Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 132 (2021) 108–134

stochastic processes and their applications

www.elsevier.com/locate/spa

Quasi-stationary distributions for subcritical superprocesses

Rongli Liu[a,1], Yan-Xia Ren[b,2], Renming Song[c,3], Zhenyao Sun[d,*]

[a] School of Science, Beijing Jiaotong University, Beijing 100044, PR China
[b] LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing 100871, PR China
[c] Department of Mathematics, University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA
[d] Faculty of Industrial Engineering and Management Technion, Isreal Institute of Technology, Haifa 3200003, Isreal

Received 19 January 2020; received in revised form 5 September 2020; accepted 22 October 2020
Available online 1 November 2020

Abstract

Suppose that X is a subcritical superprocess. Under some asymptotic conditions on the mean semigroup of X, we prove the Yaglom limit of X exists and identify all quasi-stationary distributions of X.

MSC: primary 60J68; 60F05; secondary 60J80; 60J25

Keywords: Superprocess; Yaglom limit; Quasi-stationary distribution

1. Introduction

1.1. Background

Denote $\mathbb{Z}_+ := \{1, 2, \ldots\}$ and $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$. Suppose that $Z = \{(Z_n)_{n\in\mathbb{N}}; (P_z)_{z\in\mathbb{N}}\}$ is a Galton–Watson process with offspring distribution $(p_n)_{n\in\mathbb{N}}$. Let $m := \sum_{n=1}^{\infty} np_n$ be the mean

* Corresponding author.
E-mail addresses: rlliu@bjtu.edu.cn (R. Liu), yxren@math.pku.edu.cn (Y.-X. Ren), rsong@illinois.edu (R. Song), zhenyao.sun@gmail.com (Z. Sun).
1 The research of Rongli Liu is supported in part by NSFC (Grant No. 11301261), and the Fundamental Research Funds for the Central Universities (Grant No. 2017RC007).
2 The research of Yan-Xia Ren is supported in part by NSFC (Grant Nos. 11671017 and 11731009) and LMEQF.
3 The research of Renming Song is supported in part by a grant from the Simons Foundation (#429343, Renming Song).

https://doi.org/10.1016/j.spa.2020.10.007
0304-4149/

of the offspring distribution. It is well known that when $m \le 1$ and $p_1 < 1$, the process Z becomes extinct in finite time almost surely, that is,

$$P_z(Z_n = 0 \text{ for some } n \in \mathbb{N}) = 1, \quad z \in \mathbb{N}.$$

Let $\zeta := \inf\{n \ge 0 : Z_n = 0\}$ be the extinction time of Z. If ν is a distribution on $\mathbb{Z}_+$ such that for any $z \in \mathbb{Z}_+$ and subset A of $\mathbb{Z}_+$,

$$\lim_{n\to\infty} P_z \left(Z_n \in A \middle| \zeta > n\right) = \nu(A),$$

then we say that ν is the Yaglom limit of Z. Yaglom [34] showed that such limit exists when $m < 1$ and the offspring distribution has finite second moment. This was generalized to the case without the second moment assumption in [10,13]. See also [2, pp. 64–65] for an alternative analytical approach; and [23] for a probabilistic proof. If ν is a distribution on $\mathbb{Z}_+$ such that for any subset A of $\mathbb{Z}_+$,

$$\sum_{z=1}^{\infty} \nu(z) P_z \left(Z_n \in A \middle| \zeta > n\right) = \nu(A), \quad n \in \mathbb{N},$$

then we say ν is a quasi-stationary distribution of Z. Hoppe and Seneta [12] studied the quasi-stationary distributions of $(Z_n)_{n\in\mathbb{N}}$. Recently, Maillard [24] characterized all λ-invariant measures of $(Z_n)_{n\in\mathbb{N}}$. If a λ-invariant measure is a probability measure, then it is equivalent to a quasi-stationary distribution. Multitype analogs for the Yaglom limit results can be found in [11,12,14].

Now suppose that $Z = \{(Z_t)_{t\ge0}; (P_x)_{x\ge0}\}$ is a continuous-state branching process on $[0,\infty)$ where 0 is an absorbing state. Let $\zeta := \inf\{t \ge 0 : Z_t = 0\}$ be the extinction time of Z. If ν is a distribution on $(0,\infty)$ such that for any $x > 0$ and Borel subset A of $(0,\infty)$,

$$\lim_{t\to\infty} P_x \left(Z_t \in A \middle| \zeta > t\right) = \nu(A),$$

then ν is called the Yaglom limit of Z. If ν is a distribution on $(0,\infty)$ such that for any Borel subset A of $(0,\infty)$,

$$\int_{(0,\infty)} \nu(dx) P_x(Z_t \in A | \zeta > t) = \nu(A), \quad t \ge 0,$$

then we say ν is a quasi-stationary distribution for Z. The Yaglom limits of continuous-state branching processes were studied in [20], where conditioning of the type $\{\zeta > t + r\}$ for any finite $r > 0$ instead of $\{\zeta > t\}$ was also considered. Lambert [19] also studied Yaglom limits using a different method, and characterized all the quasi-stationary distributions for Z. Seneta and Vere-Jones [32] studied some similar type of conditional limits for discrete-time continuous-state branching processes. Recently [18] considered quasi-stationary distributions for continuous-state branching processes conditioned on non-explosion.

