 3\|g\|_\infty C_1(U,\mathbb{R},b,\gamma_0) \\ &< \infty\end{aligned}$$

as desired for this lemma. □

As mentioned earlier, we want to show that $(\tilde{Z}_t)_{t>0}$ has a càdlàg modification. The idea is to consider, for each $g \in \mathcal{C}_c^\infty(\mathbb{R})$, the following "super-martingale":

$$e^{\Phi'(0+)t}\tilde{Z}_t(g) - \frac{1}{2}\int_0^t e^{\Phi'(0+)s}\tilde{Z}_s(g)\mathrm{d}s, \quad t \geq 0. \tag{5.13}$$

Two technical problems arise:

(5.14) To utilize the regularization theory of martingales/super-martingales, e.g. [Kal21, Theorem 9.28], one typically needs to work with a filtration satisfying the usual hypothesis rather than the natural filtration $(\mathcal{F}_t^{\tilde{Z}})_{t>0}$.

(5.15) The integral term on the right hand side of (5.13) is not well-defined yet, because it is not clear whether $\tilde{Z}_s(g)$ is measurable in s or not.

Let the σ-field $\mathcal{F}$ and filtration $(\mathcal{F}_t)_{t>0}$ be the usual augmentation of $\mathcal{F}^{\tilde{Z}}$ and $(\mathcal{F}_t^{\tilde{Z}})_{t>0}$ w.r.t. the probability $\mathbb{P}_{(\Lambda,\mu)}$ in the sense of [Kal21, Lemma 9.8]. We will fix the first technical problem (5.14) by showing that $(\tilde{Z}_t)_{t>0}$ is a Markov process w.r.t. the filtration $(\mathcal{F}_t)_{t>0}$ in Proposition 5.12 below. We will fix the second problem (5.15) by showing that $(\tilde{Z}_s(g))_{s>0}$ has a measurable version in Proposition 5.13 below. Here, we say a real-valued process $(A_t)_{t>0}$ defined on the probability space $(\Omega,\mathcal{F},\mathbb{P}_{(\Lambda,\mu)})$ is measurable, if $(\omega,t) \mapsto A_t(\omega)$ is a measurable map from the product measurable space $(\Omega\times(0,\infty),\mathcal{F}\otimes\mathcal{B}_{(0,\infty)})$ to $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Our proofs of Propositions 5.12 and 5.13 rely on some preliminary results saying that $(\tilde{Z}_t)_{t>0}$ is stochastically right-continuous. We will establish those results in Lemmas 5.10 and 5.11 below.

Lemma 5.10. *Suppose that F is a closed interval containing $\cup_{i\in\mathbb{N}}(x_i-1,x_i+1)$ and that U is an open interval. Suppose that $U\cap F$ is bounded. Let g be a bounded continuous function on $\mathbb{R}$ such that the support of g is contained in U. Then $(\tilde{Z}_t(g))_{t>0}$ is stochastically right-continuous, i.e., for any $\epsilon > 0$ and $s > 0$,*

$$\lim_{t\downarrow s}\mathbb{P}_{(\Lambda,\mu)}\left(\left|\tilde{Z}_t(g) - \tilde{Z}_s(g)\right| > \epsilon\right) = 0.$$

Proof. Without loss of generality, we assume that g is non-negative. Define

$$F_U(\nu) := \nu(U) + \sum_{i\in\mathbb{Z}}\frac{\nu((i-1,i+1))}{(|i|+1)^2}, \quad \nu \in \mathcal{N}. \tag{5.16}$$

Step 1. We will show in this step that $\mathbb{E}_{(\Lambda,\mu)}[F_U(\tilde{Z}_t)] < \infty$ for every $t > 0$. Let $\gamma \in (0,1)$ and denote $\gamma_0 := \gamma\Psi'(0+)/(2\beta_c)$. From Proposition 5.8, we have

$$\begin{aligned}&\mathbb{E}_{(\Lambda,\mu)}\Big[\tilde{Z}_t((i-1,i+1))\Big]\\ &\leq \frac{e^{\lambda_o t}}{1-\gamma}\bigg(\int_{i-1}^{i+1} v_t^{(\Lambda,\mu)}(y)\mathrm{d}y + \frac{\Psi'(0+)e^{\lambda_o t}}{2}\mathcal{V}_t^{(\Lambda,\mu,F)}\bigg) + 3C_1((i-1,i+1),F,t,\gamma_0).\end{aligned}$$

From (4.8), (4.9) and how $C_1(\cdot,\cdot,\cdot,\cdot)$ is defined in Lemma 4.7, we have

$$\begin{aligned}&\mathbb{E}_{(\Lambda,\mu)}\Big[\tilde{Z}_t((i-1,i+1))\Big]\\ &\leq \frac{e^{\lambda_o t}}{1-\gamma}\bigg(\frac{4}{\Psi'(0+)t} + \frac{\Psi'(0+)e^{\lambda_o t}}{2}\mathcal{V}_t^{(\Lambda,\mu,F)}\bigg) + 3C_1((-1,1),F,t,\gamma_0).\end{aligned}$$

This, and Lemma 4.5 (ii), implies that $\sup_{i\in\mathbb{Z}}\mathbb{E}_{(\Lambda,\mu)}[\tilde{Z}_t((i-1,i+1))] < \infty$. Also from Proposition 5.8, it holds that $\mathbb{E}_{(\Lambda,\mu)}[\tilde{Z}_t(U)] < \infty$. Therefore, the desired result for this step follows.

Step 2. Fix an arbitrary $\nu \in \mathcal{N}$ satisfying that $F_U(\nu) < \infty$, we will show that

$$\lim_{s\downarrow 0}\int e^{-\theta\tilde{\nu}(g)}\mathscr{Q}_s(\nu,\mathrm{d}\tilde{\nu}) = e^{-\theta\nu(g)}, \quad \theta \geq 0. \tag{5.17}$$

To do this, let us fix the arbitrary $\theta > 0$. By Corollary 5.7, for any $s > 0$,

$$\int e^{-\theta\tilde{\nu}(g)}\mathscr{Q}_s(\nu,\mathrm{d}\tilde{\nu}) = \tilde{\mathbb{E}}_{1-e^{-\theta g}}\Bigg[\prod_{z\in\mathbb{R}}(1-u_s(z))^{\nu(\{z\})}\Bigg]. \tag{5.18}$$

It is clear that almost surely w.r.t. probability $\tilde{\mathbb{P}}_{1-e^{-\theta g}}$, for every $z \in \mathbb{R}$, $u_s(z)$ converges to $1-e^{-\theta g(z)}$ when $s \downarrow 0$. Define the closed interval

$$K := \bigg\{z \in \mathbb{R} : \mathrm{dist}(\{z\},U) \leq 1 + \frac{|z|}{2}\bigg\}. \tag{5.19}$$

Notice that $U \subset K$ and $K \setminus U$ is bounded. Since $\mathrm{supp}(g) \subset U \subset K$, we see that $\nu(g) = (\mathbf{1}_K\cdot\nu)(g)$. From the condition $F_U(\nu) < \infty$, we have $\nu(U) < \infty$. Therefore $\nu(K) = \nu(U) + \nu(K\setminus U) < \infty$. In particular, $\mathbf{1}_K\cdot\nu$ is a finite integer-valued measure on $\mathbb{R}$. Therefore, by the bounded convergence theorem

$$\lim_{s\to 0}\tilde{\mathbb{E}}_{1-e^{-\theta g}}\Bigg[\prod_{z\in\mathbb{R}}(1-u_s(z))^{(\mathbf{1}_K\cdot\nu)(\{z\})}\Bigg] = e^{-\theta(\mathbf{1}_K\cdot\nu)(g)} = e^{-\theta\nu(g)}. \tag{5.20}$$

By Lemmas 4.1 and 4.2, for every $s > 0$,

$$\begin{aligned}&\tilde{\mathbb{E}}_{1-e^{-\theta g}}\Bigg[\Bigg|\prod_{z\in\mathbb{R}}(1-u_s(z))^{\nu(\{z\})} - \prod_{z\in\mathbb{R}}(1-u_s(z))^{(\mathbf{1}_K\cdot\nu)(\{z\})}\Bigg|\Bigg]\\ &\leq \tilde{\mathbb{E}}_{1-e^{-\theta g}}\Bigg[\Bigg|\prod_{z\in\mathbb{R}}(1-u_s(z))^{(\mathbf{1}_{K^c}\cdot\nu)(\{z\})} - 1\Bigg|\Bigg]\end{aligned}$$

$$\leq \sum_{z\in\mathbb{R}} \tilde{\mathbb{E}}_{1-e^{-\theta g}}[u_s(z)](\mathbf{1}_{K^c}\cdot\nu)(\{z\}) \leq e^{\lambda_o s}\sum_{z\in\mathbb{R}} \mathbf{E}_z\big[1-e^{-\theta g(B_s)}\big]\mathbf{1}_{K^c}(z)\nu(\{z\}).$$

From $1-e^{-\theta g} \leq \mathbf{1}_U$, (5.19), and Markov's inequality, we have for every $s>0$,

$$\tilde{\mathbb{E}}_{1-e^{-\theta g}}\left[\left|\prod_{z\in\mathbb{R}}(1-u_s(z))^{\nu(\{z\})} - \prod_{z\in\mathbb{R}}(1-u_s(z))^{(\mathbf{1}_K\cdot\nu)(\{z\})}\right|\right] \tag{5.21}$$

$$\leq e^{\lambda_o s}\sum_{z\in\mathbb{R}} \mathbf{P}_z(B_s\in U)\mathbf{1}_{\left\{\mathrm{dist}(\{z\},U)>1+\frac{|z|}{2}\right\}}\nu(\{z\})$$

$$\leq e^{\lambda_o s}\sum_{z\in\mathbb{R}} \mathbf{P}_0\left(|B_s|\geq 1+\frac{|z|}{2}\right)\nu(\{z\}) \leq 4se^{\lambda_o s}\sum_{z\in\mathbb{R}}\frac{\nu(\{z\})}{(2+|z|)^2}$$

$$\leq 4se^{\lambda_o s}\sum_{i\in\mathbb{Z}}\sum_{z\in\mathbb{R}}\frac{\nu(\{z\})}{(2+|z|)^2}\mathbf{1}_{\{z\in(i-1,i+1)\}} \leq 4se^{\lambda_o s}\sum_{i\in\mathbb{Z}}\frac{\nu((i-1,i+1))}{(1+|i|)^2}.$$

Therefore, from the condition that $F_U(\nu)<\infty$, the left hand side of (5.21) converges to 0 when $s\to 0$. Combine this with (5.18) and (5.20), we get (5.17) as desired.

Step 3. We will finish the proof. Fixing an arbitrary $\theta>0$. According to Step 2, for any $\nu\in\mathcal{N}$ with $F_U(\nu)<\infty$,

$$\lim_{s\downarrow 0}\int\left(e^{-\theta\tilde{\nu}(g)}-e^{-\theta\nu(g)}\right)^2\mathscr{Q}_s(\nu,\mathrm{d}\tilde{\nu})$$

$$=\lim_{s\downarrow 0}\int e^{-2\theta\tilde{\nu}(g)}\mathscr{Q}_s(\nu,\mathrm{d}\tilde{\nu}) - 2e^{-\theta\nu(g)}\lim_{s\downarrow 0}\int e^{-\theta\tilde{\nu}(g)}\mathscr{Q}_s(\nu,\mathrm{d}\tilde{\nu}) + e^{-2\theta\nu(g)} = 0.$$

Therefore, from Step 1, for every $s>0$ almost surely,

$$\lim_{r\downarrow 0}\int\left(e^{-\theta\tilde{\nu}(g)}-e^{-\theta\tilde{Z}_s(g)}\right)^2\mathscr{Q}_r(\tilde{Z}_s,\mathrm{d}\tilde{\nu})=0.$$

By the Markov property of the process $(\tilde{Z}_\cdot)$, we conclude from the bounded convergence theorem that, for any $s>0$,

$$\lim_{t\downarrow s}\mathbb{E}_{(\Lambda,\mu)}\left[\left(e^{-\theta\tilde{Z}_t(g)}-e^{-\theta\tilde{Z}_s(g)}\right)^2\right]$$

$$=\lim_{t\downarrow s}\mathbb{E}_{(\Lambda,\mu)}\left[\int\left(e^{-\theta\tilde{\nu}(g)}-e^{-\theta\tilde{Z}_s(g)}\right)^2\mathscr{Q}_{t-s}(\tilde{Z}_s,\mathrm{d}\tilde{\nu})\right]=0.$$

This says that, for any $s>0$, the random variable $e^{-\theta\tilde{Z}_t(g)}$ converges to $e^{-\theta\tilde{Z}_s(g)}$ when $t\downarrow s$ in L^2, and therefore, in probability. From the continuous mapping theorem (e.g. [Kal21, Lemma 5.3]), we get the desired result for this lemma. □

To pass the stochastic right-continuity of $(\tilde{Z}_t(g))_{t>0}$ to $(\tilde{Z}_t)_{t>0}$, we will use a metrization of $\mathcal{N}$. One can argue that, c.f. [Yan04, Theorem 6.5.8], there exists a sequence $(h_i)_{i\in\mathbb{N}}$ in $\mathcal{C}_c^\infty(\mathbb{R})$ such that the vague topology of $\mathcal{N}$ is compatible with the complete metric

$$d_{\mathcal{N}}(\nu_0,\nu_1):=\sum_{i=1}^\infty\frac{1}{2^i}(1\wedge|\nu_0(h_i)-\nu_1(h_i)|),\quad \nu_0,\nu_1\in\mathcal{N}.$$

From now on, we will fix this sequence $(h_i)_{i\in\mathbb{N}}$ and treat $\mathcal{N}$ as a complete seperable metric space.

Lemma 5.11. *The process $(\tilde{Z}_t)_{t>0}$ is stochastically right-continuous, i.e., for any $\epsilon > 0$ and $s > 0$,*

$$\lim_{t\downarrow s}\mathbb{P}_{(\Lambda,\mu)}\Big(d_{\mathcal{N}}(\tilde{Z}_t,\tilde{Z}_s) > \epsilon\Big) = 0.$$

Proof. From [Kal21, Lemma 5.2] and the fact that $d_{\mathcal{N}}$ is bounded by 1, we only have to show that, for any $s > 0$,

$$\lim_{t\downarrow s}\mathbb{E}_{(\Lambda,\mu)}\Big[d_{\mathcal{N}}(\tilde{Z}_t,\tilde{Z}_s)\Big] = \lim_{t\downarrow s}\sum_{i=1}^{\infty}2^{-i}\mathbb{E}_{(\Lambda,\mu)}\Big[1\wedge\Big|\tilde{Z}_t(h_i)-\tilde{Z}_s(h_i)\Big|\Big] = 0.$$

Note that the above holds since, from [Kal21, Lemma 5.2] again and Lemma 5.10, we have, for any $s > 0$ and $i \in \mathbb{N}$,

$$\lim_{t\downarrow s}\mathbb{E}_{(\Lambda,\mu)}\Big[1\wedge\Big|\tilde{Z}_t(h_i)-\tilde{Z}_s(h_i)\Big|\Big] = 0. \qquad \square$$

As mentioned earlier, the first technical problem (5.14) will be handled with the help of the next proposition.

Proposition 5.12. *$(\tilde{Z}_t)_{t>0}$ is a Markov process with transition kernels $(\mathscr{Q}_s)_{s\geq 0}$ w.r.t. the filtration $(\mathcal{F}_t)_{t>0}$, i.e. $(\tilde{Z}_t)_{t>0}$ is $(\mathcal{F}_t)_{t>0}$-adapted and*

$$\mathbb{P}_{(\Lambda,\mu)}\Big(\tilde{Z}_{t+s}\in A\Big|\mathcal{F}_t\Big) = \mathscr{Q}_s(\tilde{Z}_t,A), \quad a.s., A\in\mathcal{B}_{\mathcal{N}}, s\geq 0, t>0 \tag{5.22}$$

where $\mathcal{B}_{\mathcal{N}}$ is the Borel σ-field of $\mathcal{N}$ generated by the vague topology.

Proof. It is clear that $(\tilde{Z}_t)_{t>0}$ is $(\mathcal{F}_t)_{t>0}$-adapted. So we only have to verify (5.22). Note that (5.22) is trivial when $s = 0$. So, let us fix $s > 0$ and $t > 0$. Take a non-negative continuous function g on $\mathbb{R}$ with compact support satisfying that $\|1-e^{-g}\|_\infty \leq \Psi'(0+)/(4\beta_{\mathrm{c}})$. Let $\mathscr{N} := \{B \in \mathcal{F} : \mathbb{P}_{(\Lambda,\mu)}(B) = 0\}$ be the collection of the null subsets of Ω. Define $\bar{\mathcal{F}}^{\tilde{Z}}_r := \sigma(\mathcal{F}^{\tilde{Z}}_r, \mathscr{N})$ for every $r > 0$. From [Kal21, Lemma 9.8], we have $\mathcal{F}_t = \cap_{k=1}^{\infty}\bar{\mathcal{F}}^{\tilde{Z}}_{t+1/k}$. From Proposition 5.3, we have almost surely for each $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}_{(\Lambda,\mu)}\Big[e^{-\tilde{Z}_{t+1/k+s}(g)}\Big|\bar{\mathcal{F}}^{\tilde{Z}}_{t+1/k}\Big] &= \mathbb{E}_{(\Lambda,\mu)}\Big[e^{-\tilde{Z}_{t+1/k+s}(g)}\Big|\mathcal{F}^{\tilde{Z}}_{t+1/k}\Big]\\ &= \int e^{-\nu(g)}\mathscr{Q}_s(\tilde{Z}_{t+1/k},\mathrm{d}\nu) = H_s^{1-e^{-g}}(\tilde{Z}_{t+1/k}),\end{aligned} \tag{5.23}$$

where $\nu \mapsto H_s^{1-e^{-g}}(\nu)$ is the bounded continuous function on $\mathcal{N}$ given as in (5.9). Taking $k \uparrow \infty$, from the continuous mapping theorem (e.g. [Kal21, Lemma 5.3]) and Lemma 5.11, we know that the most right hand side of (5.23) converges to $H_s^{1-e^{-g}}(\tilde{Z}_t)$ in probability when $k \uparrow \infty$. From [Dur19, Theorem 4.7.3], we know that

$$\mathbb{E}_{(\Lambda,\mu)}\Big[e^{-\tilde{Z}_{t+s}(g)}\Big|\bar{\mathcal{F}}^{\tilde{Z}}_{t+1/k}\Big] \xrightarrow[k\uparrow\infty]{L^1} \mathbb{E}_{(\Lambda,\mu)}\Big[e^{-\tilde{Z}_{t+s}(g)}\Big|\mathcal{F}_t\Big]. \tag{5.24}$$

Also, from Jensen's inequality, Lemma 5.11, and the bounded convergence theorem, we have that

$$\begin{aligned}
&\mathbb{E}_{(\Lambda,\mu)}\left[\left|\mathbb{E}_{(\Lambda,\mu)}\left[e^{-\tilde{Z}_{t+1/k+s}(g)}\middle|\bar{\mathcal{F}}^{\tilde{Z}}_{t+1/k}\right]-\mathbb{E}_{(\Lambda,\mu)}\left[e^{-\tilde{Z}_{t+s}(g)}\middle|\bar{\mathcal{F}}^{\tilde{Z}}_{t+1/k}\right]\right|\right]\\
&\le \mathbb{E}_{(\Lambda,\mu)}\left[\mathbb{E}_{(\Lambda,\mu)}\left[\left|e^{-\tilde{Z}_{t+1/k+s}(g)}-e^{-\tilde{Z}_{t+s}(g)}\right|\middle|\bar{\mathcal{F}}^{\tilde{Z}}_{t+1/k}\right]\right]\\
&= \mathbb{E}_{(\Lambda,\mu)}\left[\left|e^{-\tilde{Z}_{t+1/k+s}(g)}-e^{-\tilde{Z}_{t+s}(g)}\right|\right]\xrightarrow[k\uparrow\infty]{} 0.
\end{aligned}$$

Combine this with (5.24), we know that the most left hand side of (5.23) converges to the right hand side of (5.24) in L^1 when $k\uparrow\infty$. Now, taking $k\uparrow\infty$ in (5.23), we obtain that almost surely

$$\mathbb{E}_{(\Lambda,\mu)}\left[e^{-\tilde{Z}_{t+s}(g)}\middle|\mathcal{F}_t\right]=H_s^{1-e^{-g}}(\tilde{Z}_t)=\int e^{-\nu(g)}\mathscr{Q}_s(\tilde{Z}_t,\mathrm{d}\nu).$$

From [Kal21, Theorem 8.5] and [Kal17, Theorem 2.2], we can verify the desired result for this proposition. □

The second technical problem (5.15) will be handled by the next proposition.

Proposition 5.13. *Suppose that F is a closed interval containing $\cup_{i\in\mathbb{N}}(x_i-1,x_i+1)$ and that U is an open interval. Suppose that $U\cap F$ is bounded. Let g be a bounded continuous function on $\mathbb{R}$ such that the support of g is contained in U. Then under $\mathbb{P}_{(\Lambda,\mu)}$, there exists a measurable version $(\tilde{Y}^g_t)_{t>0}$ of the process $(\tilde{Z}_t(g))_{t>0}$.*

We omit the proof of the above proposition, because it follows the standard argument similar to that of [Bor95, Theorem 6.2.3] noticing, from Lemma 5.10, that $(\tilde{Z}_t(g))_{t\ge0}$ is stochastically right continuous. (A detailed proof is included in the Supplement Material [HS25].)

5.3. **Càdlàg realization.** In this subsection, we complete the proof of Theorem 1.2.

Lemma 5.14. *Fix a smooth function g with bounded derivatives of any orders. Assume that U is an open interval containing the support of g. Let $\nu\in\mathcal{N}$ satisfy that $F_U(\nu)<\infty$ where $F_U(\nu)$ is given as in* (5.16).

(i) *It holds that*

$$\left|\mathbb{E}_{(\emptyset,\nu)}\left[\tilde{Z}_t(g)\right]-\mathbb{E}_{(\emptyset,\mathbf{1}_U\nu)}\left[\tilde{Z}_t(g)\right]\right|\le e^{-\Phi'(0+)t}\int_{U^c}\mathbf{E}_x[|g(B_t)|]\nu(\mathrm{d}x),\quad t>0.$$

Here, for each $x\in\mathbb{R}$, $(B_t)_{t\ge0}$ is a Brownian motion initiated at position x w.r.t. the expectation operator $\mathbf{E}_x$.

(ii) $(\mathbb{E}_{(\emptyset,\nu)}[\tilde{Z}_t(g)])_{t>0}$ *is continuous, and* $\lim_{t\downarrow0}\mathbb{E}_{(\emptyset,\nu)}[\tilde{Z}_t(g)]=\nu(g)$.

(iii) *Suppose further that g is non-negative. Then for every $t>0$,*

$$e^{\Phi'(0+)t}\mathbb{E}_{(\emptyset,\nu)}\left[\tilde{Z}_t(g)\right]-\nu(g)-\int_0^t e^{\Phi'(0+)s}\frac{1}{2}\mathbb{E}_{(\emptyset,\nu)}\left[\tilde{Z}_s(g'')\right]\mathrm{d}s\le 0.$$

Proof. (i). Fix an arbitrary $t > 0$. From $F_U(\nu) < \infty$ we have $\nu(U) < \infty$. In particular, since ν is an integer-valued measure, $\operatorname{supp}(\nu) \cap U$ is bounded. Therefore, from Corollary 5.9, $\mathbb{E}_{(\emptyset,\nu)}[\tilde{Z}_t(g)]$ is finite, and similarly, so is $\mathbb{E}_{(\emptyset,\mathbf{1}_U\nu)}[\tilde{Z}_t(g)]$. We can assume without loss of generality that g is non-negative. By the monotone convergence theorem,

$$\left|\mathbb{E}_{(\emptyset,\nu)}\left[\tilde{Z}_t(g)\right] - \mathbb{E}_{(\emptyset,\mathbf{1}_U\nu)}\left[\tilde{Z}_t(g)\right]\right| = \left|\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\Big(\mathbb{E}_{(\emptyset,\nu)}\left[1-e^{-\varepsilon\tilde{Z}_t(g)}\right] - \mathbb{E}_{(\emptyset,\mathbf{1}_U\nu)}\left[1-e^{-\varepsilon\tilde{Z}_t(g)}\right]\Big)\right|.$$

From Proposition 5.6, we see that

$$\begin{aligned}&\left|\mathbb{E}_{(\emptyset,\nu)}\left[\tilde{Z}_t(g)\right] - \mathbb{E}_{(\emptyset,\mathbf{1}_U\nu)}\left[\tilde{Z}_t(g)\right]\right| = \left|\lim_{\varepsilon}\frac{1}{\varepsilon}\Big(\mathbb{E}_{(\emptyset,\nu)}\left[e^{-\varepsilon\tilde{Z}_t(g)}\right] - \mathbb{E}_{(\emptyset,\mathbf{1}_U\nu)}\left[e^{-\varepsilon\tilde{Z}_t(g)}\right]\Big)\right| \\ &= \left|\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\tilde{\mathbb{E}}_{1-e^{-\varepsilon g}}\left[\left(\prod_{x\in U}(1-u_t(x))^{\nu(\{x\})}\right)\left(\prod_{x\in U^c}(1-u_t(x))^{\nu(\{z\})}-1\right)\right]\right| \\ &\le \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\tilde{\mathbb{E}}_{1-e^{-\varepsilon g}}\left[\left|\prod_{x\in U^c}(1-u_t(x))^{\nu(\{z\})}-1\right|\right]\end{aligned}$$

From Lemmas 4.1 and 4.2, Fubini's theorem, and inequality $1-e^{-|x|} \le |x|$, we get that

$$\begin{aligned}&\left|\mathbb{E}_{(\emptyset,\nu)}\left[\tilde{Z}_t(g)\right] - \mathbb{E}_{(\emptyset,\mathbf{1}_U\nu)}\left[\tilde{Z}_t(g)\right]\right| \le \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\tilde{\mathbb{E}}_{1-e^{-\varepsilon g}}\left[\int_{U^c}u_t(x)\nu(\mathrm{d}x)\right] \\ &\le e^{-\Phi'(0+)t}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{U^c}\mathbf{E}_x\left[1-e^{-\varepsilon g(B_t)}\right]\nu(\mathrm{d}x) \le e^{-\Phi'(0+)t}\int_{U^c}\mathbf{E}_x[g(B_t)]\nu(\mathrm{d}x).\end{aligned}$$

(ii). Note that for each $m \in \mathbb{N}$, $U_m := \{z : \operatorname{dist}(\{z\},U) < m\}$ is an open interval containing U and $\nu(U_m) < \infty$. Fixing an arbitrary $m \in \mathbb{N}$ in the rest of this paragraph. Let $(\hat{I}_t)_{t\ge 0} := (I_t^{(n)})_{t\ge 0}$, $(\hat{X}_t^\alpha)_{\alpha\in\hat{I}_t,t\ge 0} := (X_t^{(n),\alpha})_{\alpha\in I_t^{(n)},t\ge 0}$ and $(\hat{Z}_t)_{t\ge 0} := (Z_t^{(n)})_{t\ge 0}$ be notations given as in Subsection 1.2 (right after Proposition 1.1) for an SBBM with ordinary branching rate β_{o}, ordinary offspring law $(p_k)_{k=0}^\infty$, catalytic branching rate β_{c}, and catalytic offspring law $(q_k)_{k=0}^\infty$, given as in (1.2)–(1.5), and an initial configuration $(\hat{x}_i)_{i=1}^{\nu(U_m)}$ satisfying that $\mathbf{1}_{U_m}\nu = \sum_{i=1}^{\nu(U_m)}\delta_{\hat{x}_i}$. It is clear from Propositions 3.3 and 5.6 that

$$\mathbb{E}_{(\emptyset,\mathbf{1}_{U_m}\nu)}\left[\prod_{x\in\mathbb{R}}(1-f(x))^{\tilde{Z}_t(\{x\})}\right] = \mathbb{E}\left[\prod_{x\in\mathbb{R}}(1-f(x))^{\hat{Z}_t(\{x\})}\right], \quad t>0.$$

Since f is arbitrary, it is not hard to see that,

(5.25) for each $t > 0$, $\tilde{Z}_t$ under the probability measure $\mathbb{P}_{(\emptyset,\mathbf{1}_{U_m}\nu)}$ has the same law as $\hat{Z}_t$ under $\mathbb{P}$.

Define

$$R_t^{(m)}(g) := \mathbb{E}\left[\hat{Z}_t(g)\right] = \mathbf{1}_{\{0\}}(t)\nu(\mathbf{1}_{U_m}g) + \mathbf{1}_{(0,\infty)}(t)\mathbb{E}_{(\emptyset,\mathbf{1}_{U_m}\nu)}\left[\tilde{Z}_t(g)\right], \quad t\ge 0.$$

From Proposition 2.6, (2.8) and (2.9), we have for every $t \ge 0$,

$$e^{\Phi'(0+)t}\mathbb{E}\left[\hat{Z}_t(g)\right] = \nu(\mathbf{1}_{U_m}g) + \mathbb{E}\left[\int_0^t e^{\Phi'(0+)s}\frac{1}{2}\hat{Z}_s(g'')\mathrm{d}s\right] \tag{5.26}$$

$$-\frac{1}{2}\Psi'(0+)\mathbb{E}\left[\int_0^t e^{\Phi'(0+)s}\sum_{\{\alpha,\beta\}\subset\hat{I}_{s-}:\alpha\neq\beta} g(\hat{X}_s^\alpha)\mathrm{d}\hat{L}_s^{\{\alpha,\beta\}}\right].$$

Here, $(\hat{L}_t^{\{\alpha,\beta\}})_{t\geq 0}$ represents the intersection local time between any two particles labelled by α and β. Combining (5.26), (2.8), (2.9) and the dominated convergence theorem, we see that $(R_t^{(m)}(g))_{t\geq 0}$ is continuous.

Define

$$R_t(g) := \mathbf{1}_{\{0\}}(t)\nu(g) + \mathbf{1}_{(0,\infty)}(t)\mathbb{E}_{(\emptyset,\nu)}\left[\tilde{Z}_t(g)\right], \quad t\geq 0.$$

We want to approximate $(R_t(g))_{t\geq 0}$ by $(R_t^{(m)}(g))_{t\geq 0}$. Notice that there exists a $c_0>0$ such that for every $x\in U_1^{\mathrm{c}}$, $\mathrm{dist}(\{x\},U)\geq c_0(|x|+2)$. Also note that for every $m\geq 3$ and $i\in\mathbb{Z}$ with $(i-1,i+1)\not\subseteq U_m$, we have $(i-1,i+1)\subset U_1^{\mathrm{c}}$. Therefore, for every $m\geq 3$, $i\in\mathbb{Z}$ with $(i-1,i+1)\not\subseteq U_m$, and $x\in(i-1,i+1)$, it holds that $\mathrm{dist}(\{x\},U)\geq c_0(|x|+2)\geq c_0(|i|+1)$. It also holds that $U_m^{\mathrm{c}}\subset\bigcup_{i\in\mathbb{Z}:(i-1,i+1)\not\subseteq U_m}(i-1,i+1)$ for every $m\in\mathbb{N}$. Now, by Lemma 5.14 (i), for $t\geq 0$ and $m\geq 3$,

$$\begin{aligned}
e^{\Phi'(0+)t}\left|R_t(g)-R_t^{(m)}(g)\right| &\leq \int_{U_m^{\mathrm{c}}}\mathbf{E}_x[|g(B_t)|]\nu(\mathrm{d}x)\\
&\leq \sum_{i\in\mathbb{Z}:(i-1,i+1)\not\subseteq U_m}\nu((i-1,i+1))\sup_{x\in(i-1,i+1)}\mathbf{E}_x[|g(B_t)|]\\
&\leq \|g\|_\infty\sum_{i\in\mathbb{Z}:(i-1,i+1)\not\subseteq U_m}\nu((i-1,i+1))\sup_{x\in(i-1,i+1)}\mathbf{P}_x(B_t\in U)\\
&\leq \|g\|_\infty\sum_{i\in\mathbb{Z}:(i-1,i+1)\not\subseteq U_m}\nu((i-1,i+1))\mathbf{P}_0(|B_t|\geq c_0(|i|+1)).
\end{aligned}$$

Together with Markov's inequality, we have

$$e^{\Phi'(0+)t}\left|R_t(g)-R_t^{(m)}(g)\right|\leq\frac{t}{c_0^2}\|g\|_\infty\sum_{i\in\mathbb{Z}:(i-1,i+1)\not\subseteq U_m}\frac{\nu((i-1,i+1))}{(|i|+1)^2},\quad t\geq 0.$$

By the condition $F_U(\nu)<\infty$ and the monotone convergence theorem, we have

$$\lim_{m\to\infty}\sup_{t\in[0,T]}\left|R_t(g)-R_t^{(m)}(g)\right|=0. \tag{5.27}$$

By the uniform limit theorem, $(R_t(g))_{t\geq 0}$ must be continuous.

(iii). Fix an arbitrary $m\in\mathbb{N}$, and let $(\hat{Z}_t)_{t\geq 0}$ be the SBBM considered in (ii). From (5.25), (5.26) and the condition that g is non-negative, we have for every $t>0$

$$\begin{aligned}
&e^{\Phi'(0+)t}\mathbb{E}_{(\emptyset,\mathbf{1}_{U_m}\nu)}\left[\tilde{Z}_t(g)\right]-\nu(\mathbf{1}_{U_m}g)-\int_0^t e^{\Phi'(0+)s}\frac{1}{2}\mathbb{E}_{(\emptyset,\mathbf{1}_{U_m}\nu)}\left[\tilde{Z}_s(g'')\right]\mathrm{d}s\\
&= e^{\Phi'(0+)t}\mathbb{E}\left[\hat{Z}_t(g)\right]-\nu(\mathbf{1}_{U_m}g)-\int_0^t e^{\Phi'(0+)s}\frac{1}{2}\mathbb{E}\left[\hat{Z}_s(g'')\right]\mathrm{d}s\leq 0.
\end{aligned}$$

Now, for any $t > 0$, we have

$$\begin{aligned}
& e^{\Phi'(0+)t}\mathbb{E}_{(\emptyset,\nu)}\Big[\tilde{Z}_t(g)\Big] - \nu(g) - \int_0^t e^{\Phi'(0+)s}\frac{1}{2}\mathbb{E}_{(\emptyset,\nu)}\Big[\tilde{Z}_s(g'')\Big]\mathrm{d}s \\
& \leq e^{\Phi'(0+)t}\mathbb{E}_{(\emptyset,\mathbf{1}_{U_m}\nu)}\Big[\tilde{Z}_t(g)\Big] - \nu(\mathbf{1}_{U_m}g) - \int_0^t e^{\Phi'(0+)s}\frac{1}{2}\mathbb{E}_{(\emptyset,\mathbf{1}_{U_m}\nu)}\Big[\tilde{Z}_s(g'')\Big]\mathrm{d}s \\
& \qquad + e^{\Phi'(0+)t}\Big|R_t(g) - R_t^{(m)}(g)\Big| + \nu(\mathbf{1}_{U_m^{\mathrm{c}}}g) + \int_0^t e^{\Phi'(0+)s}\big|R_s(g'') - R_s^{(m)}(g'')\big|\mathrm{d}s \\
& \leq e^{\Phi'(0+)t}\Big|R_t(g) - R_t^{(m)}(g)\Big| + \nu(\mathbf{1}_{U_m^{\mathrm{c}}}g) + \int_0^t e^{\Phi'(0+)s}\big|R_s(g'') - R_s^{(m)}(g'')\big|\mathrm{d}s. \qquad (5.28)
\end{aligned}$$

Noticing from $\nu(g) < \infty$, (5.27), and the fact that (5.27) also holds with g being replaced by g'', the right hand side of (5.28) converges to 0 when $m \to \infty$. Therefore, the desired result in (iii) of this lemma holds. □

Lemma 5.15. *Let g be a non-negative smooth function on $\mathbb{R}$ with bounded derivatives of any order whose support is contained in an open interval U. Suppose that $U \cap F$ is bounded where F is a closed interval containing $\cup_{i=1}^{\infty}(x_i - 1, x_i + 1)$. Let $a > 0$. Let $(\tilde{Y}_t^{g''})_{t>0}$ be a measurable version of the process $(\tilde{Z}_t(g''))_{t>0}$ given as in Proposition 5.13. Then*

$$M_g(t;a) := e^{\Phi'(0+)t}\tilde{Z}_t(g) - \int_a^t e^{\Phi'(0+)s}\frac{1}{2}\tilde{Y}_s^{g''}\mathrm{d}s, \quad t \geq a, \qquad (5.29)$$

is a super-martingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq a}, \mathbb{P}_{(\Lambda,\mu)})$. In particular, $(M_g(t;a))_{t\geq a}$ has a càdlàg modification.

Proof. From Proposition 5.13 and Corollary 5.9, we know that the second term on the right hand side of (5.29) is a well-defined random variable with finite mean. Clearly the process $(M_g(t;a))_{t\geq a}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq a}$. Notice that almost surely, for every $t \geq a$ and $r \geq 0$,

$$M_g(t+r;a) - M_g(t;a) = e^{\Phi'(0+)t}\bigg(e^{\Phi'(0+)r}\tilde{Z}_{t+r}(g) - \tilde{Z}_t(g) - \int_0^r e^{\Phi'(0+)s}\frac{1}{2}\tilde{Y}_{t+s}^{g''}\mathrm{d}s\bigg).$$

Let us fix an arbitrary $t \geq a$ and an arbitrary bounded continuous function ϕ on $\mathbb{R}$ whose support is contained in U. From Proposition 5.12, [Kal21, Theorem 8.5], and Remark 5.4, for each $r \geq 0$ and $m \in \mathbb{N}$, we have almost surely,

$$\mathbb{E}_{(\Lambda,\mu)}\Big[\mathcal{X}_m\Big(\tilde{Z}_{t+r}(\phi)\Big)\Big|\mathcal{F}_t\Big] = \int \mathcal{X}_m(\nu(\phi))\mathscr{Q}_r(\tilde{Z}_t, \mathrm{d}\nu) = \mathbb{E}_{(\emptyset,\tilde{Z}_t)}\Big[\mathcal{X}_m\Big(\tilde{Z}_r(\phi)\Big)\Big],$$

where the truncation function $\mathcal{X}_m(z) := z\mathbf{1}_{[-m,m]}(z)$ for every $z \in \mathbb{R}$. Taking $m \uparrow \infty$, it can be verified from the dominated convergence theorem and Corollary 5.9 that, for each $r \geq 0$, almost surely, $\mathbb{E}_{(\Lambda,\mu)}[\tilde{Z}_{t+r}(\phi)|\mathcal{F}_t] = \mathbb{E}_{(\emptyset,\tilde{Z}_t)}[\tilde{Z}_r(\phi)]$. Combine this with Proposition 5.13, we have, for each $r \geq 0$, almost surely, $\mathbb{E}_{(\Lambda,\mu)}[\tilde{Y}_{t+r}^{\phi}|\mathcal{F}_t] = \mathbb{E}_{(\emptyset,\tilde{Z}_t)}[\tilde{Z}_r(\phi)]$. Note from Lemma 5.14 (ii), almost surely, $\mathbb{E}_{(\emptyset,\tilde{Z}_t)}[\tilde{Z}_r(\phi)]$ is continuous, and therefore measurable, in

$r > 0$. From Corollary 5.9 and Fubini's theorem, we can verify that, for each $r \geq 0$, almost surely,

$$\mathbb{E}_{(\Lambda,\mu)}\left[\int_0^r e^{\Phi'(0+)s}\frac{1}{2}\tilde{Y}^{\phi}_{t+s}\mathrm{d}s\middle|\mathcal{F}_t\right] = \int_0^r e^{\Phi'(0+)s}\frac{1}{2}\mathbb{E}_{(\emptyset,\tilde{Z}_t)}\left[\tilde{Z}_s(\phi)\right]\mathrm{d}s.$$

Now, we can verify that for each $r \geq 0$, almost surely

$$\begin{aligned}&\mathbb{E}_{(\Lambda,\mu)}[M_g(t+r;a) - M_g(t;a)|\mathcal{F}_t]\\&= e^{\Phi'(0+)t}\left(e^{\Phi'(0+)r}\mathbb{E}_{(\emptyset,\tilde{Z}_t)}\left[\tilde{Z}_r(g)\right] - \tilde{Z}_t(g) - \int_0^r e^{\Phi'(0+)s}\frac{1}{2}\mathbb{E}_{(\emptyset,\tilde{Z}_t)}\left[\tilde{Z}_s(g'')\right]\mathrm{d}s\right).\end{aligned}$$

From the Step 1 of the proof of Lemma 5.10, we know that $F_U(\tilde{Z}_t) < \infty$ almost surely w.r.t. $\mathbb{P}_{(\Lambda,\mu)}$. Therefore, from Lemma 5.14 (iii), we know that, for each $r \geq 0$, almost surely, $\mathbb{E}_{(\Lambda,\mu)}[M_g(t+r;a) - M_g(t;a)|\mathcal{F}_t] \leq 0$. Since $t \geq a$ is chosen arbitrarilly, we have that $(M_g(t;a))_{t\geq a}$ is a super-martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\geq a}$. It can also be verified from Lemma 5.14 (ii) and dominated convergence theorem that $(\mathbb{E}_{(\Lambda,\mu)}[M_g(t;a)])_{t\geq a}$ is a continuous process. Therefore, from [Kal21, Theorem 9.28], $(M_g(t;a))_{t\geq a}$ has a càdlàg modification. We are done. □

The following lemma is standard. (Its proof is induced in the Supplementary Material [HS25].)

Lemma 5.16. *Suppose that $a \geq 0$ and $(\tilde{Z}_t)_{t\geq a}$ is a $\mathcal{N}$-valued stochastic process such that $(\tilde{Z}_t(g))_{t\geq a}$ admits a càdlàg modification for every smooth function g on $\mathbb{R}$ with compact support. Then $(\tilde{Z}_t)_{t\geq a}$ itself admits a càdlàg modification.*

Proof of Theorem 1.2. Since the convergence in finite dimensional distributions of the processes $(Z_t^{(n)})_{t>0}$ is already established in Proposition 5.3, we only have to show that the corresponding limit, i.e. the $\mathcal{N}$-valued process $(\tilde{Z}_t)_{t>0}$, has a càdlàg modification. It is suffice to show that $(\tilde{Z}_t)_{t\geq a}$ has a càdlàg modification for an arbitrarily fixed $a > 0$. Note from Lemma 5.15 that, for any smooth function g on $\mathbb{R}$ with compact support, the process $(\tilde{Z}_t(g))_{t\geq a}$ has a càdlàg modification. Now the desired result follows from Lemma 5.16. □

6. Coming down from infinity: Proof of Theorem 1.3

Let $(x_i)_{i=1}^{\infty}$, β_{o}, $(p_k)_{k=0}^{\infty}$, β_{c} and $(q_k)_{k=0}^{\infty}$ be given as in (1.2)–(1.5). Assume that (1.10), (1.11) and (1.16) hold. Let $(\Lambda,\mu) \in \mathcal{T}$ be given as in (1.12) and (1.13). Let Φ and Ψ be given as in (1.14) and (1.15) respectively. Let $(Z_t)_{t>0}$ be an SBBM with initial trace (Λ,μ), ordinary branching mechanism Φ, and catalytic branching mechanism Ψ. That is to say, $(Z_t)_{t>0}$ is the unique in law $\mathcal{N}$-valued càdlàg Markov process given as in Theorem 1.2. Note that the entrance law $(\mathscr{P}_t^{(\Lambda,\mu)})_{t>0}$ and the transition kernels $(\mathscr{Q}_t)_{t\geq 0}$ of the process $(Z_t)_{t>0}$ are given as in Proposition 5.3. For every $(\tilde{\Lambda},\tilde{\mu}) \in \mathcal{T}$, let $(v_t^{(\tilde{\Lambda},\tilde{\mu})}(x))_{t>0,x\in\mathbb{R}} \in \mathcal{C}^{1,2}((0,\infty)\times\mathbb{R})$ be the unique non-negative solution to the equation (1.17).