Asmussen and Hering [1] studied limit behaviors of subcritical branching Markov processes. They proved that the Yaglom limits for a class of subcritical branching Markov processes exist under some conditions on the mean semigroup, and characterized all of their quasi-stationary distributions, see [1, Chapter 5] and the references therein.

In this paper, we are interested in a class of subcritical (ξ, ψ)-superprocesses. We will prove the existence of the Yaglom limit and identify all quasi-stationary distributions under some asymptotic conditions on its mean semigroup. Our superprocesses are general in the sense that the spatial motion ξ can be a general Borel right process taking values in a Polish space, and the branching mechanism ψ can be spatially inhomogeneous. Precise statements of the assumptions and the results are presented in the next subsection.

As far as we know, there are no results on Yaglom limit and quasi-stationary distributions for general superprocesses in the literature. Here we list some papers dealing with superprocesses conditioning on various kinds of survivals under different settings: [3,6–8,22,26–28,33].

1.2. Main result

We first recall some basics about superprocesses. Let E be a Polish space. Let ∂ be an isolated point not contained in E and $E_\partial := E \cup \{\partial\}$. Denote by $\mathcal{B}(E, D)$ the collection of Borel maps from E to some measurable space D. If D is a subset of $\mathbb{R}$, we denote by $\mathcal{B}_b(E, D)$ the bounded measurable functions from E to D. Assume that *the underlying process* $\xi = \{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in E}\}$ is an E_∂-valued Borel right process with ∂ as an absorbing state. Denote by $\zeta := \inf\{t > 0 : \xi_t = \partial\}$ the lifetime of ξ. Let *the branching mechanism* ψ be a function on $E \times [0,\infty)$ given by

$$\psi(x,z) = -\beta(x)z + \sigma(x)^2 z^2 + \int_{(0,\infty)} (e^{-zu} - 1 + zu)\pi(x, du), \quad x \in E, z \geq 0,$$

where $\beta, \sigma \in \mathcal{B}_b(E,\mathbb{R})$ and $(u \wedge u^2)\pi(x, du)$ is a bounded kernel from E to $(0,\infty)$. Let $\mathcal{M}_f(E)$ denote the space of all finite Borel measures on E equipped with the topology of weak convergence. Denote by $\mathcal{B}(\mathcal{M}_f(E))$ the Borel σ-field generated by this topology. For any $\mu \in \mathcal{M}_f(E)$ and $g \in \mathcal{B}(E, [0,\infty))$, we use $\mu(g)$ to denote the integration of g with respect to μ whenever the integration is well defined. We will use $\|\mu\|$ to denote $\mu(1)$. For any $f \in \mathcal{B}_b(E, [0,\infty))$, there is a unique locally bounded non-negative map $(t, x) \mapsto V_t f(x)$ on $[0,\infty) \times E$ such that

$$V_t f(x) + \Pi_x\left[\int_0^{t\wedge\zeta} \psi\left(\xi_s, V_{t-s} f(\xi_s)\right) ds\right] = \Pi_x\left[f(\xi_t)\mathbf{1}_{t<\zeta}\right], \quad t \geq 0, x \in E. \tag{1.1}$$

Here, the local boundedness of the map $(t, x) \mapsto V_t f(x)$ means that $\sup_{0\leq t\leq T, x\in E} V_t f(x) < \infty$ for $T > 0$. Moreover, there exists an $\mathcal{M}_f(E)$-valued Borel right process $X = \{(X_t)_{t\geq 0}; (\mathbb{P}_\mu)_{\mu\in\mathcal{M}_f(E)}\}$ such that

$$\mathbb{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad t \geq 0, \ \mu \in \mathcal{M}_f(E), f \in \mathcal{B}_b(E, [0,\infty)).$$

We call X a (ξ,ψ)-*superprocess*. See [21] for more details.

The *mean semigroup* $(P_t^\beta)_{t\geq 0}$ of X is defined by

$$P_t^\beta f(x) := \Pi_x\left[e^{\int_0^t \beta(\xi_r)dr} f(\xi_t)\mathbf{1}_{t<\zeta}\right], \quad f \in \mathcal{B}_b(E,\mathbb{R}), t \geq 0, x \in E.$$

It is well-known (see [21, Proposition 2.27]) that

$$\mathbb{P}_\mu[X_t(f)] = \mu(P_t^\beta f), \quad \mu \in \mathcal{M}_f(E), t \geq 0, f \in \mathcal{B}_b(E,\mathbb{R}). \tag{1.2}$$