In this section, we will prove Theorem 1.3. We assume without loss of generality that $(Z_t)_{t>0}$ is the canonical process of $\mathbb{D}((0,\infty),\mathcal{N})$, the space of $\mathcal{N}$-valued càdlàg paths indexed by $(0,\infty)$. More precisely, for any $\omega \in \mathbb{D}((0,\infty),\mathcal{N})$ and $t>0$, $Z_t(\omega)=w_t$. Note that this setup is different from Section 5.2 where $(Z_t)_{t>0}$ was the càdlàg modification of the canonical process of the path space $\mathcal{N}^{(0,\infty)}$. Correspondingly, we redefine our probability space $\Omega := \mathbb{D}((0,\infty,\mathcal{N}))$. Let $\mathcal{F}^Z$ and $(\mathcal{F}^Z_t)_{t>0}$ be the natural σ-field and the natural filtration generated by the process $(Z_t)_{t>0}$. For any closed subset $\tilde{\Lambda}$ of $\mathbb{R}$ and integer-valued locally finite measure $\tilde{\mu}$ on $\tilde{\Lambda}^{\mathrm{c}}$, denote by $\mathbb{P}_{(\tilde{\Lambda},\tilde{\mu})}$ the law of an SBBM with initial trace $(\tilde{\Lambda},\tilde{\mu})$ induced on $(\Omega,\mathcal{F}^Z)$. Let the σ-field $\mathcal{F}$ and the filtration $(\mathcal{F}_t)_{t>0}$ be the usual augmentation of $\mathcal{F}^Z$ and $(\mathcal{F}^Z_t)_{t>0}$ w.r.t. the probability $\mathbb{P}_{(\Lambda,\mu)}$.

Intuitively speaking, Z_t behaves similarly to its mean field counterpart v_t when t is small. The next lemma gives the integrability of v_t on a given interval U in term of the intersection between U and the initial trace.

Lemma 6.1. *Let U be an open interval of $\mathbb{R}$.*

(i) *Let $t>0$. Then $\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x<\infty$ if and only if $U\cap \operatorname{supp}(\Lambda,\mu)$ is bounded.*

(ii) *Suppose that $U\cap\operatorname{supp}(\Lambda,\mu)$ is bounded. Then*

$$\bar{U}\cap\Lambda=\emptyset \implies \limsup_{t\downarrow 0}\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x<\infty$$

and

$$\bar{U}\cap\Lambda\neq\emptyset \implies \lim_{t\downarrow 0}\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x=\infty.$$

Proof. (i). The sufficiency of the boundedness of $U\cap\operatorname{supp}(\Lambda,\mu)$ follows from Lemma 4.5 (i). To show its necessity, let us assume otherwise that $U\cap\operatorname{supp}(\Lambda,\mu)$ is unbounded. In this case, we can find a sequence $(z_i)_{i\in\mathbb{N}}$ in $U\cap\operatorname{supp}(\Lambda,\mu)$ such that $\{(z_i-1,z_i+1): i\in\mathbb{N}\}$ is a family of disjoint subinterval of U. Then, by (4.7) and [BMS24a, (2.4)], we can verify

$$\begin{aligned}\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x &\geq \sum_{i=1}^{\infty}\int_{z_i-1}^{z_i+1} v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\\ &\geq \sum_{i=1}^{\infty}\int_{z_i-1}^{z_i+1} v_t^{(\emptyset,\delta_{z_i})}(x)\mathrm{d}x=\sum_{i=1}^{\infty}\int_{-1}^{1} v_t^{(\emptyset,\delta_0)}(x)\mathrm{d}x=\infty.\end{aligned}$$

(ii). Observing (4.7), this is done in [BMS24a, Lemma 3.2]. □

The next two lemmas demenstrate how the 'density' of Z_t is comparable to the solution $v_t^{(\Lambda,\mu)}$ of the MFE.

Lemma 6.2. *Let U be an open interval such that $U\cap\operatorname{supp}(\Lambda,\mu)$ is bounded. Suppose that $\bar{U}\cap\Lambda\neq\emptyset$. Then*

$$\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)^{-1}\mathbb{E}_{(\Lambda,\mu)}[Z_t(U)]\xrightarrow{t\downarrow 0}1.$$

We omit the proof of the above lemma because it is similar to that of [BMS24a, Lemma 3.3]. (We include its proof in the Supplementary Material [HS25].)

Lemma 6.3. *Let U be an open interval such that $U \cap \operatorname{supp}(\Lambda,\mu)$ is bounded. Suppose that $\bar{U} \cap \Lambda \neq \emptyset$. Then*

$$\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)^{-1} Z_t(U) \xrightarrow{t\downarrow 0} 1, \quad \textit{in probability.}$$

We omit the proof of Lemma 6.3 because it is similar to that of [BMS24a, Lemma 3.4]. (We include its proof in the Supplementary Material [HS25].)

The following technical lemma will be used in the proof of Theorem 1.3.

Lemma 6.4. *Let $m \in \mathbb{N}$, $(t_i)_{i=1}^m$ be a list in $(0,\infty)$, and $(f_i)_{i=1}^m$ be a list of non-negative elements in $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$. Then*

$$\nu_0 \mapsto \int_{\mathcal{N}^m} \prod_{i=1}^m \left(e^{-\nu_i(f_i)} \mathscr{Q}_{t_i}(\nu_{i-1}, \mathrm{d}\nu_i)\right)$$

is a continuous function on $\mathcal{N}$.

Proof. Let $(\nu^{(N)})_{N\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{N}$ converging to some element $\nu \in \mathcal{N}$. From Corollaries 5.2 and 5.7, we know that

$$\lim_{N\to\infty} \int_{\mathcal{N}} e^{-\nu_1(f)} \mathscr{Q}_{t_1}(\nu^{(N)}, \mathrm{d}\nu_1) = \int_{\mathcal{N}} e^{-\nu_1(f)} \mathscr{Q}_{t_1}(\nu, \mathrm{d}\nu_1)$$

for every non-negative $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$ satisfying $\|1 - e^{-f}\|_\infty \leq \Psi'(0+)/(4\beta_{\mathrm{c}})$. From this and [Kal17, Theorem 4.11 (iii)], we know that the probability measures $\mathscr{Q}_{t_1}(\nu^{(N)}, \cdot)$ converges weakly to $\mathscr{Q}_{t_1}(\nu, \cdot)$ as $N \uparrow \infty$. Therefore, for any bounded continuous function G on $\mathcal{N}$,

$$\lim_{N\to\infty} \int_{\mathcal{N}} G(\nu_1) \mathscr{Q}_{t_1}(\nu^{(N)}, \mathrm{d}\nu_1) = \int_{\mathcal{N}} G(\nu_1) \mathscr{Q}_{t_1}(\nu, \mathrm{d}\nu_1).$$

Since the converging sequence $(\nu^{(N)})_{N\in\mathbb{N}}$ in $\mathcal{N}$ is chosen arbitrarily, the above says that

(6.1) the map $\nu_0 \mapsto \int_{\mathcal{N}} G(\nu_1)\mathscr{Q}_{t_1}(\nu_0, \mathrm{d}\nu_1)$ from $\mathcal{N}$ to $\mathbb{R}$ is continuous for every bounded continuous function G on $\mathcal{N}$.

In particular, the desired result of this lemma holds when $m = 1$.

Let us now assume, for the sake of induction, that the desired result of this lemma holds when m is replaced by $m - 1$. Under this assumption,

$$\tilde{G}(\nu_1) := \int_{\mathcal{N}^{m-1}} \prod_{i=2}^m (e^{-\nu_i(f_i)} \mathscr{Q}_{t_i}(\nu_{i-1}, \mathrm{d}\nu_i)), \quad \nu_1 \in \mathcal{N}$$

is a bounded continuous function on $\mathcal{N}$. Taking $G(\nu_1) := e^{-\nu_1(f_1)}\tilde{G}(\nu_1)$ in the statement (6.1), we obtain the desired result for this lemma. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3 (i). Step 1. Let T be an arbitrary $(0,\infty]$-valued optional time w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$. In this step, we show that for any $m \in \mathbb{N}$, list of distinct times $(s_i)_{i=1}^m$ in $(0,\infty)$, and list of non-negative functions $(f_i)_{i=1}^m$ in $\mathcal{C}_c(\mathbb{R})$,

$$\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T<\infty\}}\exp\left\{-\sum_{i=1}^m Z_{T+s_i}(f_i)\right\}\right]=\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T<\infty\}}\mathbb{E}_{(\emptyset,Z_T)}\left[\exp\left\{-\sum_{i=1}^m Z_{s_i}(f_i)\right\}\right]\right].$$

Without loss of generality, we assume that $0=:s_0<s_1<\cdots<s_m$. Fix an arbitrary $k\in\mathbb{N}$. Define optional time

$$T^{(k)}:=\frac{\lfloor 2^kT\rfloor+1}{2^k}\mathbf{1}_{\{T<\infty\}}+\infty\mathbf{1}_{\{T=\infty\}}$$

which takes values in the discrete space $2^{-k}\mathbb{N}\cup\{\infty\}$. Notice that $T^{(k)}\downarrow T$ as $k\uparrow\infty$ and that $\{T^{(k)}=d\}\in\mathcal{F}_d$ for every $d\in 2^{-k}\mathbb{N}$. Using the Markov property of the process $(Z_t)_{t>0}$ w.r.t. the augmented filtration $(\mathcal{F}_t)_{t>0}$, c.f. Proposition 5.12, we have

$$\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T^{(k)}=d\}}\exp\left\{-\sum_{i=1}^m Z_{d+s_i}(f_i)\right\}\right]=\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T^{(k)}=d\}}G(Z_d)\right],\quad d\in 2^{-k}\mathbb{N},\tag{6.2}$$

where

$$G(\nu_0):=\int_{\mathcal{N}^m}\prod_{i=1}^m\left(e^{-\nu_i(f_i)}\mathscr{Q}_{s_i-s_{i-1}}(\nu_{i-1},\mathrm{d}\nu_i)\right)=\mathbb{E}_{(\emptyset,\nu_0)}\left[\exp\left\{-\sum_{i=1}^m Z_{s_i}(f_i)\right\}\right],\quad \nu_0\in\mathcal{N}$$

is a bounded continuous function on $\mathcal{N}$, thanks to Lemma 6.4. Summing over $d\in 2^{-k}\mathbb{N}$ in (6.2), we get by Fubuni's theorem that

$$\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T<\infty\}}\exp\left\{-\sum_{i=1}^m Z_{T^{(k)}+s_i}(f_i)\right\}\right]=\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T<\infty\}}G(Z_{T^{(k)}})\right].$$

Taking $k\uparrow\infty$, from the fact that the process $(Z_t)_{t>0}$ is right-continuous, we obtain

$$\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T<\infty\}}\exp\left\{-\sum_{i=1}^m Z_{T+s_i}(f_i)\right\}\right]=\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T<\infty\}}G(Z_T)\right]$$

as desired for this step.

Step 2. Let T be an arbitrary $(0,\infty]$-valued optional time w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$. In this step, we will show that for any $0<a<b<\infty$, $x>0$ and non-negative $f\in\mathcal{C}_c(\mathbb{R})$,

$$\mathbb{P}_{(\Lambda,\mu)}\left(T<\infty,\sup_{s\in[a,b]}Z_{T+s}(f)\leq x\right)=\mathbb{E}_{(\Lambda,\mu)}\left[1_{\{T<\infty\}}\mathbb{P}_{(\emptyset,Z_T)}\left(\sup_{s\in[a,b]}Z_s(f)\leq x\right)\right].$$

To do this, let $(s_i)_{i=1}^\infty$ be a sequential arrangement of the elements in $[a,b]\cap\mathbb{Q}$. From Step 1, we see that for any $m\in\mathbb{N}$,

$$\mathbb{P}_{(\Lambda,\mu)}\left(T<\infty,\sup_{1\leq i\leq m}Z_{T+s_i}(f)\leq x\right)=\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T<\infty\}}\mathbb{P}_{(\emptyset,Z_T)}\left(\sup_{1\leq i\leq m}Z_{s_i}(f)\leq x\right)\right].\tag{6.3}$$

Notice that $\sup_{s\in[a,b]}w_s(f)=\sup_{s\in[a,b]\cap\mathbb{Q}}w_s(f)$ for every $w\in\mathbb{D}((0,\infty),\mathcal{N})$. Therefore, $\mathbb{P}_{(\Lambda,\mu)}$-almost surely on the event $\{T<\infty\}$,

$$\mathbf{1}_{\left\{\sup_{s\in[a,b]}Z_{T+s}(f)\leq x\right\}}=\lim_{m\to\infty}\mathbf{1}_{\left\{\sup_{1\leq i\leq m}Z_{T+s_i}(f)\leq x\right\}};$$

and similarly, for any $\nu \in \mathcal{N}$, $\mathbb{P}_{(\emptyset,\nu)}$-almost surely,

$$\mathbf{1}_{\left\{\sup_{s\in[a,b]} Z_s(f)\le x\right\}} = \lim_{m\to\infty} \mathbf{1}_{\left\{\sup_{1\le i\le m} Z_{s_i}(f)\le x\right\}}.$$

Taking $m \uparrow \infty$ in (6.3), we get the desired result.

Step 3. Let T be an arbitrary $(0,\infty]$-valued optional time w.r.t. the filtration $(\mathcal{F}_t)_{t\ge 0}$. In this step, we will show that for any $0 < a < b < \infty$ and $x > 0$,

$$\mathbb{P}_{(\Lambda,\mu)}\left(T < \infty, \sup_{s\in[a,b]} Z_{T+s}(U) \le x\right) = \mathbb{E}_{(\Lambda,\mu)}\left[1_{\{T<\infty\}}\mathbb{P}_{(\emptyset,Z_T)}\left(\sup_{s\in[a,b]} Z_s(U) \le x\right)\right]. \quad (6.4)$$

Let $(g_N)_{N\in\mathbb{N}}$ be an increasing sequence of compactly supported non-negative continuous functions on $\mathbb{R}$ approximating the indicator function $\mathbf{1}_U$. We claim that for every $\omega \in \Omega = \mathbb{D}((0,\infty),\mathcal{N})$,

(6.5) $\sup_{s\in[a,b]} w_s(g_N)$ increasingly converges to $\sup_{s\in[a,b]} w_s(U)$ as $N \uparrow \infty$.

Indeed, it is obvious from the monotonicity that the large N limit of $\sup_{s\in[a,b]} w_s(g_N)$ exists and is bounded by $\sup_{s\in[a,b]} w_s(U)$. On the other hand:

- If $\sup_{s\in[a,b]} w_s(U) < \infty$, then for any $\varepsilon > 0$, there exists $s_0 \in [a,b]$ such that $w_{s_0}(U) > \sup_{s\in[a,b]} w_s(U) - \varepsilon$, which implies that
$$\sup_{s\in[a,b]} w_s(U) - \varepsilon < w_{s_0}(U) = \lim_{N\to\infty} w_{s_0}(g_N) \le \lim_{N\to\infty} \sup_{s\in[a,b]} w_s(g_N).$$
Taking the arbitrary $\varepsilon \downarrow 0$, we have $\sup_{s\in[a,b]} w_s(U) \le \lim_{N\to\infty} \sup_{s\in[a,b]} w_s(g_N)$.
- If $\sup_{s\in[a,b]} w_s(U) = \infty$, then for any $K > 0$, there exists $s_0 \in [a,b]$ such that $w_{s_0}(U) > K$, which implies that
$$K < w_{s_0}(U) = \lim_{N\to\infty} w_{s_0}(g_N) \le \lim_{N\to\infty} \sup_{s\in[a,b]} w_s(g_N).$$
Taking the arbitrary $K \uparrow \infty$, we have $\sup_{s\in[a,b]} w_s(U) \le \lim_{N\to\infty} \sup_{s\in[a,b]} w_s(g_N)$.

Thus (6.5) is valid. From (6.5), we see that $\mathbb{P}_{(\Lambda,\mu)}$-almost surely on the event $\{T < \infty\}$,

$$\mathbf{1}_{\left\{\sup_{s\in[a,b]} Z_{T+s}(U)\le x\right\}} = \lim_{N\to\infty} \mathbf{1}_{\left\{\sup_{s\in[a,b]} Z_{T+s}(g_N)\le x\right\}}, \quad x > 0;$$

and similarly, for any $\nu \in \mathcal{N}$, $\mathbb{P}_{(\emptyset,\nu)}$-almost surely,

$$\mathbf{1}_{\left\{\sup_{s\in[a,b]} Z_s(U)\le x\right\}} = \lim_{N\to\infty} \mathbf{1}_{\left\{\sup_{s\in[a,b]} Z_s(g_N)\le x\right\}}, \quad x > 0.$$

From Step 2, for every $N \in \mathbb{N}$ and $x > 0$,

$$\mathbb{P}_{(\Lambda,\mu)}\left(T < \infty, \sup_{s\in[a,b]} Z_{T+s}(g_N) \le x\right) = \mathbb{E}_{(\Lambda,\mu)}\left[1_{\{T<\infty\}}\mathbb{P}_{(\emptyset,Z_T)}\left(\sup_{s\in[a,b]} Z_s(g_N) \le x\right)\right].$$

Taking $N \uparrow \infty$ we get the desired result for this step.

Step 4. Fix an arbitrary $k \in \mathbb{N}$ and define $T_k := \inf\{t > 1/k : Z_t(U) \le k\}$ which is an optional time w.r.t. $(\mathcal{F}_t)_{t>0}$. Let $0 < a < b < \infty$. We verify in this step that $\mathbb{P}_{(\Lambda,\mu)}$-almost surely on the event $\{T_k < \infty\}$,

$$\mathbb{P}_{(\emptyset, Z_{T_k})}\left(\sup_{s\in[a,b]} Z_s(U) < \infty\right) = 1.$$

Note that $\mathbb{P}_{(\Lambda,\mu)}$-almost surely there exists a decreasing sequence $(t_m)_{m\in\mathbb{N}}$ in $\mathbb{R}_+$ converging to 0 such that $Z_{T_k+t_m}(U) \leq k$. Let $(g_N)_{N\in\mathbb{N}}$ be an increasing sequence of compactly supported continuous functions on $\mathbb{R}$ approximating the indicator function $\mathbf{1}_U$. Then, by the fact that $(Z_t)_{t>0}$ is an $\mathcal{N}$-valued càdlàg process, $\mathbb{P}_{(\Lambda,\mu)}$-almost surely on the event $\{T_k < \infty\}$,

$$Z_{T_k}(U) = \lim_{N\to\infty} Z_{T_k}(g_N) = \lim_{N\to\infty}\lim_{m\to\infty} Z_{T_k+t_m}(g_N) \leq k. \tag{6.6}$$

Fix an arbitrary $\nu \in \mathcal{N}$ such that $\nu(U) < \infty$. Let

$$U_1 := \{x \in \mathbb{R} : |x - x_0| \leq 1 \text{ for some } x_0 \in U\}$$

be the unit enlargement of the open interval U, and let g be a smooth function with bounded derivatives of all orders satisfying that $\mathbf{1}_U \leq g \leq \mathbf{1}_{U_1}$. Pay attention that g'' is compactly supported. Note that $U_1 \cap \tilde{F}$ is bounded if $\tilde{F}$ is the smallest closed interval containing each $(x-1, x+1)$ such that $\nu(\{x\}) > 0$. Therefore, c.f. Lemma 5.15,

$$M_g(t; a) := e^{\Phi'(0+)t} Z_t(g) - \int_a^t e^{\Phi'(0+)s}\frac{1}{2} Z_s(g'')\mathrm{d}s, \quad t \geq a,$$

is a super-martingale on the filtered probability space $(\Omega, \mathcal{F}^Z, (\mathcal{F}^Z_t)_{t\geq a}, \mathbb{P}_{(\emptyset,\nu)})$. From this and [CW05, Theorem 1 of Section 1.4.], we can verify that $\mathbb{P}_{(\emptyset,\nu)}$-a.s.,

$$\sup_{q\in[a,b+1]\cap\mathbb{Q}} Z_q(U) \leq \sup_{q\in[a,b+1]\cap\mathbb{Q}} Z_q(g) < \infty.$$

Therefore, $\mathbb{P}_{(\emptyset,\nu)}$-a.s.,

$$\begin{aligned}\sup_{s\in[a,b]} Z_s(U) &= \sup_{s\in[a,b]}\lim_{N\to\infty} Z_s(g_N) = \sup_{s\in[a,b]}\lim_{N\to\infty}\lim_{q\downarrow s, q\in\mathbb{Q}} Z_q(g_N) \\ &\leq \sup_{s\in[a,b]}\lim_{N\to\infty}\limsup_{q\downarrow s, q\in\mathbb{Q}} Z_q(U) \leq \sup_{q\in[a,b+1]\cap\mathbb{Q}} Z_q(U) < \infty.\end{aligned}$$

Now, since we have shown

$$\mathbb{P}_{(\emptyset,\nu)}\left(\sup_{s\in[a,b]} Z_s(U) < \infty\right) = 1$$

for the arbitrary $\nu \in \mathcal{N}$ satisfying $\nu(U) < \infty$, the desired result for this step follows from (6.6).