In this paper, we will always assume that there exist a constant $\lambda < 0$, a function $\phi \in \mathcal{B}_b(E, (0,\infty))$ and a probability measure ν with full support on E such that for each $t \geq 0$, $P_t^\beta \phi = e^{\lambda t}\phi$, $\nu P_t^\beta = e^{\lambda t}\nu$ and $\nu(\phi) = 1$. The assumption $\lambda < 0$ says that the mean of $(X_t(\phi))_{t\geq 0}$ decays exponentially with rate λ, and in this case the superprocess X is called subcritical. Denote by $L_1^+(\nu)$ the collection of non-negative Borel functions on E which are integrable with respect to the measure ν. We further assume that the following two conditions hold:

$$\begin{aligned}&\text{For all } t>0,\ x\in E, \text{ and } f\in L_1^+(\nu), \text{ it holds that}\\ &\qquad P_t^\beta f(x)=e^{\lambda t}\phi(x)\nu(f)(1+H_{t,x,f})\\ &\text{for some real } H_{t,x,f} \text{ with}\\ &\qquad \sup_{x\in E, f\in L_1^+(\nu)}|H_{t,x,f}|<\infty \text{ and } \lim_{t\to\infty}\sup_{x\in E, f\in L_1^+(\nu)}|H_{t,x,f}|=0.\end{aligned}\tag{H1}$$

$$\text{There exists } T\geq 0 \text{ such that } \mathbb{P}_\nu(\|X_t\|=0)>0 \text{ for all } t>T.\tag{H2}$$

Note that $L_1^+(\nu)$ in (H1) can be replaced by the collection of all non-negative Borel functions f with $\nu(f)=1$. In fact, for any $f\in L_1^+(\nu)$ and $k\in(0,\infty)$, it is easy to see that $H_{t,x,f}=H_{t,x,kf}$.

(H1) is mainly concerned with the spatial motion and (H2) is mainly about the branching mechanism of the superprocess. In Section 1.3, we will give examples satisfying these two assumptions.

We mention here that quantities like $H_{t,x,f}$ in this paper might depend on the underlying process ξ and the branching mechanism ψ. Since ξ and ψ are fixed, dependence on them will not be explicitly specified.

Denote by $\mathbf{0}$ the null measure on E. Write $\mathcal{M}_f^o(E):=\mathcal{M}_f(E)\backslash\{\mathbf{0}\}$. Any probability measure $\mathbf{P}$ on $\mathcal{M}_f^o(E)$ will also be understood as its unique extension on $\mathcal{M}_f(E)$ with $\mathbf{P}(\{\mathbf{0}\})=0$. Since ϕ is strictly positive, we have

$$\mathbb{P}_\mu[X_t(\phi)]\overset{(1.2)}{=}\mu(P_t^\beta\phi)=e^{\lambda t}\mu(\phi)>0,\quad t\geq 0, \mu\in\mathcal{M}_f^o(E).$$

Thus,

$$\mathbb{P}_\mu(\|X_t\|>0)>0,\quad t\geq 0, \mu\in\mathcal{M}_f^o(E).\tag{1.3}$$

Hence we can condition the superprocess X on survival up to time t if the distribution of X_0 is not concentrated on $\{\mathbf{0}\}$. Our first main result is the following.

Theorem 1.1. *If (H1) and (H2) hold, then there exists a probability measure $\mathbf{Q}_\lambda$ on $\mathcal{M}_f^o(E)$ such that*

$$\mathbb{P}_\mu\left(X_t\in\cdot\middle|\|X_t\|>0\right)\xrightarrow[t\to\infty]{w}\mathbf{Q}_\lambda(\cdot),\quad \mu\in\mathcal{M}_f^o(E),$$

where $\overset{w}{\to}$ stands for weak convergence.

Now we introduce the concepts of *quasi-limiting distribution (QLD)* and *quasi-stationary distribution (QSD)* for our superprocess X. For any probability measure $\mathbf{P}$ on $\mathcal{M}_f(E)$, define $(\mathbf{P}\mathbb{P})[\cdot]:=\int_{\mathcal{M}_f(E)}\mathbb{P}_\mu[\cdot]\mathbf{P}(d\mu)$. We say a probability measure $\mathbf{Q}$ on $\mathcal{M}_f^o(E)$ is a QLD of X, if there exists a probability measure $\mathbf{P}$ on $\mathcal{M}_f^o(E)$ such that

$$(\mathbf{P}\mathbb{P})\left(X_t\in B\middle|\|X_t\|>0\right)\xrightarrow[t\to\infty]{}\mathbf{Q}(B),\quad B\in\mathcal{B}(\mathcal{M}_f^o(E)).$$

We say a probability measure $\mathbf{Q}$ on $\mathcal{M}_f^o(E)$ is a QSD of X, if

$$(\mathbf{Q}\mathbb{P})\left(X_t\in B\middle|\|X_t\|>0\right)=\mathbf{Q}(B),\quad t\geq 0, B\in\mathcal{B}(\mathcal{M}_f^o(E)).$$