Step 5. In this step, we show that

$$\mathbb{P}_{(\Lambda,\mu)}(Z_t(U) = \infty, \forall t \in \mathbb{Q} \cap (0,\infty)) = 1.$$

To do this, for any measurable function f on $\mathbb{R}$ which can be approximated by the elements of $\mathcal{C}(\mathbb{R}, [0, z^*])$ monotonically from below, let $(u_t)_{t>0}$ be a $\mathcal{C}(\mathbb{R}, [0, z^*])$-valued continuous process given as in Proposition 3.6 with initial value f on a probability space whose expectation operator will be denoted by $\tilde{\mathbb{E}}_f$. Let F be the smallest closed interval containing $\cup_{i=1}^{\infty}(x_i-1,x_i+1)$. From the condition $U\cap \mathrm{supp}(\Lambda,\mu)$ is unbounded, we have $U\cap F$ is unbounded. From Proposition 5.6, Lemmas 4.4, 4.6 and 6.1 (i), for any $\varepsilon\in(0,1/2)$ and $t>0$,

$$\begin{aligned}
&\mathbb{E}_{(\Lambda,\mu)}\left[(1-\varepsilon)^{Z_t(U)}\right]=\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1-u_t(x_i))\right]\\
&\leq \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\left\{-\sum_{i=1}^{\infty}u_t(x_i)\right\}\right]+\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\left(\sup_{s\leq t,y\in F}u_s(y)>\frac{1}{2}\right)\\
&\leq \exp\left\{-\varepsilon\kappa(1/2)e^{-\beta_{\mathrm{o}}t}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\right\}+2\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\left(\sup_{s\leq t,y\in F}u_s(y)>\frac{1}{2}\right)+\varepsilon\beta_{\mathrm{c}}e^{\lambda_{\mathrm{o}}t}\mathcal{V}_t^{(\Lambda,\mu,F)}\\
&=2\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\left(\sup_{s\leq t,y\in F}u_s(y)>\frac{1}{2}\right)+\varepsilon\beta_{\mathrm{c}}e^{\lambda_{\mathrm{o}}t}\mathcal{V}_t^{(\Lambda,\mu,F)}.
\end{aligned}$$

Here, $\mathcal{V}_t^{(\Lambda,\mu,F)}$, λ_{o} and $\kappa(\cdot)$ are given as in (4.13), (2.2) and (4.2) respectively. From Lemma 4.5 (ii) and Lemma 4.7, taking $\varepsilon\downarrow 0$, we obtain that for any $t>0$, $\mathbb{P}_{(\Lambda,\mu)}(Z_t(U)=\infty)=1$. The desired result for this step follows.

Step 6. Let $k\in\mathbb{N}$ be arbitrary, and let T_k be the optional time given as in Step 4. Let $0<a<b<\infty$ and $x>0$ be arbitrary. From Step 3, we know that (6.4) holds with T being replaced by T_k, which, by taking $x\uparrow\infty$, implies that

$$\mathbb{P}_{(\Lambda,\mu)}\left(T_k<\infty,\sup_{s\in[a,b]}Z_{T_k+s}(U)<\infty\right)=\mathbb{E}_{(\Lambda,\mu)}\left[\mathbf{1}_{\{T_k<\infty\}}\mathbb{P}_{(\emptyset,Z_{T_k})}\left(\sup_{s\in[a,b]}Z_s(U)<\infty\right)\right].\tag{6.7}$$

From Step 5, we know that the left hand side of (6.7) equals to 0. From Step 4, we know that the right hand side of (6.7) equals to $\mathbb{P}_{(\Lambda,\mu)}(T_k<\infty)$. Therefore, $\mathbb{P}_{(\Lambda,\mu)}(T_k<\infty)=0$. As a consequence,

$$1=\mathbb{P}_{(\Lambda,\mu)}(T_k=\infty)=\mathbb{P}_{(\Lambda,\mu)}(Z_t(U)>k,\forall t>1/k).$$

Taking the arbitrary $k\uparrow\infty$, by the monotone convergence theorem, we get the desired result (i) of the Theorem 1.3. □

Proof of Theorem 1.3 (ii). Let the open interval $U_1:=\{x\in\mathbb{R}:\exists x_0\in U,|x-x_0|\leq 1\}$ be the unit enlargement of U, and let g be a smooth function with bounded derivatives of all orders satisfying that $\mathbf{1}_U\leq g\leq \mathbf{1}_{U_1}$. Pay attention that g'' is compactly supported. Note that, from the condition that $U\cap\mathrm{supp}(\Lambda,\mu)$ is bounded, we have $U_1\cap F$ is bounded where F is the smallest closed interval containing $\cup_{i\in\mathbb{N}}(x_i-1,x_i+1)$. Therefore, c.f. Lemma 5.15, for any $a>0$,

$$M_g(t;a):=e^{\Phi'(0+)t}Z_t(g)-\int_a^t e^{\Phi'(0+)s}\frac{1}{2}Z_s(g'')\mathrm{d}s,\quad t\geq a$$

is a super-martingale w.r.t. the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq a}, \mathbb{P}_{(\Lambda,\mu)})$. From this and [CW05, Theorem 1 of Section 1.4.], we can verify that $\mathbb{P}_{(\emptyset,\nu)}$-a.s., for any $b > a > 0$,

$$\sup_{q\in[a,b]\cap\mathbb{Q}} Z_q(U) \leq \sup_{q\in[a,b]\cap\mathbb{Q}} Z_q(g) < \infty.$$

Let $(g_N)_{N\in\mathbb{N}}$ be an increasing sequence of compactly supported continuous functions on $\mathbb{R}$ approximating the indicator function $\mathbf{1}_U$. Then, $\mathbb{P}_{(\emptyset,\nu)}$-almost surely, for every $t > 0$,

$$Z_t(U) = \lim_{N\uparrow\infty} Z_t(g_N) = \lim_{N\uparrow\infty} \lim_{q\downarrow t, q\in\mathbb{Q}} Z_q(g_N) \leq \sup_{q\in\mathbb{Q}\cap[t,t+1]} Z_q(U) < \infty$$

as desired. □

Proof of Theorem 1.3 (iii)–(v). (iii) follows from Proposition 5.8.

(iv) follows from Proposition 5.8, Lemmas 6.1 (i), 4.5 (ii), and 4.7.

(v) follows from Lemmas 6.2 and 6.3 and [Kal21, Theorem 5.12]. □

References

[ABBP19] L. Addario-Berry, J. Berestycki, and S. Penington, *Branching Brownian motion with decay of mass and the nonlocal Fisher-KPP equation*, Comm. Pure Appl. Math. **72** (2019), no. 12, 2487–2577, DOI 10.1002/cpa.21827. MR4020312

[Ald99] D. J. Aldous, *Deterministic and stochastic models for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists*, Bernoulli **5** (1999), no. 1, 3–48, DOI 10.2307/3318611. MR1673235

[ABL12] O. Angel, N. Berestycki, and V. Limic, *Global divergence of spatial coalescents*, Probab. Theory Related Fields **152** (2012), no. 3-4, 625–679, DOI 10.1007/s00440-010-0332-5. MR2892958

[Ath98] S. R. Athreya, *Probability and semilinear partial differential equations*, ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)–University of Washington. MR2698239

[AT00] S. Athreya and R. Tribe, *Uniqueness for a class of one-dimensional stochastic PDEs using moment duality*, Ann. Probab. **28** (2000), no. 4, 1711–1734, DOI 10.1214/aop/1019160504. MR1813840

[BDS24] S. Baguley, L. Döring, and Q. Shi, *The structure of entrance and exit at infinity for time-changed Lévy processes*, arXiv preprint (2024), available at `https://arxiv.org/abs/2410.07664`.

[BMS24a] C. Barnes, L. Mytnik, and Z. Sun, *On the coming down from infinity of coalescing Brownian motions*, Ann. Probab. **52** (2024), no. 1, 67–92, DOI 10.1214/23-aop1640. MR4698025

[BMS24b] C. Barnes, L. Mytnik, and Z. Sun, *Wright-Fisher stochastic heat equations with irregular drifts*, arXiv preprint (2024), available at `https://arxiv.org/abs/2402.11160`.

[Ber14] J. Berestycki, *Topics on branching Brownian motion*, Lecture notes (2014), available at `https://www.stats.ox.ac.uk/~berestyc/articles.html`.

[BBL10] J. Berestycki, N. Berestycki, and V. Limic, *The Λ-coalescent speed of coming down from infinity*, Ann. Probab. **38** (2010), no. 1, 207–233, DOI 10.1214/09-AOP475. MR2599198

[Bor95] V. S. Borkar, *Probability Theory: An Advanced Course*, Springer, New York, 1995. MR1367959

[Bra78] M. D. Bramson, *Maximal displacement of branching Brownian motion*, Comm. Pure Appl. Math. **31** (1978), no. 5, 531–581, DOI 10.1002/cpa.3160310502. MR0494541

[CW05] K. L. Chung and J. B. Walsh, *Markov processes, Brownian motion, and time symmetry*, 2nd ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 249, Springer, New York, 2005. MR2152573

[DPZ14] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 152, Cambridge University Press, Cambridge, 2014. MR3236753

[Dur19] R. Durrett, *Probability—theory and examples*, 5th ed., Cambridge Series in Statistical and Probabilistic Mathematics, vol. 49, Cambridge University Press, Cambridge, 2019. MR3930614

[DF16] R. Durrett and W.-T. Fan, *Genealogies in expanding populations*, Ann. Appl. Probab. **26** (2016), no. 6, 3456–3490, DOI 10.1214/16-AAP1181. MR3582808

[Fan21] W.-T. L. Fan, *Stochastic PDEs on graphs as scaling limits of discrete interacting systems*, Bernoulli **27** (2021), no. 3, 1899–1941, DOI 10.3150/20-bej1296. MR4278798

[FT23] W.-T. L. Fan and O. Tough, *Quasi-stationary behavior of the stochastic FKPP equation on the circle*, arXiv preprint (2023), available at `https://arxiv.org/abs/2309.10998`.

[Fou19] C. Foucart, *Continuous-state branching processes with competition: duality and reflection at infinity*, Electron. J. Probab. **24** (2019), Paper No. 33, 38, DOI 10.1214/19-EJP299. MR3940763

[FLZ21] C. Foucart, P.-S. Li, and X. Zhou, *Time-changed spectrally positive Lévy processes started from infinity*, Bernoulli **27** (2021), no. 2, 1291–1318, DOI 10.3150/20-bej1274. MR4255235

[HOV21] M. Hammer, M. Ortgiese, and F. Völlering, *Entrance laws for annihilating Brownian motions and the continuous-space voter model*, Stochastic Process. Appl. **134** (2021), 240–264, DOI 10.1016/j.spa.2021.01.002. MR4212931

[HT05] T. Hobson and R. Tribe, *On the duality between coalescing Brownian particles and the heat equation driven by Fisher-Wright noise*, Electron. Comm. Probab. **10** (2005), 136–145, DOI 10.1214/ECP.v10-1143. MR2162813

[HS25] H. Hou and Z. Sun, *Supplement to "On the subcritical self-catalytic branching Brownian motions"* (2025). Attached.

[IW89] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, 2nd ed., North-Holland Mathematical Library, vol. 24, North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989. MR1011252

[Kal17] O. Kallenberg, *Random measures, theory and applications*, Probability Theory and Stochastic Modelling, vol. 77, Springer, Cham, 2017. MR3642325

[Kal21] O. Kallenberg, *Foundations of modern probability*, 3rd ed., Probability Theory and Stochastic Modelling, vol. 99, Springer, Cham, 2021. MR4226142

[KS88] N. Konno and T. Shiga, *Stochastic partial differential equations for some measure-valued diffusions*, Probab. Theory Related Fields **79** (1988), no. 2, 201–225, DOI 10.1007/BF00320919. MR0958288

[Lam05] A. Lambert, *The branching process with logistic growth*, Ann. Appl. Probab. **15** (2005), no. 2, 1506–1535, DOI 10.1214/105051605000000098. MR2134113

[LG96] J.-F. Le Gall, *A probabilistic approach to the trace at the boundary for solutions of a semilinear parabolic partial differential equation*, J. Appl. Math. Stochastic Anal. **9** (1996), no. 4, 399–414, DOI 10.1155/S1048953396000354. MR1429263

[LZ24] B. Li and X. Zhou, *Entrance laws for continuous-state nonlinear branching processes coming down from infinity*, 2021–2022 MATRIX annals, MATRIX Book Ser., vol. 5, Springer, Cham, 2024, pp. 491–503, DOI 10.1007/978-3-031-47417-0_25. MR4807290

[LYZ19] P.-S. Li, X. Yang, and X. Zhou, *A general continuous-state nonlinear branching process*, Ann. Appl. Probab. **29** (2019), no. 4, 2523–2555, DOI 10.1214/18-AAP1459. MR3983343

[Li11] Z. Li, *Measure-valued branching Markov processes*, Probability and its Applications (New York), Springer, Heidelberg, 2011. MR2760602

[LS06] V. Limic and A. Sturm, *The spatial Λ-coalescent*, Electron. J. Probab. **11** (2006), no. 15, 363–393, DOI 10.1214/EJP.v11-319. MR2223040

[Mai16] P. Maillard, *Speed and fluctuations of N-particle branching Brownian motion with spatial selection*, Probab. Theory Related Fields **166** (2016), no. 3-4, 1061–1173, DOI 10.1007/s00440-016-0701-9. MR3568046

[MP24] P. Maillard and S. Penington, *Branching random walk with non-local competition*, J. Lond. Math. Soc. (2) **109** (2024), no. 6, Paper No. e12919, 78 pp, DOI 10.1112/jlms.12919. MR4751871

[MW17] J.-C. Mourrat and H. Weber, *The dynamic Φ^4_3 model comes down from infinity*, Comm. Math. Phys. **356** (2017), no. 3, 673–753, DOI 10.1007/s00220-017-2997-4. MR3719541

[MMQ11] C. Mueller, L. Mytnik, and J. Quastel, *Effect of noise on front propagation in reaction-diffusion equations of KPP type*, Invent. Math. **184** (2011), no. 2, 405–453, DOI 10.1007/s00222-010-0292-5. MR2793860

[MMR21] C. Mueller, L. Mytnik, and L. Ryzhik, *The speed of a random front for stochastic reaction-diffusion equations with strong noise*, Comm. Math. Phys. **384** (2021), no. 2, 699–732, DOI 10.1007/s00220-021-04084-0. MR4259374

[McK75] H. P. McKean, *Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov*, Comm. Pure Appl. Math. **28** (1975), no. 3, 323–331, DOI 10.1002/cpa.3160280302. MR0400428

[OP24] G. B. Ojeda and J. C. Pardo, *Branching processes with pairwise interactions*, arXiv preprint (2024), available at `https://arxiv.org/abs/2009.11820`.

[Sha88] M. Sharpe, *General theory of Markov processes*, Pure and Applied Mathematics, vol. 133, Academic Press, Inc., Boston, MA, 1988. MR0958914

[Sch00] J. Schweinsberg, *A necessary and sufficient condition for the Λ-coalescent to come down from infinity*, Electron. Comm. Probab. **5** (2000), 1–11, DOI 10.1214/ECP.v5-1013. MR1736720

[Pai16] M. Pain, *Velocity of the L-branching Brownian motion*, Electron. J. Probab. **21** (2016), Paper No. 28, 28, DOI 10.1214/16-EJP4639. MR3492932

[Qua12] J. Quastel, *Introduction to KPZ*, Current developments in mathematics, 2011, Int. Press, Somerville, MA, 2012, pp. 125–194. MR3098078

[Rud87] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987. MR0924157

[Shi94] T. Shiga, *Two contrasting properties of solutions for one-dimensional stochastic partial differential equations*, Canad. J. Math. **46** (1994), no. 2, 415–437, DOI 10.4153/CJM-1994-022-8. MR1271224

[Shi88] T. Shiga, *Stepping stone models in population genetics and population dynamics*, Stochastic processes in physics and engineering (Bielefeld, 1986), Math. Appl., vol. 42, Reidel, Dordrecht, 1988, pp. 345–355. MR0948717

[Tri95] R. Tribe, *Large time behavior of interface solutions to the heat equation with Fisher-Wright white noise*, Probab. Theory Related Fields **102** (1995), no. 3, 289–311, DOI 10.1007/BF01192463. MR1339735

[Wal86] J. B. Walsh, *An introduction to stochastic partial differential equations*, École d'été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986, pp. 265–439, DOI 10.1007/BFb0074920. MR0876085

[Yan04] J.-A. Yan, *Lectures on measure theory*, Lecture Notes of Chinese Academy of Sciences, Science Press, Beijing, 2004 (in Chinese).

SUPPLEMENT TO "ON THE SUBCRITICAL SELF-CATALYTIC BRANCHING BROWNIAN MOTIONS"

HAOJIE HOU AND ZHENYAO SUN

This document serves as supplementary material for our paper titled "On the Subcritical Self-Catalytic Branching Brownian Motions." It provides detailed proofs of several propositions and lemmas presented in the main paper. The numerical labels and references used in this supplementary material correspond to those in the main paper, while labels with an alphabetic prefix are newly introduced and defined herein.

A. Proofs of Lemmas 3.1, 3.5, and Proposition 3.6

Proof of Lemma 3.1. From the continuity of the function $\Psi(\cdot)$, $\Psi(1) = \beta_c q_0 \geq 0$ and that

$$\Psi(2) = \beta_c \left(\sum_{k=0}^{\infty} q_k (-1)^k - 1 \right) \leq \beta_c \left(\sum_{k=0}^{\infty} q_k - 1 \right) = 0,$$

we have $z^* \in [1, 2]$ and $\Psi(z^*) = 0$. Observe that for any $k \in \mathbb{Z}_+$, $z^k - z = z(z^{k-1} - 1) \geq 0$ for every $z \in [-1, 0]$. Therefore, for every $z \in [1, 2]$, we have

$$\Phi(z) = \beta_o \left(p_0 z + \sum_{k=1}^{\infty} p_k \big((1-z)^k - (1-z) \big) \right) \geq 0.$$

In particular, $\Phi(z^*) \geq 0$. The fact $\Phi(0) = \Psi(0) = 0$ can be verified directly from their expressions. From the definition of z^*, the continuity of the function $\Psi(\cdot)$ and the fact that $\Psi(1) \geq 0$, we have $\Psi(z) \geq 0$ for every $z \in [1, z^*]$. Finally observe that, for every $z \in [0, 1)$, since $x \mapsto z^x$ is a convex function on $\mathbb{R}$, by Jensen's inequality and (1.11),

$$\sum_{k=0}^{\infty} q_k z^k \geq z^{\sum_{k=0}^{\infty} k q_k} \geq z^2.$$

This proves that $\Psi(z) \geq 0$ for every $z \in (0, 1]$.

If the additional assumption (1.16) holds, then just take an odd number k_0 with $q_{k_0} > 0$, we see that

$$\Psi(2) = \beta_c \left(\sum_{k=0}^{\infty} q_k (-1)^k - 1 \right) \leq \beta_c \left(\sum_{k \neq k_0}^{\infty} q_k - q_{k_0} - 1 \right) = -2\beta_c q_{k_0} < 0,$$

which implies that $z^* < 2$. We are done. ☐

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China

E-mail addresses: houhaojie@bit.edu.cn, zhenyao.sun@gmail.com, houhaojie@bit.edu.cn, zhenyao.sun@gmail.com.

Proof of Lemma 3.5. Let $(\sigma_m)_{m=1}^\infty$ be a sequence of non-negative Lipschitz functions on $[0,z^*]$ converging to $\sqrt{\Psi}$ uniformly, satisfying that $\sigma_m(0)=\sigma_m(z^*)=0$ for each $m\in\mathbb{N}$. For each $i\in\{1,2\}$ and $m\in\mathbb{N}$, let $u^{(i,m)}$ be a $\mathcal{C}(\mathbb{R},[0,z^*])$-valued solution to the SPDE (3.1) with f and $\sqrt{\Psi}$ being replaced by $f^{(i)}$ and σ_m respectively. For each $i\in\{1,2\}$ and $m\in\mathbb{N}$, the existence of the $\mathcal{C}([0,\infty),\mathcal{C}(\mathbb{R},[0,z^*]))$-valued random element $u^{(i,m)}$ is guaranteed by the standard theory of 1-d stochastic heat equation with Lipschitz coefficients [Shi94].

By the strong comparison principle [Shi94, Corollary 2.4], we can assume without loss of generality that, for each $m\in\mathbb{N}$, $u^{(1,m)}$ and $u^{(2,m)}$ are defined on the same probability space satisfying that almost surely $u_t^{(1,m)}(x)\leq u_t^{(2,m)}(x)$ for every $t\geq 0$ and $x\in\mathbb{R}$. For each $i\in\{1,2\}$, it is also standard to argue, c.f. [Shi94, Proof of Theorem 2.6], that the family of $\mathcal{C}([0,\infty),\mathcal{C}(\mathbb{R},[0,z^*]))$-valued random elements $\{u^{(i,m)}:m\in\mathbb{N}\}$ is tight, and any sub-sequential convergence-in-distribution limit has the law $\mathscr{L}_{f^{(i)}}$. This implies that the family of $\mathcal{C}([0,\infty),\mathcal{C}(\mathbb{R},[0,z^*]))^2$-valued random elements $\{(u^{(1,m)},u^{(2,m)}):m\in\mathbb{N}\}$ is tight.

Let $(u^{(1)},u^{(2)})$ be one of its sub-sequential convergence-in-distribution limit. Clearly, the law of $u^{(1)}$ and $u^{(2)}$ are given by $\mathscr{L}_{f^{(1)}}$ and $\mathscr{L}_{f^{(2)}}$ respectively; and almost surely, $u_t^{(1)}(x)\leq u_t^{(2)}(x)$ for every $t\geq 0$ and $x\in\mathbb{R}$. By the disintegration theorem [Kal21, Theorem 8.5], there exists a probability kernel $\mathscr{K}_{f^{(1)},f^{(2)}}$ on $\mathcal{C}([0,\infty),\mathcal{C}(\mathbb{R},[0,z^*]))$ such that $\mathscr{K}_{f^{(1)},f^{(2)}}(u^{(1)},\cdot)$ is the regular conditional distribution of $u^{(2)}$ conditioned on given $u^{(1)}$. It is then straightforward to verify that $\mathscr{K}_{f^{(1)},f^{(2)}}$ satisfies all the properties desired for this Lemma. □

Proof of Proposition 3.6. Construct a $\mathcal{C}([0,\infty),\mathcal{C}(\mathbb{R},[0,z^*]))$-valued Markov chain $(u^{(m)})_{m\in\mathbb{N}}$, on a probability space with its probability measure denoted by $\tilde{\mathbb{P}}_g$, such that the initial value $u^{(1)}$ has the law $\mathscr{L}_{f^{(1)}}$, and that, for each $m\in\mathbb{N}$, the transition kernel from steps m to $m+1$ is $\mathscr{K}_{f^{(m)},f^{(m+1)}}$. Here, $\mathscr{L}_\cdot$ and $\mathscr{K}_{\cdot,\cdot}$ are the same as in Lemma 3.5. It is clear that $u^{(m)}$ has the law $\mathscr{L}_{f^{(m)}}$ for each $m\in\mathbb{N}$; and almost surely $u_t^{(m)}(x)\leq u_t^{(m+1)}(x)$ for every $t\geq 0,x\in\mathbb{R}$ and $m\in\mathbb{N}$. This allows us to define a random field $\bar{u}=(\bar{u}_t(x))_{t\geq 0,x\in\mathbb{R}}$ as the pointwisely non-decreasing limit of $u^{(m)}$ when $m\uparrow\infty$.

It is standard (c.f. [BMS24a, p. 82]) to verify from the mild formulation (3.2), Burkholder-Davis-Gundy inequality, Jensen's inequality, the property of the heat kernels, and the fact that $(u^{(m)})_{m\in\mathbb{N}}$ are bounded random fields, that, for any $T>0$ and $l>2$,

$$\tilde{\mathbb{E}}_g\left[\left|U_t^{(m)}(x)-U_s^{(m)}(y)\right|^{2l}\right]\lesssim\left(|x-y|+\sqrt{|t-s|}\right)^l$$

uniformly in $m\in\mathbb{N}$, $x,y\in\mathbb{R}$ and $t,s\in[0,T]$. Here, for any $t>0$ and $x\in\mathbb{R}$,

$$U_t^{(m)}(x):=u_t^{(m)}(x)-\int p_t(x-y)f^{(m)}(y)\mathrm{d}y+\iint_0^t p_{t-s}(x-y)\Phi\big(u_s^{(m)}(y)\big)\mathrm{d}s\mathrm{d}y$$

and $U_0^{(m)}(x) := 0$. Taking $m \uparrow \infty$, we know from the bounded convergence theorem that for any $T > 0$ and $l > 2$,

$$\tilde{\mathbb{E}}_g\Big[\big|\bar{U}_t(x) - \bar{U}_s(y)\big|^{2l}\Big] \lesssim \Big(|x-y| + \sqrt{|t-s|}\Big)^l \tag{A.1}$$

uniformly in $x, y \in \mathbb{R}$ and $t, s \in [0, T]$, where, for each $t > 0$ and $x \in \mathbb{R}$,

$$\bar{U}_t(x) := \bar{u}_t(x) - \int p_t(x-y) g(y) \mathrm{d}y + \iint_0^t p_{t-s}(x-y)\Phi(\bar{u}_s(y))\mathrm{d}s\mathrm{d}y$$

and $\bar{U}_0(x) := 0$. From (A.1), we can verify (c.f. [Shi94, Lemma 6.3 (i)]) that there exists a jointly continuous modification $(U_t(x))_{t\geq 0, x\in\mathbb{R}}$ of $(\bar{U}_t(x))_{t\geq 0, x\in\mathbb{R}}$. Define, for every $t > 0$ and $x \in \mathbb{R}$,

$$u_t(x) := U_t(x) + \int p_t(x-y) g(y) \mathrm{d}y - \iint_0^t p_{t-s}(x-y)\Phi(\bar{u}_s(y))\mathrm{d}s\mathrm{d}y$$

and $u_0(x) := g(x)$. It is then not hard to see that $(u_t(x))_{t\geq 0, x\in\mathbb{R}}$ is a modification of $(\bar{u}_t(x))_{t\geq 0, x\in\mathbb{R}}$ and that $(u_t)_{t>0}$ is a $\mathcal{C}(\mathbb{R}, [0, z^*])$-valued continuous process. In particular, from Fubini's theorem, almost surely, for almost every $t > 0$ and $x \in \mathbb{R}$ w.r.t. the Lebesgue measure, $u_t(x) = \bar{u}_t(x)$.

Notice from Proposition 3.3 that, for each $m \in \mathbb{N}$, (3.3) holds with f and u being replaced by $f^{(m)}$ and $u^{(m)}$ respectively. Taking $m \uparrow \infty$, by the bounded convergence theorem, we have

$$\tilde{\mathbb{E}}_g\left[\prod_{i=1}^n (1 - \bar{u}_t(x_i))\right] = \mathbb{E}\left[\prod_{\alpha\in I_t^{(n)}}\Big(1 - g\Big(X_t^{(n,\alpha)}\Big)\Big)\right], \quad t \geq 0.$$

Now, since u is a modification of $\bar{u}$, the desired statement (3.13) holds.

Let us fix an arbitrary testing function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. From [Shi94, Theorem 2.1], almost surely for every $t \geq 0$ and $m \in \mathbb{N}$,

$$\int u_t^{(m)}(x)\phi(x)\mathrm{d}x = \int f^{(m)}(x)\phi(x)\mathrm{d}x + \iint_0^t u_s^{(m)}(y)\frac{\phi''(y)}{2}\mathrm{d}s\mathrm{d}y - \iint_0^t \Phi\big(u_s^{(m)}(y)\big)\phi(y)\mathrm{d}s\mathrm{d}y + M_t^{(m,\phi)} \tag{A.2}$$

where $M_\cdot^{(m,\phi)}$ is a $(\mathcal{G}_t^{(m)})_{t\geq 0}$-adapted continuous martingale with quadratic variation

$$\big\langle M_\cdot^{(m,\phi)}\big\rangle_t := \iint_0^t \Psi\big(u_s^{(m)}(y)\big)\phi(y)^2\mathrm{d}s\mathrm{d}y, \quad t \geq 0 \tag{A.3}$$

and $(\mathcal{G}_t^{(m)})_{t\geq 0}$ is the natural filtration of the process $(u_t^{(m)})_{t\geq 0}$. In particular, for any $0 \leq s \leq t$, $N \in \mathbb{N}$, bounded continuous map G from $\mathbb{R}^N$ to $\mathbb{R}$, and list $((s_i, y_i))_{i=1}^N$ in $[0, s] \times \mathbb{R}$, we have

$$\tilde{\mathbb{E}}_g\Big[G\big(u_{s_1}^{(m)}(y_1), \ldots, u_{s_N}^{(m)}(y_N)\big)M_t^{(m,\phi)}\Big] = \tilde{\mathbb{E}}_g\big[G\big(u_{s_1}^{(m)}(y_1), \ldots, u_{s_N}^{(m)}(y_N)\big)M_s^{(m,\phi)}\big] \tag{A.4}$$

and

$$\begin{aligned}&\tilde{\mathbb{E}}_g\Big[G\big(u^{(m)}_{s_1}(y_1),\dots,u^{(m)}_{s_N}(y_N)\big)\Big(\Big(M^{(m,\phi)}_t\Big)^2-\big\langle M^{(m,\phi)}_\cdot\big\rangle_t\Big)\Big] \\ &=\tilde{\mathbb{E}}_g\Big[G\big(u^{(m)}_{s_1}(y_1),\dots,u^{(m)}_{s_N}(y_N)\big)\Big(\big(M^{(m,\phi)}_s\big)^2-\big\langle M^{(m,\phi)}_\cdot\big\rangle_s\Big)\Big].\end{aligned} \tag{A.5}$$

Taking $m\uparrow\infty$ in (A.2), by the dominated convergence theorem, we know that $M_t^{(m,\phi)}$ converges to $M_t^{(\phi)}$ almost surely for every $t\geq 0$, where the continuous process $(M_t^{(\phi)})_{t\geq 0}$ is defined through

$$\begin{aligned}\int u_t(x)\phi(x)\mathrm{d}x=\int g(x)\phi(x)\mathrm{d}x+\iint_0^t u_s(y)\frac{\phi''(y)}{2}\mathrm{d}s\mathrm{d}y-\\ \iint_0^t\Phi(u_s(y))\phi(y)\mathrm{d}s\mathrm{d}y+M_t^{(\phi)},\quad t\geq 0.\end{aligned}$$

Taking $m\uparrow\infty$ in (A.3), we know that $\langle M^{(m,\phi)}_\cdot\rangle_t$ converges to $\langle M^{(\phi)}_\cdot\rangle_t$ almost surely for every $t\geq 0$, where

$$\big\langle M^{(\phi)}_\cdot\big\rangle_t:=\iint_0^t\Psi(u_s(y))\phi(y)^2\mathrm{d}s\mathrm{d}y,\quad t\geq 0.$$

Note that, for each $t\geq 0$, the families of random variables $(M_t^{(m,\phi)}:m\in\mathbb{N})$ and $((M_t^{(m,\phi)})^2-\langle M^{(m,\phi)}_\cdot\rangle_t:m\in\mathbb{N})$ are bounded by a deterministic constant depending only on Φ, Ψ, t and ϕ. Therefore, taking $m\uparrow\infty$ in (A.4) and (A.5), for any $0\leq s\leq t$, $N\in\mathbb{N}$, bounded continuous map G from $\mathbb{R}^N$ to $\mathbb{R}$, and list $((s_i,y_i))_{i=1}^N$ in $[0,s]\times\mathbb{R}$, we have

$$\tilde{\mathbb{E}}_g\Big[G(u_{s_1}(y_1),\dots,u_{s_N}(y_N))M_t^{(\phi)}\Big]=\tilde{\mathbb{E}}_g\big[G(u_{s_1}(y_1),\dots,u_{s_N}(y_N))M_s^{(\phi)}\big]$$

and

$$\begin{aligned}&\tilde{\mathbb{E}}_g\Big[G(u_{s_1}(y_1),\dots,u_{s_N}(y_N))\Big(\Big(M_t^{(\phi)}\Big)^2-\big\langle M^{(\phi)}_\cdot\big\rangle_t\Big)\Big]\\ &=\tilde{\mathbb{E}}_g\Big[G(u_{s_1}(y_1),\dots,u_{s_N}(y_N))\Big(\big(M_s^{(\phi)}\big)^2-\big\langle M^{(\phi)}_\cdot\big\rangle_s\Big)\Big].\end{aligned}$$

These imply that $(M_t^{(\phi)})_{t\geq 0}$ is an $(\mathcal{G}_t)_{t\geq 0}$-adapted continuous martingale with quadratic variation $(\langle M^{(\phi)}_\cdot\rangle_t)_{t\geq 0}$. We are done. □

B. Proof of Lemmas 4.1, 4.4 and 4.6

Proof of Lemma 4.1. If $z_i\in[0,1]$ for all $i\in\mathbb{N}$, then the desired inequality follows from Bernoulli's inequality. If there is $i\neq j$ such that $z_i,z_j>1$, then $0\leq 1-\prod_{i=1}^N(1-z_i)\leq 2<\sum_{i=1}^N z_i$. Otherwise, there exists only one i_0 such that $z_j\in[0,1]$ for all $j\neq i_0$ and that $z_{i_0}\in(1,2]$. Without loss of generality, we assume in this case $i_0=1$. Then by Bernoulli's inequality,

$$\prod_{i=1}^N(1-z_i)=-\frac{z_1}{2}\prod_{i=2}^N(1-z_i)+(1-\frac{z_1}{2})\prod_{i=2}^N(1-z_i)\geq-\frac{z_1}{2}+1-\frac{z_1}{2}-\sum_{i=2}^N z_i,$$

which again implies the desired result. □

Proof of Lemma 4.4. Let $t > 0$ and $\gamma \in (\varepsilon, 1)$. Define $\tau_\gamma := \inf\{s \geq 0 : \sup_{y\in F} u_s(y) > \gamma\}$. For the lower bound, noticing that $1 - w \geq e^{-\theta(\gamma)w}$ for $w \in [0, \gamma]$, we have

$$\begin{aligned}
\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1-u_t)(x_i)\right] &= \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1-u_t)(x_i)\mathbf{1}_{\{\tau_\gamma\leq t\}}\right] + \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1-u_t)(x_i)\mathbf{1}_{\{\tau_\gamma> t\}}\right] \\
&\geq -\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}(\tau_\gamma \leq t) + \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\left\{-\theta(\gamma)\sum_{i=1}^{\infty}u_t(x_i)\right\}\mathbf{1}_{\{\tau_\gamma>t\}}\right] \\
&\geq \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\left\{-\theta(\gamma)\sum_{i=1}^{\infty}u_t(x_i)\right\}\right] - 2\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}(\tau_\gamma \leq t).
\end{aligned}$$

For the upper bound, using the inequality $1 - w \leq e^{-w}$ and the fact that $1 - u_t(x_i) \geq 0$ for every $i \in \mathbb{N}$ almost surely on the event $\{\tau_{1/2} > t\}$, we have

$$\begin{aligned}
\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1-u_t)(x_i)\right] &= \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1-u_t)(x_i)\mathbf{1}_{\{\tau_{1/2}\leq t\}}\right] + \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1-u_t)(x_i)\mathbf{1}_{\{\tau_{1/2}> t\}}\right] \\
&\leq \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\big(\tau_{1/2} \leq t\big) + \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\left\{-\sum_{i=1}^{\infty}u_t(x_i)\right\}\right].
\end{aligned}$$

We are done. □

Proof of Lemma 4.6. Define $\tau_z := \inf\{s \geq 0 : \sup_{y\in F} u_s(y) > z\}$ for every $z \in (0, 1)$. Let $t > 0$. For every $\tilde{\gamma} \in [0, 1)$, define

$$M_s^{(\tilde{\gamma},n)} := \begin{cases} \dfrac{1}{1-\tilde{\gamma}}\displaystyle\int u_s(y)v_{t-s}^{(\emptyset,\mu^{(\tilde{\gamma},n)})}(y)\mathrm{d}y, & s \in [0,t); \\ \dfrac{1}{1-\tilde{\gamma}}\displaystyle\int u_t(y)\mu^{(\tilde{\gamma},n)}(\mathrm{d}y), & s = t, \end{cases}$$

with $\mu^{(\tilde{\gamma},n)} := (1-\tilde{\gamma})\theta(\tilde{\gamma})\sum_{i=1}^{n}\delta_{x_i}$. From stochastic Fubini's theorem [DPZ14, Theorem 4.33], one can verify that almost surely for every $\tilde{\gamma} \in [0, 1)$ and $s \in [0, t]$,

$$\begin{aligned}
&M_s^{(\tilde{\gamma},n)} - M_0^{(\tilde{\gamma},n)} \qquad\qquad \text{(B.1)}\\
&= \frac{1}{1-\tilde{\gamma}}\iint_0^s\left(-v_{t-r}^{(\emptyset,\mu^{(\tilde{\gamma},n)})}(y)\Phi(u_r(y)) + \frac{\Psi'(0+)}{2}v_{t-r}^{(\emptyset,\mu^{(\tilde{\gamma},n)})}(y)^2u_r(y)\right)\mathrm{d}r\mathrm{d}y \\
&\quad + \frac{1}{1-\tilde{\gamma}}\iint_0^s v_{t-r}^{(\emptyset,\mu^{(\tilde{\gamma},n)})}(y)\sqrt{\Psi(u_r(y))}W(\mathrm{d}r\mathrm{d}y).
\end{aligned}$$

In particular, $(M_s^{(\tilde{\gamma},n)})_{s\in[0,t]}$ is a continuous semi-martingale. By (B.1) and Itô's formula, it is easy to verify that almost surely for every $s \in [0, t]$,

$$\exp\{-e^{\lambda_\circ(t-s)}M_s^{(\gamma,n)}\} - \exp\left\{-e^{\lambda_\circ t}M_0^{(\gamma,n)}\right\} \qquad \text{(B.2)}$$

$$= \iint_0^s \exp\{-e^{\lambda_0(t-r)} M_r^{(\gamma,n)}\} \frac{e^{\lambda_0(t-r)}}{1-\gamma} \times \Bigg(- v_{t-r}^{(\emptyset,\mu^{(\gamma,n)})}(y)\sqrt{\Psi(u_r(y))} W(\mathrm{d}r\mathrm{d}y) + v_{t-r}^{(\emptyset,\mu^{(\gamma,n)})}(y)(\Phi(u_r(y)) + \lambda_{\mathrm{o}} u_r(y))\mathrm{d}r\mathrm{d}y - \frac{1}{2} v_{t-r}^{(\emptyset,\mu^{(\gamma,n)})}(y)^2 \Big(\Psi'(0+)u_r(y) - \frac{e^{\lambda_0(t-r)}}{1-\gamma}\Psi(u_r(y))\Big)\mathrm{d}r\mathrm{d}y\Bigg).$$

Set

$$\gamma_0 := \frac{\gamma}{2\beta_{\mathrm{c}}}\Psi'(0+) = \frac{\gamma}{2}\left(2 - \sum_{k=0}^{\infty} k q_k\right) \in (0,\gamma).$$

We see that for any $w \in (0, \gamma_0]$,

$$\Psi(w) = \int_0^w \Psi'(z)\mathrm{d}z = \beta_{\mathrm{c}} \int_0^w \left(2(1-z) - \sum_{k=1}^{\infty} k q_k (1-z)^{k-1}\right)\mathrm{d}z \qquad \text{(B.3)}$$
$$\geq \beta_{\mathrm{c}} w \left(2(1-\gamma_0) - \sum_{k=1}^{\infty} k q_k\right) = (1-\gamma)\Psi'(0+)w.$$

Note the fact that almost surely for $r \leq \tau_{\gamma_0} \wedge t$ and $y \in F$,

$$\Psi'(0+)u_r(y) - \frac{e^{\lambda_{\mathrm{o}}(t-r)}}{1-\gamma}\Psi(u_r(y)) \leq \Psi'(0+)u_r(y) - \frac{1}{1-\gamma}\Psi(u_r(y)) \leq 0,$$

where the last inequality follows from (B.3). Combining this with (4.1), $\lambda_{\mathrm{o}} > 0$, and $M_r^{(\gamma,n)} \geq 0$, we get by taking expectation for (B.2) with $s = \tau_{\gamma_0} \wedge t$ that

$$\begin{aligned}
&\tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\Big[\exp\Big\{-e^{\lambda_{\mathrm{o}}(t-\tau_{\gamma_0}\wedge t)} M_{\tau_{\gamma_0}\wedge t}^{(\gamma,n)}\Big\}\Big] \\
&\geq \exp\Big\{-e^{\lambda_{\mathrm{o}} t} M_0^{(\gamma,n)}\Big\} - \\
&\quad \frac{1}{2(1-\gamma)^2}\tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\Big[\int_0^{\tau_{\gamma_0}\wedge t}\int_{F^c} \exp\{-e^{\lambda_{\mathrm{o}}(t-r)} M_r^{(\gamma,n)}\} e^{\lambda_{\mathrm{o}}(t-r)} v_{t-r}^{(\emptyset,\mu^{(\gamma,n)})}(y)^2 \times \\
&\qquad \big((1-\gamma)\Psi'(0+)u_r(y) - \Psi(u_r(y))e^{\lambda_{\mathrm{o}}(t-r)}\big)\mathrm{d}y\mathrm{d}r\Big] \\
&\geq \exp\Big\{-e^{\lambda_{\mathrm{o}} t} M_0^{(\gamma,n)}\Big\} - \frac{\Psi'(0+)e^{\lambda_{\mathrm{o}} t}}{2(1-\gamma)}\int_0^t \int_{F^c} v_{t-r}^{(\emptyset,\mu^{(\gamma,n)})}(y)^2 \tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}[u_r(y)]\mathrm{d}y\mathrm{d}r.
\end{aligned}$$

From Lemma 4.2, we get that

$$\begin{aligned}
&\tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\Big[\exp\Big\{-e^{\lambda_{\mathrm{o}}(t-\tau_{\gamma_0}\wedge t)} M_{\tau_{\gamma_0}\wedge t}^{(\gamma,n)}\Big\}\Big] \\
&\geq \exp\Big\{-e^{\lambda_{\mathrm{o}} t} M_0^{(\gamma,n)}\Big\} - \frac{\varepsilon \Psi'(0+)e^{2\lambda_{\mathrm{o}} t}}{2(1-\gamma)}\int_0^t \int_{F^c} v_{t-r}^{(\emptyset,\mu^{(\gamma,n)})}(y)^2 \mathrm{d}y\mathrm{d}r.
\end{aligned}$$

By the definition of $M_t^{(\gamma,n)}$ and that

$$\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[e^{-M_t^{(\gamma,n)}}\right] \geq \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\Big\{-e^{\lambda_o(t-\tau_{\gamma_0}\wedge t)}M^{(\gamma,n)}_{\tau_{\gamma_0}\wedge t}\Big\}\right] - \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}(\tau_{\gamma_0}<t),$$

we have

$$\begin{aligned}
&\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\left\{-\theta(\gamma)\sum_{i=1}^n u_t(x_i)\right\}\right] \qquad\qquad \text{(B.4)}\\
&\geq \exp\left\{-\frac{\varepsilon e^{\lambda_o t}}{1-\gamma}\int_U v_t^{(\emptyset,\mu^{(\gamma,n)})}(y)\mathrm{d}y\right\} - \frac{\varepsilon\Psi'(0+)e^{2\lambda_o t}}{2(1-\gamma)}\int_0^t\int_{F^c} v_{t-r}^{(\emptyset,\mu^{(\gamma,n)})}(y)^2\mathrm{d}r\mathrm{d}y - \\
&\quad \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\left(\sup_{s\leq t, y\in F} u_s(y) > \gamma_0\right).
\end{aligned}$$

It has been explained in [BMS24a, (2.19)] that

$$v_{r,y}^{(\emptyset,\mu^{(\gamma,n)})} \xrightarrow[n\uparrow\infty]{\text{increasingly}} v_{r,y}^{(\Lambda,(1-\gamma)\theta(\gamma)\mu)} \leq v_{r,y}^{(\Lambda,\mu)}, \quad r>0, y\in\mathbb{R}.$$

Now we get the first desired result (4.15) by monotone convergence theorem while taking $n\uparrow\infty$ in (B.4).