It follows from [25, Proposition 1] that, for any Markov process on $[0,\infty)$ with 0 as an absorbing state, its QLDs and QSDs are equivalent. We claim that this is also the case for our $\mathcal{M}_f(E)$-valued Markov process X, for which the null measure $\mathbf{0}$ is an absorbing state. In fact, since E is a Polish space, $\mathcal{M}_f(E)$ is again Polish [16, Lemma 4.3]. So is $\mathcal{M}_f^o(E)$ [15, Theorem A1.2]. Thus $\mathcal{M}_f^o(E)$ is Borel isomorphic to $(0,\infty)$ [15, Theorem A.1.6]. That is, there exists a bijection $\tau : \mathcal{M}_f^o(E) \to (0,\infty)$ such that both τ and its inverse τ^{-1} are Borel measurable. Extend τ uniquely so that it is a bijection between $\mathcal{M}_f(E)$ and $[0,\infty)$. Then, it is easy to verify that τ is a Borel isomorphism between $\mathcal{M}_f(E)$ and $[0,\infty)$ which maps $\mathbf{0}$ to 0. Now for any $\mathcal{M}_f(E)$-valued Markov process with $\mathbf{0}$ as an absorbing state, its image under τ is a $[0,\infty)$-valued Markov process with 0 as an absorbing state. Therefore we can apply [25, Proposition 1] to $(\tau(X_t))_{t\geq 0}$ which gives that a probability $\mathbf{Q}$ on $\mathcal{M}_f^o(E)$ is a QLD for X if and only if it is a QSD for X. Similarly, we can apply [25, Proposition 2] to X which says that

if a probability measure $\mathbf{Q}$ on $\mathcal{M}_f^o(E)$ is a QSD of X, then there exists an $r \in (-\infty, 0)$ such that $(\mathbf{Q}\mathbb{P})(\|X_t\| > 0) = e^{rt}$ for all $t \geq 0$. In this case, we call r the mass decay rate of $\mathbf{Q}$. (1.4)

Theorem 1.2. *Suppose that (H1) and (H2) hold. Then (1) for each $r \in [\lambda, 0)$, there exists a unique QSD for X with mass decay rate r; and (2) for each $r \in (-\infty, \lambda)$, there is no QSD for X with mass decay rate r.*

1.3. Examples

In this subsection, we will give some examples satisfying (H1) and (H2).

We first give an example satisfying (H2). Suppose that ψ is bounded from below by a spatially independent branching mechanism, that is, there is a function $\widetilde{\psi}$ of the form

$$\widetilde{\psi}(z) = \widetilde{\beta} z + \widetilde{\sigma}^2 z^2 + \int_0^\infty (e^{-zu} - 1 + zu)\widetilde{\pi}(du), \quad z \geq 0$$

with $\widetilde{\beta} \in \mathbb{R}$, $\widetilde{\sigma} \geq 0$ and $\widetilde{\pi}$ is a measure on $(0,\infty)$ satisfying $\int_0^\infty (u \wedge u^2)\widetilde{\pi}(du) < \infty$ such that

$$\psi(x,z) \geq \widetilde{\psi}(z), \quad x \in E, z \geq 0.$$

If $\widetilde{\psi}(\infty) = \infty$ and $\int^\infty 1/\widetilde{\psi}(z)dz < \infty$, then by [28, Lemma 2.3], for any $t > 0$,

$$\inf_{x\in E} \mathbb{P}_{\delta_x}(\|X_t\| = 0) > 0.$$

Using this and (2.4) below one can easily get that $\mathbb{P}_\nu(\|X_t\| = 0) > 0$ for all $t > 0$. Thus (H2) is satisfied with $T = 0$.

Now we give conditions that imply (H1). We assume that ξ is a Hunt process and there exist an σ-finite measure m with full support on E and a family of strictly positive, bounded continuous functions $\{p_t(\cdot,\cdot) : t > 0\}$ on $E \times E$ such that

$$\Pi_x[f(\xi_t)\mathbf{1}_{t<\zeta}] = \int_E p_t(x,y) f(y) m(dy), \quad t > 0, x \in E, f \in \mathcal{B}_b(E,\mathbb{R});$$

$$\int_E p_t(x,y)m(dx) \leq 1, \quad t > 0, y \in E;$$

$$\int_E \int_E p_t(x,y)^2 m(dx)m(dy) < \infty, \quad t > 0;$$

and the functions $x \mapsto \int_E p_t(x,y)^2 m(dy)$ and $y \mapsto \int_E p_t(x,y)^2 m(dx)$ are both continuous. Choose an arbitrary $\mathfrak{b} \in \mathcal{B}_b(E,\mathbb{R})$. Denote by $(P_t^{\mathfrak{b}})_{t\geq 0}$ a semigroup of operators on $\mathcal{B}_b(E,\mathbb{R})$ given by

$$P_t^{\mathfrak{b}} f(x) := \Pi_x[e^{\int_0^t \mathfrak{b}(\xi_s)ds} f(\xi_t)\mathbf{1}_{t<\zeta}], \quad f \in \mathcal{B}_b(E,\mathbb{R}), t \geq 0, x \in E.$$

Let us write $\langle f, g\rangle_m := \int_E f(x)g(x)m(dx)$ for the inner product of the Hilbert space $L^2(E,m)$. Then it is proved in [28,29] that there exists a family of strictly positive, bounded continuous functions $\{p_t^{\mathfrak{b}} : t > 0\}$ on $E \times E$ such that

$$e^{-\|\mathfrak{b}\|_\infty t} p_t(x,y) \leq p_t^{\mathfrak{b}}(x,y) \leq e^{\|\mathfrak{b}\|_\infty t} p_t(x,y), \quad t > 0, x, y \in E \tag{1.5}$$

and that

$$P_t^{\mathfrak{b}} f(x) = \int_E p_t^{\mathfrak{b}}(x,y) f(y) m(dy), \quad t > 0, x \in E.$$

Define the dual semigroup $(\widehat{P_t^{\mathfrak{b}}})_{t\geq 0}$ by

$$\widehat{P_0^{\mathfrak{b}}} = I; \quad \widehat{P_t^{\mathfrak{b}}} f(x) := \int_E p_t^{\mathfrak{b}}(y,x) f(y) m(dy), \quad t > 0, x \in E, f \in \mathcal{B}_b(E,\mathbb{R}).$$

It is proved in [28,29] that both $(P_t^{\mathfrak{b}})_{t\geq 0}$ and $(\widehat{P_t^{\mathfrak{b}}})_{t\geq 0}$ are strongly continuous semigroups of compact operators on $L^2(E,m)$. Let $L^{\mathfrak{b}}$ and $\widehat{L^{\mathfrak{b}}}$ be the generators of the semigroups of compact operators on $(P_t^{\mathfrak{b}})_{t\geq 0}$ and $(\widehat{P_t^{\mathfrak{b}}})_{t\geq 0}$, respectively. Denote by $\sigma(L^{\mathfrak{b}})$ and $\sigma(\widehat{L^{\mathfrak{b}}})$ the spectra of $L^{\mathfrak{b}}$ and $\widehat{L^{\mathfrak{b}}}$, respectively. According to Theorem 29 of [31], $\lambda_{\mathfrak{b}} := \sup \Re(\sigma(L^{\mathfrak{b}})) = \sup \Re(\sigma(\widehat{L^{\mathfrak{b}}}))$ is a common eigenvalue of multiplicity 1 for both $L^{\mathfrak{b}}$ and $\widehat{L^{\mathfrak{b}}}$. By the argument in [28] and [29], the eigenfunctions $h_{\mathfrak{b}}$ of $L^{\mathfrak{b}}$ and $\widehat{h}_{\mathfrak{b}}$ of $\widehat{L^{\mathfrak{b}}}$ associated with the eigenvalue $\lambda_{\mathfrak{b}}$ can be chosen to be strictly positive and continuous everywhere on E. Setting $\langle h_{\mathfrak{b}}, h_{\mathfrak{b}}\rangle_m = \langle h_{\mathfrak{b}}, \widehat{h}_{\mathfrak{b}}\rangle_m = 1$ so that $h_{\mathfrak{b}}$ and $\widehat{h}_{\mathfrak{b}}$ are uniquely determined pointwisely.

We assume further that $h_0 := h_{\mathfrak{b}}|_{\mathfrak{b}\equiv 0}$ is bounded, and the semigroup $(P_t)_{t\geq 0}$ is intrinsically ultracontractive in the following sense: for all $t > 0$ and $x, y \in E$, it holds that $p_t(x,y) = c_{t,x,y} h_0(x)\widehat{h}_0(y)$ for some positive $c_{t,x,y}$ with $\sup_{x,y\in E} c_{t,x,y} < \infty$. Here, $\widehat{h}_0 := \widehat{h}_{\mathfrak{b}}|_{\mathfrak{b}\equiv 0}$. Then, it is proved in [28,29] that, for arbitrary $\mathfrak{b} \in \mathcal{B}_b(E,\mathbb{R})$, $h_{\mathfrak{b}}$ is also bounded; and $(P_t^{\mathfrak{b}})_{t\geq 0}$ is also intrinsically ultracontractive, in the sense that for any $t > 0$ and $x, y \in E$ we have

$$p_t^{\mathfrak{b}}(x,y) = C^1_{\mathfrak{b},t,x,y} h_{\mathfrak{b}}(x)\widehat{h}_{\mathfrak{b}}(y) \tag{1.6}$$

for some positive $C^1_{\mathfrak{b},t,x,y}$ with $\sup_{x,y\in E} C^1_{\mathfrak{b},t,x,y} < \infty$. It follows from [17, Proposition 2.5 and Theorem 2.7], when (1.6) holds, $C^1_{\mathfrak{b},t,x,y}$ can be chosen so that

$$\sup_{x,y\in E} (C^1_{\mathfrak{b},t,x,y})^{-1} < \infty, \quad t > 0, \tag{1.7}$$

and that for any $t > 0, x, y \in E$,

$$C^1_{\mathfrak{b},t,x,y} = e^{t\lambda_{\mathfrak{b}}}(1 + C^2_{\mathfrak{b},t,x,y}) \tag{1.8}$$

for some real $C^2_{\mathfrak{b},t,x,y}$ with $\lim_{t\to\infty} \sup_{x,y\in E} C^2_{\mathfrak{b},t,x,y} = 0$. Therefore,

$$\begin{aligned} m(\widehat{h}_{\mathfrak{b}}) &\overset{(1.6)}{=} \int_E p_t^{\mathfrak{b}}(x,y) h_{\mathfrak{b}}(x)^{-1} (C^1_{\mathfrak{b},t,x,y})^{-1} m(dy), \quad x \in E, \\ &\leq h_{\mathfrak{b}}(x)^{-1} \left(\sup_{z\in E} (C^1_{\mathfrak{b},t,x,z})^{-1} \right) \int_E p_t^{\mathfrak{b}}(x,y) m(dy) \\ &< \infty \quad \text{by (1.5) and (1.7) and the strict positivity of } h_{\mathfrak{b}}. \end{aligned}$$