Let us now prove the upper bound (4.16). Similarly using (B.1) and Itô's formula, we see that almost surely for every $s\in[0,t]$,

$$\begin{aligned}
&\exp\big\{-\kappa(\gamma)e^{-\beta_o(t-s)}M_s^{(0,n)}\big\} - \exp\Big\{-\kappa(\gamma)e^{-\beta_o t}M_0^{(0,n)}\Big\}\\
&= \kappa(\gamma)\iint_0^s \exp\big\{-\kappa(\gamma)e^{\beta_0(t-r)}M_r^{(0,n)}\big\}e^{-\beta_o(t-r)}\times\\
&\qquad\Bigg(-v_{t-r}^{(\emptyset,\mu^{(0,n)})}(y)\sqrt{\Psi(u_r(y))}W(\mathrm{d}r\mathrm{d}y)) + \\
&\qquad v_{t-r}^{(\emptyset,\mu^{(0,n)})}(y)(\Phi(u_r(y))-\beta_o u_r(y))\mathrm{d}r\mathrm{d}y - \\
&\qquad \frac{1}{2}v_{t-r}^{(\emptyset,\mu^{(\gamma,n)})}(y)^2\big(\Psi'(0+)u_r(y)-\kappa(\gamma)\Psi(u_r(y))e^{-\beta_o(t-r)}\big)\mathrm{d}r\mathrm{d}y\Bigg).
\end{aligned}$$

Replacing s with $\tau_\gamma\wedge t$ and taking the expectation of the above equation, combining (4.1) and the fact that almost surely for all $r\leq t\wedge\tau_\gamma$ and $y\in F$,

$$\Psi'(0+)u_r(y)-\kappa(\gamma)\Psi(u_r(y))e^{-\beta_o(t-r)} \geq \Psi'(0+)u_r(y)-\kappa(\gamma)\Psi(u_r(y)) \geq 0,$$

we have

$$\begin{aligned}
&\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\Big\{-\kappa(\gamma)e^{-\beta_o(t-t\wedge\tau_\gamma)}M^{(0,n)}_{\tau_\gamma\wedge t}\Big\}\right]\\
&\leq \exp\Big\{-\kappa(\gamma)e^{-\beta_o t}M_0^{(0,n)}\Big\} - \\
&\quad \frac{\kappa(\gamma)}{2}\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\Bigg[\int_0^{\tau_\gamma\wedge t}\int_{F^c}\exp\big\{-\kappa(\gamma)e^{-\beta_o(t-r)}M_r^{(0,n)}\big\}e^{-\beta_o(t-r)}v_{t-r}^{(\emptyset,\mu^{(0,n)})}(y)^2\times
\end{aligned}$$

$$\left(\Psi'(0+)u_r(y) - \kappa(\gamma)\Psi(u_r(y))e^{-\beta_{\mathrm{o}}(t-r)}\right)\mathrm{d}r\mathrm{d}y\Big]$$

$$\leq \exp\Big\{-\kappa(\gamma)e^{-\beta_{\mathrm{o}}t}M_0^{(0,n)}\Big\} + \frac{1}{2}\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\Big[\int_0^t\int_{F^c} v_{t-r,y}^{(\emptyset,\mu^{(0,n)})}(y)^2\Psi(u_r(y))\mathrm{d}r\mathrm{d}y\Big],$$

where in the last inequality we used the fact that $\kappa(\gamma) \leq 1$.

According to (4.1) and Lemma 4.2, we concluded from above inequality that

$$\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\Big[\exp\Big\{-\kappa(\gamma)e^{-\beta_{\mathrm{o}}(t-t\wedge\tau_\gamma)}M_{\tau_\gamma\wedge t}^{(0,n)}\Big\}\Big]$$
$$\leq \exp\Big\{-\kappa(\gamma)e^{-\beta_{\mathrm{o}}t}M_0^{(0,n)}\Big\} + \varepsilon\beta_{\mathrm{c}}e^{\lambda_{\mathrm{o}}t}\int_0^t\int_{F^c} v_{t-r}^{(\emptyset,\mu^{(0,n)})}(y)^2\mathrm{d}r\mathrm{d}y.$$

Finally, noticing that

$$0 \leq \kappa(\gamma)e^{-\beta_{\mathrm{o}}(t-t\wedge\tau_\gamma)}M_{\tau_\gamma\wedge t}^{(0,n)} \leq e^{-\beta_{\mathrm{o}}(t-t\wedge\tau_\gamma)}M_{\tau_\gamma\wedge t}^{(0,n)}$$

and that

$$\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\Big[\exp\Big\{-e^{-\beta_{\mathrm{o}}(t-t\wedge\tau_\gamma)}M_{\tau_\gamma\wedge t}^{(0,n)}\Big\}\Big] \geq \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\Big[\exp\Big\{-M_t^{(0,n)}\Big\}\Big] - \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}(\tau_\gamma \leq t),$$

we conclude that

$$\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\Big[\exp\Big\{-\sum_{i=1}^n u_t(x_i)\Big\}\Big] = \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\Big[\exp\Big\{-M_t^{(0,n)}\Big\}\Big]$$
$$\leq \exp\Big\{-\kappa(\gamma)e^{-\beta_{\mathrm{o}}t}M_0^{(0,n)}\Big\} + \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}(\tau_\gamma \leq t) + \varepsilon\beta_{\mathrm{c}}e^{\lambda_{\mathrm{o}}t}\int_0^t\int_{F^c} v_{t-r}^{(\emptyset,\mu^{(0,n)})}(y)^2\mathrm{d}r\mathrm{d}y.$$

Combining the above inequality and the fact that $v_t^{(0,\mu^{(0,n)})}$ converges to $v_t^{(\Lambda,\mu)}$ (c.f. the argument below [BMS24a, (2.18)]), we get (4.16). □

C. Proof of Lemma 4.7

Let β_{o}, $(p_k)_{k=0}^\infty$, β_{c} and $(q_k)_{k=0}^\infty$ be given as in (1.2)–(1.5). Suppose that (1.10), (1.11) and (1.16) hold. Let Φ and Ψ be given as in (1.14) and (1.15) respectively. Let f be a measurable function on $\mathbb{R}$ which can be approximated by the elements of $\mathcal{C}(\mathbb{R}, [0, z^*])$ monotonically from below. Let $(u_t)_{t>0}$ be the continuous $\mathcal{C}(\mathbb{R}, [0, z^*])$-valued process given as in Proposition 3.6, with initial value $u_0 = f$, on a probability space whose probability measure will be denoted by $\tilde{\mathbb{P}}_f$.

In this section, we prove Lemma 4.7 following the standard strategy of [Tri95]. From (3.12), we can assume without loss of generality, c.f. [KS88, Proof of Lemma 2.4], that there exists a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}_f, W)$ such that $(u_t)_{t>0}$ is an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-adapted $\mathcal{C}(\mathbb{R}, [0, z^*])$-valued continuous process, and that, for each $\phi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R})$, almost surely,

$$\int u_t(x)\phi(x)\mathrm{d}x = \int f(x)\phi(x)\mathrm{d}x + \iint_0^t u_s(y)\frac{\phi''(y)}{2}\mathrm{d}s\mathrm{d}y -$$

$$\iint_0^t \Phi(u_s(y))\phi(y)\mathrm{d}s\mathrm{d}y + \iint_0^t \Psi(u_s(y))W(\mathrm{d}s\mathrm{d}y), \quad t \geq 0.$$

It is then standard, c.f. [Shi94, Theorem 2.1], that the mild form (3.2) holds almost surely for every $(t,x) \in (0,\infty)\times\mathbb{R}$. Define continuous random fields $(M(s,y))_{s\geq 0, y\in\mathbb{R}}$ and $(N(s,y))_{s\geq 0,y\in\mathbb{R}}$ such that for any $s>0$ and $y \in \mathbb{R}$,

$$M(s,y) := \iint_0^s p_{s-r}(y-z)\Phi(u_r(z))\mathrm{d}r\mathrm{d}z,$$

$$N(s,y) := u_s(y) - \int p_s(y-z)f(z)\mathrm{d}z + \iint_0^t p_{s-r}(y-z)\Phi(u_r(z))\mathrm{d}r\mathrm{d}z$$

and $M_0(y) := N_0(y) = 0$. Note that, for every $s>0$ and $y\in\mathbb{R}$, almost surely

$$N(s,y) = \iint_0^s p_{s-r}(y-z)\sqrt{\Psi(u_r(z))}W(\mathrm{d}r\mathrm{d}z). \tag{C.1}$$

In the following, we take $p_s(x) := 0$ when $s\leq 0$ and $x\in\mathbb{R}$ for the sake of notation conveniences.

Lemma C.1. *For every $t>0$, uniformly for every $(s,y),(s',y') \in [0,t]\times\mathbb{R}$,*

$$\mathcal{K}^{(1)}_{s,y;s'y'} := \iint_0^\infty |p_{s-r}(y-z) - p_{s'-r}(y'-z)|\,\mathrm{d}z\mathrm{d}r \lesssim |y-y'| + \sqrt{|s-s'|} \tag{C.2}$$

and

$$\mathcal{K}^{(2)}_{s,y;s'y'} := \iint_0^\infty (p_{s-r}(y-z) - p_{s'-r}(y'-z))^2\mathrm{d}z\mathrm{d}r \lesssim |y-y'| + \sqrt{|s-s'|}. \tag{C.3}$$

Proof. For (C.3), see [Shi94, Lemma 6.2 (i)]. We now prove (C.2). Let $t>0$ and let $(s,y),(s',y')\in[0,t]\times\mathbb{R}$ be arbitrary. Without loss of generality, let us assume that $s'\leq s$ and define $\delta_s := s-s'$ and $\delta_y := |y-y'|$. Note that $\mathcal{K}^{(1)}_{s,y;s'y'} \leq I_1 + I_2$ where

$$I_1 := \iint_0^\infty |p_{s-r}(y-z) - p_{s-r}(y'-z)|\mathrm{d}r\mathrm{d}z = \iint_0^s |p_r(z+\delta_y) - p_r(z)|\mathrm{d}r\mathrm{d}z$$

and

$$I_2 := \iint_0^\infty |p_{s-r}(y'-z) - p_{s'-r}(y'-z)|\mathrm{d}r\mathrm{d}z = \iint_{-\delta_s}^{s'} |p_{\delta_s+r}(z) - p_r(z)|\,\mathrm{d}r\mathrm{d}z.$$

For I_1, we get that uniformly for the arbitrary $(s,y),(s',y')\in[0,t]\times\mathbb{R}$,

$$I_1 \leq \iint_0^s \int_0^{\delta_y} \frac{|\xi+z|}{\sqrt{2\pi r^3}} e^{-(\xi+z)^2/(2r)}\mathrm{d}\xi\mathrm{d}r\mathrm{d}z = \delta_y \int \frac{|z|}{\sqrt{2\pi}}e^{-z^2/2}\mathrm{d}z \int_0^s \frac{1}{\sqrt{r}}\mathrm{d}r \lesssim \delta_y.$$

For I_2, we decompose it as $I_2 = I_{21} + I_{22}$ where uniformly for the arbitrary $(s,y),(s',y')\in[0,t]\times\mathbb{R}$,

$$I_{21} := \iint_{-\delta_s}^{\sqrt{\delta_s}\wedge s'} |p_{\delta_s+r}(z) - p_r(z)|\mathrm{d}r\mathrm{d}z \leq 2\delta_s + 2\sqrt{\delta_s}\wedge s' \lesssim \sqrt{\delta_s}$$

and

$$\begin{aligned}
I_{22} &:= \iint_{\sqrt{\delta_s}\wedge s'}^{s'} |p_{\delta_s+r}(z) - p_r(z)| \mathrm{d}r\mathrm{d}z = \iint_{\sqrt{\delta_s}}^{s'\vee\sqrt{\delta_s}} |p_{\delta_s+r}(z) - p_r(z)| \mathrm{d}r\mathrm{d}z \\
&\leq \iint_{\sqrt{\delta_s}}^{s'\vee\sqrt{\delta_s}} \int_r^{\delta_s+r} \left| -\frac{1}{2a} + \frac{z^2}{2a^2} \right| p_a(z) \mathrm{d}a\mathrm{d}r\mathrm{d}z \leq \int_{\sqrt{\delta_s}}^{s'\vee\sqrt{\delta_s}} \int_r^{\delta_s+r} \frac{1}{a} \mathrm{d}a\mathrm{d}r \\
&= \int_{\sqrt{\delta_s}}^{s'\vee\sqrt{\delta_s}} \log\left(\frac{\delta_s+r}{r}\right) \mathrm{d}r \leq \int_{\sqrt{\delta_s}}^{s'\vee\sqrt{\delta_s}} \frac{\delta_s}{r} \mathrm{d}r \leq \int_{\sqrt{\delta_s}}^{s'\vee\sqrt{\delta_s}} \sqrt{\delta_s} \mathrm{d}r \lesssim \sqrt{\delta_s}.
\end{aligned}$$

We are done. □

Lemma C.2. *Let U be an open interval and $\varepsilon \in (0,1)$. Let the initial value f of the process $(u_t)_{t\geq 0}$ be given by $f = \varepsilon \mathbf{1}_U$. Then for any $p > 1$ and $t > 0$, uniformly for the arbitrary open interval U, the arbitrary parameter $\varepsilon \in (0,1)$, $s, s' \in (0,t)$ and $y, y' \in \mathbb{R}$, we have*

$$\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[|M(s,y) - M(s',y')|^p\right] \lesssim \varepsilon\Big(|y-y'| + \sqrt{|s-s'|}\Big)^{p-1} (\mathbf{P}_y(B_s \in U) + \mathbf{P}_{y'}(B_{s'} \in U)),$$

and

$$\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[|N(s,y) - N(s',y')|^{2p}\right] \lesssim \varepsilon\Big(|y-y'| + \sqrt{|s-s'|}\Big)^{p-1} (\mathbf{P}_y(B_s \in U) + \mathbf{P}_{y'}(B_{s'} \in U)).$$

Proof. From (4.1) we know that the random field $\Psi(u)$ is bounded by a deterministic constant. Therefore, uniformly for the arbitrary open interval U, the arbitrary parameter $\varepsilon \in (0,1)$, $s, s' \in (0,t)$ and $y, y' \in \mathbb{R}$, we have

$$|M(s,y) - M(s',y')| \leq \iint_0^\infty |p_{s-r}(y-z) - p_{s'-r}(y'-z)| \, |\Phi(u_r(z))| \mathrm{d}r\mathrm{d}z \lesssim \mathcal{K}^{(1)}_{s,y';s',y'}.$$

From (4.1), Lemma 4.2, and the Markov property of the Brownian motion that, uniformly for the arbitrary open interval U, the arbitrary parameter $\varepsilon \in (0,1)$, and $(s,y) \in [0,t] \times \mathbb{R}$,

$$\begin{aligned}
\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}[|M(s,y)|] &\leq \iint_0^s p_{s-r}(y-z) \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}[|\Phi(u_r(z))|] \mathrm{d}r\mathrm{d}z \\
&\lesssim \iint_0^s p_{s-r}(y-z) \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}[u_r(z)] \mathrm{d}r\mathrm{d}z \lesssim \varepsilon \iint_0^s p_{s-r}(y-z) \mathbf{P}_z(B_r \in U) \mathrm{d}r\mathrm{d}z \\
&= \varepsilon \int_0^s \mathbf{E}_y\big[\mathbf{P}_{B_{s-r}}(B_r \in U)\big] \mathrm{d}r = \varepsilon s \mathbf{P}_y(B_s \in U).
\end{aligned}$$

Therefore, for every $p > 1$, from Lemma C.1, uniformly for the arbitrary open interval U, the arbitrary parameter $\varepsilon \in (0,1)$, and $(s,y), (s',y') \in [0,t] \times \mathbb{R}$,

$$\begin{aligned}
\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[|M(s,y) - M(s',y')|^p\right] &\lesssim \Big(\mathcal{K}^{(1)}_{s,y';s',y'}\Big)^{p-1} \Big(\tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}[|M(s,y)|] + \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}[|M(s',y')|]\Big) \\
&\lesssim \varepsilon\Big(|y-y'| + \sqrt{|s-s'|}\Big)^{p-1} (\mathbf{P}_y(B_s \in U) + \mathbf{P}_{y'}(B_{s'} \in U))
\end{aligned}$$

as desired for the first inequality.