This allows us to define a probability measure $\nu_{\mathfrak{b}}(dx) := m(\widehat{h}_{\mathfrak{b}})^{-1}\widehat{h}_{\mathfrak{b}}(x)m(dx), x \in E$, and an eigenfunction $\phi_{\mathfrak{b}}(x) := m(\widehat{h}_{\mathfrak{b}})h_{\mathfrak{b}}(x), x \in E$.

Finally we write $\lambda := \lambda_\beta$ and assume that $\lambda < 0$. We now show that X satisfies (H1) with $\phi := \phi_\beta$ and $\nu := \nu_\beta$. From their definitions, we see that the function $\phi \in \mathcal{B}_b(E, (0, \infty))$, and that the probability measure ν has full support on E. Further, it is easy to see that for each $t \geq 0$, $P_t^\beta \phi = e^{\lambda t}\phi$ and $\nu(\phi) = 1$. We also have that for any $t > 0$,

$$
\begin{aligned}
(\nu P_t^\beta)(dy) &= \int_{x\in E} p_t^\beta(x, y)m(dy)\nu(dx) \\
&= \int_{x\in E} p_t^\beta(x, y)m(dy)m(\widehat{h}_\beta)^{-1}\widehat{h}_\beta(x)m(dx) \\
&= m(\widehat{h}_\beta)^{-1}\left(\int_{x\in E} p_t^\beta(x, y)\widehat{h}_\beta(x)m(dx)\right)m(dy) \\
&= m(\widehat{h}_\beta)^{-1}e^{\lambda t}\widehat{h}_\beta(y)m(dy) = e^{\lambda t}\nu(dy).
\end{aligned}
$$

Therefore $\nu P_t^\beta = e^{\lambda t}\nu, t \geq 0$. Now for each $t > 0, x \in E$ and $f \in L_1^+(\nu)$, we have

$$
\begin{aligned}
P_t^\beta f(x) &= \int_E p_t^\beta(x, y)f(y)m(dy) \overset{(1.6)}{=} \int_E h_\beta(x)\widehat{h}_\beta(y)C^1_{\beta,t,x,y}f(y)m(dy) \\
&= \int_E \phi(x)C^1_{\beta,t,x,y}f(y)\nu(dy) =: e^{\lambda t}\phi(x)\nu(f)(1 + H_{t,x,f}).
\end{aligned}
$$

Finally, from (1.6) and (1.8), it is elementary to verify that $H_{t,x,f}$ satisfies the required condition (H1).

In three paragraphs above, we give some conditions that imply (H1). See [28, Section 1.4] for more than 10 concrete examples of processes satisfying these conditions.

Organization of the rest of the paper.

In Section 2.1 we will give the proof of Theorem 1.1 using Propositions 2.1–2.4. In Section 2.2 we will give the proof of Theorem 1.2 using Propositions 2.5–2.7. The proofs of Propositions 2.1–2.4 are given in Section 3. The proof of Propositions 2.5–2.7 are given in Section 4. Some technical lemmas are in the Appendix, and will be referred to as needed in the proofs.

2. Proofs of Theorems 1.1 and 1.2

2.1. Proof of Theorem 1.1

It is easy to see that the operators $(V_t)_{t\geq 0}$ given by (1.1) can be extended uniquely to a family of operators $(\overline{V}_t)_{t\geq 0}$ on $\mathcal{B}(E, [0, \infty])$ such that for all $t \geq 0$, $f_n \uparrow f$ pointwisely in $\mathcal{B}(E, [0, \infty])$ implies that $\overline{V}_t f_n \uparrow \overline{V}_t f$ pointwisely. Moreover, $(\overline{V}_t)_{t\geq 0}$ satisfies that

$$\overline{V}_t f \leq \overline{V}_t g \text{ for } t \geq 0 \text{ and } f \leq g \text{ in } \mathcal{B}(E, [0, \infty]); \tag{2.1}$$

$$\overline{V}_{t+s} = \overline{V}_t\overline{V}_s \text{ for } t, s \geq 0; \text{ and} \tag{2.2}$$

$$\mathbb{P}_\mu[e^{-X_t(f)}] = e^{-\mu(\overline{V}_t f)} \text{ for } t \geq 0,\ \mu \in \mathcal{M}_f(E), \text{ and } f \in \mathcal{B}(E, [0, \infty]). \tag{2.3}$$

With some abuse of notation, we still write $V_t = \overline{V}_t$ for $t \geq 0$, and call $(V_t)_{t\geq 0}$ *the extended cumulant semigroup* of the superprocess X. Define $v_t = V_t(\infty\mathbf{1}_E)$ for $t \geq 0$, then it holds that

$$\mathbb{P}_\mu(\|X_t\| = 0) = e^{-\mu(v_t)}, \quad \mu \in \mathcal{M}_f(E), t \geq 0. \tag{2.4}$$

From this, we can verify that

$$\mu(v_t) > 0 \text{ for all } \mu \in \mathcal{M}_f^o(E) \text{ and } t \geq 0. \tag{2.5}$$

In fact, if $\mu(v_t) = 0$, then by (2.4) we have $\mathbb{P}_\mu(\|X_t\| = 0) = 1$, which contradicts (1.3).

In the proof of Theorem 1.1, we will use the following four propositions whose proofs are postponed to Sections 3.1–3.4 respectively.

Proposition 2.1. *For any $f \in \mathcal{B}(E, [0, \infty])$, $t > T$ and $x \in E$, we have $V_t f(x) = C^3_{t,x,f}\phi(x)$ for some non-negative $C^3_{t,x,f}$ with $\lim_{t\to\infty} \sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])} C^3_{t,x,f} = 0$. In particular, we have $\lim_{t\to\infty} \mu(V_t f) = 0$ for all $\mu \in \mathcal{M}_f(E)$ and $f \in \mathcal{B}(E, [0, \infty])$.*

Proposition 2.2. *For any $f \in \mathcal{B}(E, [0, \infty])$, $t > T$ and $x \in E$, we have $V_t f(x) = \phi(x)\nu(V_t f)(1 + C^4_{t,x,f})$ for some real $C^4_{t,x,f}$ with $\lim_{t\to\infty} \sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])} |C^4_{t,x,f}| = 0$.*

For a probability measure $\mathbf{P}$ on $\mathcal{M}_f(E)$, the log-Laplace functional of $\mathbf{P}$ is defined by

$$\mathscr{L}_{\mathbf{P}} f := -\log \int_{\mathcal{M}_f(E)} e^{-\mu(f)} \mathbf{P}(d\mu), \quad f \in \mathcal{B}(E, [0, \infty]).$$

For a finite random measure $\{Y; \mathbf{P}\}$, the log-Laplace functional of its distribution is denoted as $\mathscr{L}_{Y;\mathbf{P}}$. To simplify our notation, for each $t \geq 0$, we write $\Gamma_t := \mathscr{L}_{X_t;\mathbb{P}_\nu(\cdot|\|X_t\|>0)}$.

We say a $[0, \infty]$-valued functional A defined on $\mathcal{B}(E, [0, \infty])$ is monotone concave if (1) A is a monotone functional, i.e., $f \leq g$ in $\mathcal{B}(E, [0, \infty])$ implies $Af \leq Ag$; and (2) for any $f \in \mathcal{B}(E, [0, \infty])$ with $Af < \infty$, the function $u \mapsto A(uf)$ is concave on $[0, 1]$.

Proposition 2.3. *The limit $Gf := \lim_{t\to\infty} \Gamma_t f$ exists in $[0, \infty]$ for each $f \in \mathcal{B}(E, [0, \infty])$. Moreover, G is the unique $[0, \infty]$-valued monotone concave functional on $\mathcal{B}(E, [0, \infty])$ such that $G(\infty \mathbf{1}_E) = \infty$ and that*

$$1 - e^{-GV_s f} = e^{s\lambda}(1 - e^{-Gf}), \quad s \geq 0, f \in \mathcal{B}(E, [0, \infty]). \tag{2.6}$$

Proposition 2.4. *For any $g \in \mathcal{B}_b(E, [0, \infty))$ and sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{B}_b(E, [0, \infty))$ such that $g_n \downarrow g$ pointwisely, we have $Gg_n \downarrow Gg$.*

Proof of Theorem 1.1. It follows from Lemma A.4, Propositions 2.3 and 2.4 that there exists a unique probability measure $\mathbf{Q}_\lambda$ on $\mathcal{M}_f(E)$ such that

$$\mathbb{P}_\nu(X_t \in \cdot | \|X_t\| > 0) \xrightarrow[t\to\infty]{w} \mathbf{Q}_\lambda(\cdot) \tag{2.7}$$

and that

$$\mathscr{L}_{\mathbf{Q}_\lambda} = G \quad \text{on } \mathcal{B}_b(E, [0, \infty)). \tag{2.8}$$

We claim that (2.8) can be strengthened as

$$\mathscr{L}_{\mathbf{Q}_\lambda} = G \quad \text{on } \mathcal{B}(E, [0, \infty]); \tag{2.9}$$

and as a consequence of this, $\mathscr{L}_{\mathbf{Q}_\lambda}(\infty \mathbf{1}_E) = G(\infty \mathbf{1}_E) = \infty$, which says that $\mathbf{Q}_\lambda$ is actually a probability measure on $\mathcal{M}_f^o(E)$. To see the claim is true, we first note from Proposition 2.1 that

$$\begin{aligned}&\text{there exists } T_1 > 0 \text{ such that, for all } t > T_1 \text{ and } f \in \mathcal{B}(E, [0, \infty]),\\ &V_t f \in \mathcal{B}_b(E, [0, \infty)).\end{aligned} \tag{2.10}$$