For the second inequality, by (C.1), Burkholder-Davis-Gundy's inequality, (4.1), the fact that the random field u is bounded by 2, and the trivial inequality $(a-b)^2 \leq 2(a^2+b^2)$, we verify that, for every $p > 1$, and uniformly for the arbitrary open interval U, the arbitrary parameter $\varepsilon \in (0,1)$, and $(s,y),(s',y') \in [0,t] \times \mathbb{R}$,

$$\begin{aligned}
&\tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\left[|N(s,y)-N(s',y')|^{2p}\right] \\
&= \tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\left[\left|\iint_0^\infty (p_{s-r}(y-z)-p_{s'-r}(y'-z))\sqrt{\Psi(u_r(z))}W(\mathrm{d}r\mathrm{d}z)\right|^{2p}\right] \\
&\lesssim \tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\left[\left(\iint_0^\infty (p_{s-r}(y-z)-p_{s'-r}(y'-z))^2\Psi(u_r(z))\mathrm{d}r\mathrm{d}z\right)^p\right] \\
&\lesssim \tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\left[\left(\iint_0^\infty (p_{s-r}(y-z)-p_{s'-r}(y'-z))^2 u_r(z)\mathrm{d}r\mathrm{d}z\right)^p\right] \\
&\lesssim \left(\mathcal{K}^{(2)}_{s,y;s',y'}\right)^{p-1}\iint_0^\infty (p_{s-r}(y-z)-p_{s'-r}(y'-z))^2\tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}[u_r(z)]\mathrm{d}r\mathrm{d}z \\
&\lesssim \left(\mathcal{K}^{(2)}_{s,y;s',y'}\right)^{p-1}\iint_0^\infty \big(p_{s-r}(y-z)^2+p_{s'-r}(y'-z)^2\big)\tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}[u_r(z)]\mathrm{d}r\mathrm{d}z.
\end{aligned}$$

From Lemma 4.2, we have uniformly for the arbitrary open interval U, the arbitrary parameter $\varepsilon \in (0,1)$, and $(s,y) \in [0,t] \times \mathbb{R}$,

$$\begin{aligned}
&\iint_0^\infty p_{s-r}(y-z)^2\tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}[u_r(z)]\mathrm{d}r\mathrm{d}z \lesssim \varepsilon \iint_0^\infty p_{s-r}(y-z)^2\mathbf{P}_z(B_r \in U)\mathrm{d}r\mathrm{d}z \\
&\leq \varepsilon \iint_0^\infty \frac{1}{\sqrt{2\pi(s-r)}}p_{s-r}(y-z)\mathbf{P}_z(B_r \in U)\mathrm{d}r\mathrm{d}z \\
&= \varepsilon \int_0^s \frac{1}{\sqrt{2\pi(s-r)}}\mathbf{P}_y(B_s \in U)\mathrm{d}r \lesssim \varepsilon \mathbf{P}_y(B_s \in U).
\end{aligned}$$

Now, from above and Lemma C.1, for every $p > 1$, uniformly for the arbitrary open interval U, the arbitrary parameter $\varepsilon \in (0,1)$, and $(s,y),(s',y') \in [0,t] \times \mathbb{R}$, we have

$$\begin{aligned}
&\tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\left[|N(s,y)-N(s',y')|^{2p}\right] \lesssim \varepsilon\Big(\mathcal{K}^{(2)}_{s,y;s',y'}\Big)^{p-1}(\mathbf{P}_y(B_s \in U)+\mathbf{P}_{y'}(B_{s'} \in U)) \\
&\lesssim \varepsilon\Big(|y-y'|+\sqrt{|s-s'|}\Big)^{p-1}(\mathbf{P}_y(B_s \in U)+\mathbf{P}_{y'}(B_{s'} \in U)),
\end{aligned}$$

which implies the second inequality. We are done. □

Proof of Lemma 4.7. Let us first show (4.17). Let $\varepsilon \in (0, \gamma/2)$ be arbitrary, and assume that f, the initial value of the process $(u_t)_{t\geq 0}$, is given by $\varepsilon \mathbf{1}_U$. By (3.2), we see that

$$u_s(y) \leq \varepsilon + |M(s,y)| + |N(s,y)|, \quad (s,y) \in [0,\infty) \times \mathbb{R}, \text{a.s.}$$

Therefore, we only need to prove that uniformly for the arbitrary $\varepsilon \in (0, \gamma/2)$,

$$\tilde{\mathbb{P}}_{\varepsilon \mathbf{1}_U}\left(\sup_{s\leq t, y\in F}(|M(s,y)|+|N(s,y)|) > \frac{\gamma}{2}\right) \lesssim \varepsilon. \tag{C.4}$$

Without loss of generality, we assume that $\tilde{F} = F \setminus \{\sup(F)\}$ is non-empty. In the following, we construct a dyadic approximation of the time-space region $[0,t) \times \tilde{F}$. Note that there exists an (at most) countable set F_0 and a constant $\delta \in (0,1]$ such that $\tilde{F}$ is the disjoint union $\cup_{y\in F_0}[y, y+\delta)$. Define a sequence of time-space lattices $(L_m)_{m\in\mathbb{Z}_+}$, so that $L_0 := \{(0,y) : y \in F_0\}$, and inductively,

$$L_m := \bigcup_{(s,y)\in L_{m-1}} \{(s,y), (s+t2^{-m}, y), (s, y+\delta 2^{-m}), (s+t2^{-m}, y+\delta 2^{-m})\}, \quad m \in \mathbb{Z}_+.$$

For each $m \in \mathbb{Z}_+$, define a map Γ_m from $[0,t) \times \tilde{F}$ to the lattice L_m such that for every $(s,y) \in [0,t) \times \tilde{F}$, $(s_m, y_m) = \Gamma_m(s,y)$ is the unique element in L_m satisfying that $s_m \le s < s_m + t2^{-m}$ and $y_m \le y < y_m + \delta 2^{-m}$. It is not hard to observe that

(C.5) $\Gamma_m \circ \Gamma_m = \Gamma_m$ and $\Gamma_{m-1} \circ \Gamma_m = \Gamma_{m-1}$ for each $m \in \mathbb{N}$;

and that

(C.6) $\Gamma_m(s,y)$ converges to (s,y) when $m \uparrow \infty$ for every $(s,y) \in [0,t) \times \tilde{F}$.

Moreover, for every $m \in \mathbb{N}$, and $(s,y) \in [0,t) \times \mathbb{R}$,

$$|y_m - y_{m-1}| + \sqrt{|s_m - s_{m-1}|} \le t2^{-(m-1)} + \sqrt{\delta 2^{-(m-1)}} \le (2t + \sqrt{2\delta})2^{-m/2} \tag{C.7}$$

provided $(s_m, y_m) = \Gamma_m(s,y)$ and $(s_{m-1}, y_{m-1}) = \Gamma_{m-1}(s,y)$.

Let us consider an event on which the fluctuations of the random fields M and N on the dyadic lattices $(L_m)_{m=0}^\infty$ are delicately controlled. That is, we consider the event

$$A := \bigcap_{m=1}^{\infty} \bigcap_{(s,y)\in L_m} \bigcap_{k=1}^{2} A_m^k(s,y) \tag{C.8}$$

where, for any $(s,y) \in [0,t) \times \tilde{F}$,

$$A_m^1(s,y) := \{|(M \circ \Gamma_m - M \circ \Gamma_{m-1})(s,y)| \le \gamma_0 2^{-m/10}\},$$
$$A_m^2(s,y) := \{|(N \circ \Gamma_m - N \circ \Gamma_{m-1})(s,y)| \le \gamma_0 2^{-m/10}\}$$

and $\gamma_0 > 0$ is a constant determined so that $\gamma_0 \sum_{m=1}^\infty 2^{-m/10} = \gamma/4$. Now, almost surely on the event A, from (C.6), the fact that M and N are continuous random fields, that $N \circ \Gamma_0 = M \circ \Gamma_0 = 0$ on $[0,t) \times \tilde{F}$, and (C.5), we have, for every $(s,y) \in [0,t) \times \tilde{F}$,

$$\begin{aligned} |M(s,y)| &= \left| \sum_{m=1}^{\infty} (M \circ \Gamma_m - M \circ \Gamma_{m-1})(s,y) \right| \\ &= \left| \sum_{m=1}^{\infty} (M \circ \Gamma_m - M \circ \Gamma_m) \circ \Gamma_{m-1}(s,y) \right| \le \sum_{m=1}^{\infty} \gamma_0 2^{-m/10} = \gamma/4, \end{aligned}$$

and similarly, $|N(s,y)| \le \gamma/4$. In particular, the event in (C.4) is contained in A^c. Therefore, to show (C.4), it is suffice to show that

(C.9) uniformly for the arbitrary $\varepsilon \in (0, \gamma/2)$, $\tilde{\mathbb{P}}_{\varepsilon \mathbf{1}_U}(A^c) \lesssim \varepsilon$.

Define $\Theta_U(s,y) := \mathbf{P}_y(B_s \in U)$ the probability that a Brownian motion initiated at location $y \in \mathbb{R}$ is in the interval U at time $s \geq 0$. By the Markov inequality, (C.7), and Lemma C.2, uniformly for the arbitrary open interval U, the arbitrary closed interval F satisfying that $U \cap F$ is bounded, the arbitrary parameter $\varepsilon \in (0, \gamma/2)$, $m \in \mathbb{N}$, and $(s,y) \in L_m$, we have

$$\begin{aligned}
\tilde{\mathbb{P}}_{\varepsilon \mathbf{1}_U}\big(A_m^1(s,y)^{\mathrm{c}}\big) &\leq \tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\big[|(M \circ \Gamma_m - M \circ \Gamma_{m-1})(s,y)|^{20}\big]/(\gamma_0 2^{-m/10})^{20} \\
&\lesssim \varepsilon\Big((2t + \sqrt{2\delta})2^{-m/2}\Big)^{19}(\Theta_U \circ \Gamma_m + \Theta_U \circ \Gamma_{m-1})(s,y)/(\gamma_0 2^{-m/10})^{20} \\
&\lesssim \varepsilon 2^{-15m/2}(\Theta_U \circ \Gamma_m + \Theta_U \circ \Gamma_{m-1})(s,y)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbb{P}}_{\varepsilon \mathbf{1}_U}\big(A_m^2(s,y)^{\mathrm{c}}\big) &\leq \tilde{\mathbb{E}}_{\varepsilon \mathbf{1}_U}\big[|(N \circ \Gamma_m - N \circ \Gamma_{m-1})(s,y)|^{40}\big]/(\gamma_0 2^{-m/10})^{40} \\
&\lesssim \varepsilon\Big((2t + \sqrt{2\delta})2^{-m/2}\Big)^{19}(\Theta_U \circ \Gamma_m + \Theta_U \circ \Gamma_{m-1})(s,y)/(\gamma_0 2^{-m/10})^{40} \\
&\lesssim \varepsilon 2^{-11m/2}(\Theta_U \circ \Gamma_m + \Theta_U \circ \Gamma_{m-1})(s,y).
\end{aligned}$$

Now, from (C.8), uniformly for the arbitrary open interval U, the arbitrary closed interval F satisfying that $U \cap F$ is bounded, the arbitrary parameter $\varepsilon \in (0, \gamma/2)$, we have

$$\begin{aligned}
\tilde{\mathbb{P}}_{\varepsilon \mathbf{1}_U}(A^{\mathrm{c}}) &\leq \sum_{m=1}^{\infty} \sum_{(s,y)\in L_m} \Big(\tilde{\mathbb{P}}_{\varepsilon \mathbf{1}_U}\big(A_m^1(s,y)^{\mathrm{c}}\big) + \tilde{\mathbb{P}}_{\varepsilon \mathbf{1}_U}\big(A_m^2(s,y)^{\mathrm{c}}\big)\Big) \\
&\lesssim \varepsilon \sum_{m=1}^{\infty} 2^{-11m/2} \sum_{(s,y)\in L_m} (\Theta_U \circ \Gamma_m + \Theta_U \circ \Gamma_{m-1})(s,y) \\
&\lesssim \varepsilon \sum_{m=1}^{\infty} 2^{-7m/2} \frac{1}{t2^{-m}\delta 2^{-m}} \sum_{(s,y)\in L_m} (\Theta_U \circ \Gamma_m + \Theta_U \circ \Gamma_{m-1})(s,y) \\
&= \varepsilon \sum_{m=1}^{\infty} 2^{-7m/2} \int_F \int_0^t (\Theta_U \circ \Gamma_m + \Theta_U \circ \Gamma_{m-1})(s,y)\mathrm{d}s\mathrm{d}y \\
&\lesssim \varepsilon \sum_{m=0}^{\infty} 2^{-7m/2} \int_F \int_0^t (\Theta_U \circ \Gamma_m)(s,y)\mathrm{d}s\mathrm{d}y.
\end{aligned} \tag{C.10}$$

Since $U \cap F$ is bounded, it is not hard to verify the following analytic results:

$$\int_F \int_0^t (\Theta_U \circ \Gamma_m)(s,y)\mathrm{d}s\mathrm{d}y < \infty, \quad m \in \mathbb{Z}_+$$

and

$$\lim_{m\to\infty} \int_F \int_0^t (\Theta_U \circ \Gamma_m)(s,y)\mathrm{d}s\mathrm{d}y = \int_F \int_0^t \Theta_U(s,y)\mathrm{d}s\mathrm{d}y < \infty,$$

which together imply that

$$\sum_{m=0}^{\infty} 2^{-7m/2} \int_F \int_0^t (\Theta_U \circ \Gamma_m)(s,y)\mathrm{d}s\mathrm{d}y < \infty.$$

Now, from (C.10) we have (C.9), and therefore, the desired (4.17).

Note that $C_1(U,F,t,\gamma)$ is increasing in $t>0$. Therefore, $\limsup_{t\downarrow 0} C_1(U,F,t,\gamma)<\infty$.

Finally, let us take $\tilde{U}$ to be an arbitrary open interval such that its intersection with $F_K:=[K,\infty)$ is bounded for every $K\in\mathbb{R}$. From (C.10), we can verify that, uniformly for every $K\in\mathbb{R}$,

$$C_1(\tilde{U},F_K,t,\gamma)\lesssim\sum_{m=0}^{\infty}2^{-7m/2}\int_K^{\infty}\int_0^t(\Theta_{\tilde{U}}\circ\Gamma_m)(s,y)\mathrm{d}s\mathrm{d}y<\infty.$$

By the monotone convergence theorem, we have $\lim_{K\uparrow\infty}C_1(\tilde{U},F_K,t,\gamma)=0$ as desired. We are done. □

D. Proofs of Lemmas 5.1, 5.5, Propositions 5.8, 5.13, and Lemma 5.16

Proof of Lemma 5.1. It is clear that there exists $K>0$ such that $g(x)=1$ for every $|x|>K$ and $\nu(\{-K,K\})=0$. Therefore, c.f. [Kal17, proof of Lemma 4.12], we have $\lim_{m\to\infty}\nu_m((-K,K))=\nu((-K,K))$. Furthermore, there exists a finite integer $\kappa\geq 0$, a finite list of distinct points $(z_i)_{i=1}^{\kappa}$ in $\mathbb{R}$, and a finite list $(l_i)_{i=1}^{\kappa}$ in $\mathbb{N}$, such that $\mathbf{1}_{(-K,K)}\nu=\sum_{i=1}^{\kappa}l_i\delta_{z_i}$. Note that $\nu((-K,K))=\sum_{i=1}^{\kappa}l_i$. Let $\epsilon>0$ be small enough so that, for every $i\neq j$ in $\{1,\dots,\kappa\}$, $[z_i-\epsilon,z_i+\epsilon]$ and $[z_j-\epsilon,z_j+\epsilon]$ are disjoint subsets of $(-K,K)$. Now for every $i\in\{1,\dots,\kappa\}$, since $\nu(\{z_i-\epsilon,z_i+\epsilon\})=0$, we have, c.f. [Kal17, proof of Lemma 4.12] again,

$$\lim_{m\to\infty}\nu_m([z_i-\epsilon,z_i+\epsilon])=\nu([z_i-\epsilon,z_i+\epsilon])=\nu(\{z_i\})=l_i.$$

Denoting

$$A:=(-K,K)\setminus\left(\bigcup_{i=1}^{\kappa}[z_i-\epsilon,z_i+\epsilon]\right),$$

we also have, with a similar reason, that $\lim_{m\to\infty}\nu_m(A)=\nu(A)=0$. Noticing that for any subset $B\subset\mathbb{R}$ and $m\in\mathbb{N}$, $\nu_m(B)$ takes non-negative integer values. Therefore, there exists $M_0>0$ such that for all $m\geq M_0$, we have $\nu_m(A)=0$ and

$$\nu_m([z_i-\epsilon,z_i+\epsilon])=l_i,\quad i\in\{1,\dots,\kappa\}.$$

So, for every $m\geq M_0$ and $i\in\{1,\dots,\kappa\}$, there is a finite list $(z_{i,j}^{(m)})_{j=1}^{l_i}$ in $[z_i-\epsilon,z_i+\epsilon]$ such that $\mathbf{1}_{[z_i-\epsilon,z_i+\epsilon]}\nu_m=\sum_{j=1}^{l_i}\delta_{z_{i,j}^{(m)}}$. In particular, for every $m\geq M_0$, we have $\mathbf{1}_{(-K,K)}\nu_m=\sum_{i=1}^{\kappa}\sum_{j=1}^{l_i}\delta_{z_{i,j}^{(m)}}$. Now, for every $m\geq M_0$,

$$\begin{aligned}\left|\prod_{z\in\mathbb{R}}g(z)^{\nu_m(\{z\})}-\prod_{z\in\mathbb{R}}g(z)^{\nu(\{z\})}\right|&=\left|\prod_{i=1}^{\kappa}\prod_{j=1}^{l_i}g(z_{i,j}^{(m)})-\prod_{i=1}^{\kappa}\prod_{j=1}^{l_i}g(z_i)\right|\\&=\left|\sum_{i_0=1}^{\kappa}\sum_{j_0=1}^{l_{i_0}}\left(\prod_{i=1}^{i_0}\prod_{j=1}^{j_0-1}g(z_{i,j}^{(m)})\right)\left(\prod_{i=i_0}^{\kappa}\prod_{j=j_0+1}^{l_i}g(z_i)\right)\left(g(z_{i_0,j_0}^{(m)})-g(z_{i_0})\right)\right|\end{aligned}$$

$$\leq \sum_{i=1}^{\kappa}\sum_{j=1}^{l_i}\Big|g(z_{i,j}^{(m)}) - g(z_i)\Big| \leq \sum_{i=1}^{\kappa}\sum_{j=1}^{l_i}\sup\{|g(y)-g(z_i)| : y\in[z_i-\epsilon, z_i+\epsilon]\}.$$

Taking $m\to\infty$ in the above inequality, we get

$$\limsup_{m\to\infty}\left|\prod_{z\in\mathbb{R}} g(z)^{\nu_m(\{z\})} - \prod_{z\in\mathbb{R}} g(z)^{\nu(\{z\})}\right|$$
$$\leq \sum_{i=1}^{\kappa}\sum_{j=1}^{l_i}\sup\{|g(y)-g(z_i)| : y\in[z_i-\epsilon, z_i+\epsilon]\}.$$

From the fact that $\epsilon>0$ can be taken arbitrarily small, κ and $(l_i)_{i=1}^{\kappa}$ do not depend on ϵ when $\epsilon\to 0$, and that g is a continuous function, we have

$$\limsup_{m\to\infty}\left|\prod_{z\in\mathbb{R}} g(z)^{\nu_m(\{z\})} - \prod_{z\in\mathbb{R}} g(z)^{\nu(\{z\})}\right| = 0.$$

Now the desired result (5.1) holds. □

Proof of Lemma 5.5. Recall that $z^*\in[1,2)$. Define $I := \{i\in\mathbb{N} : z_i\in(1,z^*]\}$ and $J := \{i\in\mathbb{N} : z_i = 1\}$.

Step 1. Suppose that $J\neq\emptyset$. Without loss of generality, we assume that $z_1 = 1$. Clearly, the right hand side of (5.12) is 0. Notice that

$$\left|\prod_{i=1}^{k}\Big(1-z_i^{(m)}\Big)\right| \leq |1-z_1^{(m)}|.$$

Taking $k\to\infty$ first, and then $m\to\infty$, we obtain that

$$\limsup_{m\to\infty}\left|\prod_{i=1}^{\infty}\Big(1-z_i^{(m)}\Big)\right| = 0.$$

This implies the desired (5.12) in this case.

Step 2. Suppose that $|I|=\infty$. It is clear that the right hand side of (5.12) is 0. In this case, for any $k\in\mathbb{N}$, there exists $N_k\in\mathbb{N}$ such that $|I_k| = k$ where $I_k := \{i\in\mathbb{N} : i\leq N_k, z_i\in(1,z^*]\}$. Note that, for each $k\in\mathbb{N}$, there exists an $M_k\in\mathbb{N}$, such that for any $m\geq M_k$ and $i\in I_k$, $z_i^{(m)}\in(1,z^*]$. So for any $k\in\mathbb{N}$ and $m\geq M_k$,

$$\left|\prod_{i=1}^{\infty}(1-z_i^{(m)})\right| \leq \prod_{i\in I_k}\Big|1-z_i^{(m)}\Big| \leq |1-z^*|^k.$$

Taking $m\to\infty$ first, and then $k\to\infty$, we obtain that

$$\limsup_{m\to\infty}\left|\prod_{i=1}^{\infty}(1-z_i^{(m)})\right| = 0.$$

So (5.12) holds in this case.

Step 3. Suppose that $|J| = 0$ and $|I| = K$ for some finite integer K. Without loss of generality, we assume that $I = \{1, \dots, K\}$. Then, for any $m \in \mathbb{N}$ and integer $i > K$, $0 \le z_i^{(m)} \le z_i < 1$. Therefore,

$$\prod_{i=1}^{K+l}\Big(1 - z_i^{(m)}\Big) = \left(\prod_{i=1}^{K}\Big(1 - z_i^{(m)}\Big)\right)\exp\bigg\{-\sum_{i=K+1}^{K+l} -\log\Big(1 - z_i^{(m)}\Big)\bigg\}.$$

Taking $l \to \infty$ first, and then $m \to \infty$, by the monotone convergence theorem, we have

$$\lim_{m\to\infty}\prod_{i=1}^{\infty}\Big(1 - z_i^{(m)}\Big) = \left(\prod_{i=1}^{K}(1 - z_i)\right)\exp\bigg\{-\sum_{i=K+1}^{\infty} -\log(1 - z_i)\bigg\}.$$

So (5.12) also holds in this case. We are done. □

Proof of Proposition 5.8. It is clear from (1.11) and (1.18) that $\gamma_0 \le \gamma$. Let $t > 0$ and $\varepsilon \in (0, \gamma_0 \wedge \frac{1}{2})$. From Proposition 5.6,

$$\mathbb{E}_{(\Lambda,\mu)}\Big[(1-\varepsilon)^{\tilde{Z}_t(U)}\Big] = \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1 - u_t(x_i))\right].$$

Together with Lemmas 4.4 and 4.6, we conclude from the above equality that

$$\begin{aligned}
\mathbb{E}_{(\Lambda,\mu)}\Big[(1-\varepsilon)^{\tilde{Z}_t(U)}\Big] &\ge \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\bigg\{-\theta(\gamma)\sum_{i=1}^{\infty}u_t(x_i)\bigg\}\right] - 2\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\Big(\sup_{s\le t, y\in F} u_s(y) > \gamma\Big) \\
&\ge \exp\bigg\{-\frac{\varepsilon e^{\lambda_o t}}{1-\gamma}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\bigg\} - \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\Big(\sup_{s\le t, y\in F} u_s(y) > \frac{\gamma}{2\beta_{\mathrm{c}}}\Psi'(0+)\Big) \\
&\quad - \frac{\varepsilon\Psi'(0+)e^{2\lambda_o t}}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)} - 2\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\Big(\sup_{s\le t, y\in F} u_s(y) > \gamma\Big) \\
&\ge \exp\bigg\{-\frac{\varepsilon e^{\lambda_o t}}{1-\gamma}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\bigg\} - 3\varepsilon C_1(U,F,t,\gamma_0) - \frac{\varepsilon\Psi'(0+)e^{2\lambda_o t}}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)}
\end{aligned}$$

where in the last inequality we used Lemma 4.7. It is clear from the monotone convergence theorem that $\mathbb{E}[Z] = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}[1-(1-\varepsilon)^Z]$ for any non-negative integer-valued random variable Z. Therefore,

$$\begin{aligned}
\mathbb{E}_{(\Lambda,\mu)}\Big[\tilde{Z}_t(U)\Big] &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\Big(1 - \mathbb{E}_{(\Lambda,\mu)}\Big[(1-\varepsilon)^{\tilde{Z}_t(U)}\Big]\Big) \\
&\le \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\bigg(1 - \exp\bigg\{-\frac{\varepsilon e^{\lambda_o t}}{1-\gamma}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\bigg\}\bigg) + 3C_1(U,F,t,\gamma_0) + \frac{\Psi'(0+)e^{2\lambda_o t}}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)} \\
&= \frac{e^{\lambda_o t}}{1-\gamma}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y + \frac{\Psi'(0+)e^{2\lambda_o t}}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)} + 3C_1(U,F,t,\gamma_0), \qquad \text{(D.1)}
\end{aligned}$$

which implies the desired result. From Lemma 4.5 (i) and (ii), we know that the right hand side of (D.1) is finite. □

Proof of Proposition 5.13. Without loss of generality, we assume that g is non-negative. Also, it is suffice to show that the process $(\tilde{Z}_t(g))_{t\in[a,b)}$, restircted on an fixed arbitrary interval $[a,b) \subset (0,\infty)$, has a measurable version.

Step 1. Let us consider the dyadic discretization of the interval $[a,b)$. That is, for each $t \in [a,b)$ and $m \in \mathbb{N}$, we define

$$t_m := \inf\left\{ s > t : \exists k \in \mathbb{N}, s = a + \frac{k}{2^m}(b-a) \right\}.$$

In particular, $0 < t_m - t \leq 1/2^m$. Define $X_t^{(m)} := \tilde{Z}_{t_m}(g)$ for every $t \in [a,b)$ and $m \in \mathbb{N}$. Clearly, for every $w \in \Omega$ and $m \in \mathbb{N}$, the map $t \mapsto X_t^{(m)}(\omega)$ is measurable on $[a,b)$ w.r.t. the Borel σ-field $\mathcal{B}_{[a,b)}$. This allows us to define, for each $m \in \mathbb{N}$, a measurable map $X^{(m)} : (t,\omega) \mapsto X_t^{(m)}(\omega)$ on the product measurable space $([a,b) \times \Omega, \mathcal{B}_{[a,b)} \otimes \mathcal{F})$, which will be equipped with a product probability measure

$$M_{[a,b)}(\mathrm{d}x, \mathrm{d}\omega) := \frac{\mathbf{1}_{[a,b)}(x)\mathrm{d}x}{b-a} \otimes \mathbb{P}_{(\Lambda,\mu)}(\mathrm{d}\omega), \quad (x,\omega) \in [a,b) \times \Omega.$$

Step 2. We investigate the limit of the measurable map $X^{(m)}$ when $m \uparrow \infty$. Fixing an arbitrary $\epsilon > 0$. Define $G_{l,m}(t) := \mathbb{P}_{(\Lambda,\mu)}(|X_t^{(l)} - X_t^{(m)}| > \epsilon)$ for every $t \in [a,b)$ and $l,m \in \mathbb{N}$. By Lemma 5.10, we get that for each $t \in [a,b)$,

$$\begin{aligned} &\sup_{l,m\geq N} G_{l,m}(t) \\ &\leq \sup_{l\geq N} \mathbb{P}_{(\Lambda,\mu)}\Big(|X_t^{(l)} - \tilde{Z}_t(g)| > \frac{\epsilon}{2}\Big) + \sup_{m\geq N} \mathbb{P}_{(\Lambda,\mu)}\Big(|X_t^{(m)} - \tilde{Z}_t(g)| > \frac{\epsilon}{2}\Big) \xrightarrow[N\to\infty]{} 0. \end{aligned}$$

From Fubini's theorem and bounded convergence theorem, we see that

$$\begin{aligned} &\sup_{l,m\geq N} M_{[a,b)}\big(|X^{(l)} - X^{(m)}| > \epsilon\big) = \sup_{l,m\geq N} \frac{1}{b-a}\int_a^b G_{l,n}(t)\mathrm{d}t \\ &\leq \frac{1}{b-a}\int_a^b \sup_{l,m\geq N} G_{l,m}(t)\mathrm{d}t \xrightarrow[N\to\infty]{} 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\limsup_{N\to\infty} \sup_{l,m\geq N} \int \Big(\Big| X_t^{(l)}(\omega) - X_t^{(m)}(\omega) \Big| \wedge 1 \Big) M_{[a,b)}(\mathrm{d}t, \mathrm{d}\omega) \\ &\leq \epsilon + \limsup_{N\to\infty} \sup_{l,m\geq N} M_{[a,b)}\big(|X^{(l)} - X^{(m)}| > \epsilon\big) = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrarily chosen, we have

$$\sup_{l,m\geq N} \int \Big(\Big| X_t^{(l)}(\omega) - X_t^{(m)}(\omega) \Big| \wedge 1 \Big) M_{[a,b)}(\mathrm{d}t, \mathrm{d}\omega) \xrightarrow[N\to\infty]{} 0.$$

Therefore, $(X^{(m)})_{m\in\mathbb{N}}$ is a Cauchy sequence in probability w.r.t. $M_{[a,b)}$ in the sense of [Kal21, p. 104]. The limit, denoted by $\hat{Y}^g$, is clearly a measurable function on $([a,b) \times \Omega, \mathcal{B}_{[a,b)} \otimes \mathcal{F})$.

Step 3. By Step 1 and [Kal21, Lemma 5.2], there exists a strictly increasing sequence $(m_k)_{k=1}^\infty$ in $\mathbb{N}$ such that $(X^{(m_k)})_{k\in\mathbb{N}}$ converges almost surely to $\hat{Y}^g$ w.r.t. $M_{[a,b)}$. Let Ξ be a measurable null subset of the probability space $([a,b)\times\Omega, \mathcal{B}_{[a,b)}\otimes\mathcal{F}, M_{[a,b)})$ which contains all elements (t,ω) such that $X_t^{(m_k)}(\omega)$ does not converge to $\hat{Y}^g(t,\omega)$ when $k\uparrow\infty$. By the Fubini's theorem, the Lebesgue measure of the Borel measurable set $K := \{t\in[a,b) : \int \mathbf{1}_\Xi(t,\omega)\mathbb{P}_{\Lambda,\mu}(\mathrm{d}\omega) > 0\}$ is 0. Now for each $t\in[a,b)$, define a random variable $\tilde{Y}_t^g$ on Ω such that $\tilde{Y}_t^g(\omega) = \tilde{Z}_t(g)(\omega)\mathbf{1}_K(t) + \hat{Y}^g(t,\omega)\mathbf{1}_{K^c}(t)$ for every $\omega\in\Omega$. It is clear that $(\tilde{Y}_t^g)_{t\in[a,b)}$ is a measurable process.

Step 4. We finish the proof by showing that $(\tilde{Y}_t^g)_{t\in[a,b)}$ is a version of $(\tilde{Z}_t(g))_{t\in[a,b)}$. Note that for any $t\in K$, we already have $\mathbb{P}_{(\Lambda,\mu)}(\tilde{Y}_t^g = \tilde{Z}_t(g)) = 1$ according to how $\tilde{Y}_t^g$ is defined. Therefore, we only have to consider the case when $t\in[a,b)\setminus K$. In this case, we have $\int \mathbf{1}_\Xi(t,\omega)\mathbb{P}_{(\Lambda,\mu)}(\mathrm{d}\omega) = 0$, which implies that $X_t^{(m_k)}$ converges to $\tilde{Y}_t^g$ almost surely when $k\uparrow\infty$ w.r.t. $\mathbb{P}_{(\Lambda,\mu)}$. From Lemma 5.10, we have that $X_t^{(m_k)}$ converges to $\tilde{Z}_t(g)$ in probability. So we must have $\mathbb{P}_{(\Lambda,\mu)}(\tilde{Y}_t^g = \tilde{Z}_t(g)) = 1$ as desired in this case. We are done. □

Proof of Lemma 5.16. Step 1. We will show in this step that there exists an $\mathcal{N}$-valued process $(Z_t)_{t\geq a}$ such that almost surely, for every $t\geq a$ and every strictly decreasing sequence $(q_m)_{m\in\mathbb{N}}$ in $\mathbb{Q}$ which converges to t, $(\tilde{Z}_{q_m})_{m\in\mathbb{N}}$ converges to Z_t in $\mathcal{N}$. We will also show that, almost surely, for every strictly increasing sequence $(q_m)_{m\in\mathbb{N}}$ in $\mathbb{Q}\cap[a,\infty)$, $(\tilde{Z}_{q_m})_{m\in\mathbb{N}}$ converges in $\mathcal{N}$.

Recall that $\mathcal{N}$ is equipped with the complete metric $d_\mathcal{N}$ which is defined using a sequence $(h_i)_{i\in\mathbb{N}}$ in $\mathcal{C}_\mathrm{c}^\infty(\mathbb{R})$. Note that almost surely for every $q\in\mathbb{Q}\cap[a,\infty)$ and $i\in\mathbb{N}$, $Y_q^{h_i} = \tilde{Z}_q(h_i)$. Therefore, almost surely, for every strictly decreasing, or strictly increasing, sequence $(q_m)_{m\in\mathbb{N}}$ in $\mathbb{Q}\cap[a,\infty)$,

$$\begin{aligned}\sup_{m,l\geq N} d_\mathcal{N}\Big(\tilde{Z}_{q_m},\tilde{Z}_{q_l}\Big) &= \sup_{m,l\geq N}\sum_{i=1}^\infty \frac{1}{2^i}\Big(1\wedge\Big|\tilde{Z}_{q_m}(h_i)-\tilde{Z}_{q_l}(h_i)\Big|\Big)\\ &= \sup_{m,l\geq N}\sum_{i=1}^\infty\frac{1}{2^i}\big(1\wedge\big|Y_{q_m}^{h_i}-Y_{q_l}^{h_i}\big|\big) \leq \sum_{i=1}^\infty\frac{1}{2^i}\left(1\wedge\sup_{m,l\geq N}\big|Y_{q_m}^{h_i}-Y_{q_l}^{h_i}\big|\right)\xrightarrow{N\to\infty}0.\end{aligned}$$

Here, in the last step, we used the fact that almost surely for every $i\in\mathbb{N}$, $Y_s^{h_i}$ is càdlàg in $s>0$. Now, almost surely, for every strictly decreasing, or strictly increasing, sequence $(q_m)_{m\in\mathbb{N}}$ in $\mathbb{Q}\cap[a,\infty)$, $(\tilde{Z}_{q_m})_{m\in\mathbb{N}}$ is a Cauchy sequence w.r.t. the metric $d_\mathcal{N}$. The desired result for this step must follow.

Step 2. We argue that $(Z_t)_{t\geq a}$ is an $\mathcal{N}$-valued càdlàg process. On one hand, from Step 1, we know that almost surely, for every $t\geq a$ and every strictly decreasing sequence $(t_m)_{m\in\mathbb{N}}$ in $\mathbb{R}$ which converges to t, since there exists a strictly decreasing sequence $(q_m)_{m\in\mathbb{N}}$ in $\mathbb{Q}$ such that $q_1>t_1>q_2>t_2>\dots$ and that $d_\mathcal{N}(\tilde{Z}_{q_m},Z_{t_m})\leq 1/m$ for every $m\in\mathbb{N}$, we have

$$d_\mathcal{N}(Z_{t_m},Z_t)\leq d_\mathcal{N}(Z_{t_m},\tilde{Z}_{q_m}) + d_\mathcal{N}(\tilde{Z}_{q_m},Z_t)\xrightarrow{m\to\infty}0.$$

On the other hand, from Step 1 again, almost surely, for every strictly increasing sequence $(t_m)_{m\in\mathbb{N}}$ in $[a,\infty)$, since there exists a strictly increasing sequence $(q_m)_{m\in\mathbb{N}}$ in $\mathbb{Q}$ such that $t_1 < q_1 < t_2 < q_2 < \dots$ and that $d_{\mathcal{N}}(\tilde{Z}_{q_m}, Z_{t_m}) \leq 1/m$ for every $m \in \mathbb{N}$, we have

$$\sup_{l,m\geq N} d_{\mathcal{N}}(Z_{t_m}, Z_{t_l}) \leq \sup_{l,m\geq N}\Big(d_{\mathcal{N}}(Z_{t_m}, \tilde{Z}_{q_m}) + d_{\mathcal{N}}(\tilde{Z}_{q_m}, \tilde{Z}_{q_l}) + d_{\mathcal{N}}(\tilde{Z}_{q_l}, Z_{t_l})\Big)$$
$$\leq \frac{2}{N} + \sup_{l,m\geq N} d_{\mathcal{N}}(\tilde{Z}_{q_m}, \tilde{Z}_{q_l}) \xrightarrow{N\to\infty} 0.$$

Now, the desired result for this step holds.

Step 3. We argue that the process $(Z_t)_{t\geq a}$ is a modification of $(\tilde{Z}_t)_{t\geq a}$. Let us fix an arbitrary $t \geq a$. Let $(q_m)_{m\in\mathbb{N}}$ be a strictly decreasing sequence in $\mathbb{Q}$ which converges to t. Note that almost surely for each $m, i \in \mathbb{N}$, $Y^{h_i}_{q_m} = \tilde{Z}_{q_m}(h_i)$. Combined with Step 1, almost surely, for each $i \in \mathbb{N}$, we have

$$Z_t(h_i) = \lim_{m\to\infty} \tilde{Z}_{q_m}(h_i) = \lim_{m\to\infty} Y^{h_i}_{q_m} = Y^{h_i}_t$$

where in the last step we used the fact that almost surely $Y^{h_i}_s$ is càdlàg in $s > 0$. Therefore, we have almost surely $Z_t(h_i) = \tilde{Z}_t(h_i)$ for each $i \in \mathbb{N}$, which further implies that almost surely $d_{\mathcal{N}}(Z_t, \tilde{Z}_t) = 0$. The desired result for this step now follows.

Step 4. Combining the results from Steps 2 and 3, $(\tilde{Z}_t)_{t\geq a}$ has a càdlàg modification. According to the discussion at the beginning of this proof, we are done. □

E. Proof of Lemmas 6.2 and 6.3

Proof of Lemma 6.2. Let $\gamma \in (0,1)$ be arbitrary and $\gamma_0 = \gamma\Psi'(0+)/(2\beta_{\mathrm{c}})$. Let F be the smallest closed interval containing $\cup_{i=1}^{\infty}(x_i - 1, x_i + 1)$. It is clear that $U \cap F$ is bounded. By Lemmas 4.5 (ii), 4.7 and 6.1 (ii), we have

$$\lim_{t\downarrow 0} \frac{\mathcal{V}^{(\Lambda,\mu,F)}_t}{\int_U v^{(\Lambda,\mu)}_t(x)\mathrm{d}x} = \lim_{t\downarrow 0} \frac{C_1(U,F,t,\gamma_0)}{\int_U v^{(\Lambda,\mu)}_t(x)\mathrm{d}x} = \lim_{t\downarrow 0} \frac{C_1(U,F,t,\gamma)}{\int_U v^{(\Lambda,\mu)}_t(x)\mathrm{d}x} = 0. \tag{E.1}$$

Therefore, by Proposition 5.8, we see that

$$\limsup_{t\downarrow 0} \frac{\mathbb{E}_{(\Lambda,\mu)}[Z_t(U)]}{\int_U v^{(\Lambda,\mu)}_t(x)\mathrm{d}x} \leq \frac{1}{1-\gamma} \xrightarrow{\gamma\downarrow 0} 1. \tag{E.2}$$

On the other hand, combining Proposition 5.6, Lemmas 4.4, 4.6 and 4.7, we have for any $\gamma \in (0,1/2)$, $\varepsilon \in (0,\gamma/2)$ and $t > 0$, taking $\kappa(\gamma)$ as in (4.2),

$$\mathbb{E}_{(\Lambda,\mu)}\big[(1-\varepsilon)^{Z_t(U)}\big] = \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\prod_{i=1}^{\infty}(1 - u_t(x_i))\right] \tag{E.3}$$
$$\leq \tilde{\mathbb{E}}_{\varepsilon\mathbf{1}_U}\left[\exp\left\{-\sum_{i=1}^{\infty} u_t(x_i)\right\}\right] + \tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\left(\sup_{s\leq t, y\in F} u_s(y) > \gamma\right).$$
$$\leq \exp\left\{-\varepsilon\kappa(\gamma)e^{-\beta_{\mathrm{o}}t}\int_U v^{(\Lambda,\mu)}_t(y)\mathrm{d}y\right\} + 2\tilde{\mathbb{P}}_{\varepsilon\mathbf{1}_U}\left(\sup_{s\leq t, y\in F} u_s(y) > \gamma\right) + \varepsilon\beta_{\mathrm{c}}e^{\lambda_{\mathrm{o}}t}\mathcal{V}^{(\Lambda,\mu,F)}_t$$

$$\leq \exp\left\{-\varepsilon\kappa(\gamma)e^{-\beta_{\mathrm{o}}t}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\right\} + 2\varepsilon C_1(U,F,t,\gamma) + \varepsilon\beta_{\mathrm{c}}e^{\lambda_{\mathrm{o}}t}\mathcal{V}_t^{(\Lambda,\mu,F)}.$$

Therefore, for any $\gamma \in (0,1/2)$ and $t > 0$,

$$\mathbb{E}_{(\Lambda,\mu)}[Z_t(U)] = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(1-\mathbb{E}_{(\Lambda,\mu)}\left[(1-\varepsilon)^{Z_t(U)}\right]\right)$$
$$\geq \kappa(\gamma)e^{-\beta_{\mathrm{o}}t}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y - 2C_1(U,F,t,\gamma) - \beta_{\mathrm{c}}e^{\lambda_{\mathrm{o}}t}\mathcal{V}_t^{(\Lambda,\mu,F)}.$$

Using (E.1), we conclude that

$$\liminf_{t\downarrow 0}\frac{\mathbb{E}_{(\Lambda,\mu)}[Z_t(U)]}{\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x} \geq \kappa(\gamma) \xrightarrow{\gamma\downarrow 0} 1. \tag{E.4}$$

Therefore, we arrive at the desired result by (E.2) and (E.4). □

Proof of Lemma 6.3. For any $\vartheta > 0$ and $t > 0$, define

$$\varepsilon(U,\vartheta,t) := 1 - \exp\left\{-\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)^{-1}\vartheta\right\}.$$

By Lemma 6.1 (ii), we have

$$e^{-\beta_{\mathrm{o}}t}\varepsilon(U,\vartheta,t)\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x \xrightarrow{t\downarrow 0} \vartheta, \quad \vartheta > 0.$$

Let F be the smallest closed interval containing $\cup_{i\in\mathbb{N}}(x_i-1,x_i+1)$. Note from (E.1) that for any $\vartheta > 0$,

$$\lim_{t\downarrow 0}\varepsilon(U,\vartheta,t)C_1(U,F,t,\gamma) = \lim_{t\downarrow 0}\varepsilon(U,\vartheta,t)\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)\frac{C_1(U,F,t,\gamma)}{\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x} = 0,$$

and similarly, $\lim_{t\downarrow 0}\varepsilon(U,\vartheta,t)\mathcal{V}_t^{(\Lambda,\mu,F)} = 0$. From (E.3), for every $\gamma \in (0,1/2)$, $\vartheta > 0$, and $t > 0$ small enough such that $\varepsilon(U,\vartheta,t) < \gamma/2$,

$$\mathbb{E}_{(\Lambda,\mu)}\left[\exp\left\{-\vartheta\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)^{-1}Z_t(U)\right\}\right] = \mathbb{E}_{(\Lambda,\mu)}\left[(1-\varepsilon(U,\vartheta,t))^{Z_t(U)}\right]$$
$$\leq \exp\left\{-\varepsilon(U,\vartheta,t)\kappa(\gamma)e^{-\beta_{\mathrm{o}}t}\int_U v_t^{(\Lambda,\mu)}(y)\mathrm{d}y\right\} + 2\varepsilon(U,\vartheta,t)C_1(U,F,t,\gamma)$$
$$+ \varepsilon(U,\vartheta,t)\beta_{\mathrm{c}}e^{\lambda_{\mathrm{o}}t}\mathcal{V}_t^{(\Lambda,\mu,F)}$$
$$\xrightarrow{t\downarrow 0} e^{-\vartheta\kappa(\gamma)} \xrightarrow{\gamma\downarrow 0} e^{-\vartheta}.$$

Therefore, for every $\vartheta > 0$,

$$\limsup_{t\downarrow 0}\mathbb{E}_{(\Lambda,\mu)}\left[\exp\left\{-\vartheta\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)^{-1}Z_t(U)\right\}\right] \leq e^{-\vartheta}.$$

On the other hand, by Jensen's inequality $\mathbb{E}\big[e^{-|Y|}\big] \geq e^{-\mathbb{E}[|Y|]}$ and Lemma 6.2, for every $\vartheta > 0$,

$$\begin{aligned}&\mathbb{E}_{(\Lambda,\mu)}\left[\exp\left\{-\vartheta\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)^{-1} Z_t(U)\right\}\right]\\ &\geq \exp\left\{-\vartheta\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)^{-1} \mathbb{E}_{(\Lambda,\mu)}[Z_t(U)]\right\} \xrightarrow{t\downarrow 0} e^{-\vartheta}.\end{aligned}$$

Therefore, we have

$$\lim_{t\downarrow 0} \mathbb{E}_{(\Lambda,\mu)}\left[\exp\left\{-\vartheta\left(\int_U v_t^{(\Lambda,\mu)}(x)\mathrm{d}x\right)^{-1} Z_t(U)\right\}\right] = e^{-\vartheta}, \quad \vartheta > 0.$$

We are done. $\square$