We then notice that from (2.8) and the bounded convergence theorem,

$$\text{if } \{g_n : n \in \mathbb{N}\} \cup \{g\} \subset \mathcal{B}_b(E, [0, \infty)) \text{ and } g_n \uparrow g \text{ pointwisely, then } Gg_n \uparrow Gg. \tag{2.11}$$

Now let $\{g_n : n \in \mathbb{N}\} \cup \{g\} \subset \mathcal{B}(E, [0, \infty])$ and $g_n \uparrow g$ pointwisely. Taking and fixing an $s > T_1$, we have by (2.10) and (2.11) that

$$(1 - e^{-Gg_n}) \overset{(2.6)}{=} e^{-s\lambda}(1 - e^{-GV_s g_n}) \uparrow e^{-s\lambda}(1 - e^{-GV_s g}) \overset{(2.6)}{=} (1 - e^{-Gg}).$$

In other words, we showed that $Gg_n \uparrow Gg$. The desired claim follows from this and (2.8).

Let us now prove that the probability $\mathbf{Q}_\lambda$ on $\mathcal{M}_f^o(E)$ satisfies the requirement for the desired result. It follows from Proposition 2.2 that there exists $T_2 > 0$ such that $\sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])} |C^4_{t,x,f}| < \infty$ for $t > T_2$. Thus for $f \in \mathcal{B}(E, [0, \infty])$, $t > T_2$ and $\mu \in \mathcal{M}_f^o(E)$, we have

$$\begin{aligned}\mu(V_t f) &\overset{\text{Proposition 2.2}}{=} \int_E \phi(x)\nu(V_t f)(1 + C^4_{t,x,f})\mu(dx)\\ &= \nu(V_t f)\mu(\phi)(1 + C^5_{\mu,t,f})\end{aligned} \tag{2.12}$$

for some real $C^5_{\mu,t,f}$ with $\lim_{t\to\infty} \sup_{f\in\mathcal{B}(E,[0,\infty])} |C^5_{\mu,t,f}| = 0$. Also note that for $f \in \mathcal{B}(E, [0, \infty])$, $t > T_2$ and $\mu \in \mathcal{M}_f^o(E)$,

$$\begin{aligned}&\mathbb{P}_\mu\left[1 - e^{-X_t(f)} \middle| \|X_t\| > 0\right] \overset{(2.3),\,(2.4)}{=} \frac{1 - e^{-\mu(V_t f)}}{1 - e^{-\mu(v_t)}}\\ &= \frac{\mu(V_t f)}{\mu(v_t)}(1 + C^6_{\mu,t,f})\end{aligned} \tag{2.13}$$

for some real $C^6_{\mu,t,f}$ with $\lim_{t\to\infty} |C^6_{\mu,t,f}| = 0$. Here in the last equality we used (2.5), Proposition 2.1 and the fact that $(1 - e^{-x})/x \xrightarrow[x\to 0]{} 1$. Thus, for each $\mu \in \mathcal{M}_f^o(E)$ and $f \in C_b(E, [0, \infty))$, we have

$$\begin{aligned}&\mathbb{P}_\mu\left[1 - e^{-X_t(f)} \middle| \|X_t\| > 0\right] \overset{(2.12),\,(2.13)}{=} \frac{\nu(V_t f)}{\nu(v_t)} \frac{1 + C^5_{\mu,t,f}}{1 + C^5_{\mu,t,\infty\mathbf{1}_E}}(1 + C^6_{\mu,t,f})\\ &\overset{(2.13)}{=} \mathbb{P}_\nu\left[1 - e^{-X_t(f)} \middle| \|X_t\| > 0\right](1 + C^6_{\nu,t,f})^{-1} \frac{1 + C^5_{\mu,t,f}}{1 + C^5_{\mu,t,\infty\mathbf{1}_E}}(1 + C^6_{\mu,t,f})\\ &\xrightarrow[t\to\infty]{} \int_{\mathcal{M}_f(E)} (1 - e^{-w(f)})\mathbf{Q}_\lambda(dw),\end{aligned}$$

where in the last line above, we used (2.7). Therefore, according to [21, Theorem 1.18],

$$\mathbb{P}_\mu\left(X_t \in \cdot \middle| \|X_t\| > 0\right) \xrightarrow[t\to\infty]{w} \mathbf{Q}_\lambda(\cdot). \quad \square$$

2.2. *Proof of Theorem 1.2*

In this subsection, we give the proof of Theorem 1.2 using the following three Propositions 2.5–2.7 whose proofs are postponed to Sections 4.1–4.3, respectively.

Proposition 2.5. *(1) The Yaglom limit $\mathbf{Q}_\lambda$ given by Theorem 1.1 is a QSD of X with mass decay rate λ; and (2) for any $r \in (\lambda, 0)$, there exists a probability measure $\mathbf{Q}_r$ on $\mathcal{M}_f^o(E)$ such that $\mathbf{Q}_r$ is a QSD of X with mass decay rate r.*

Proposition 2.6. *Suppose that $r \in (-\infty, 0)$ and that $\mathbf{Q}_r^*$ is a QSD for X with mass decay rate r. Then we have that (1) $r \geq \lambda$; and (2) $\mathscr{L}_{\mathbf{Q}_r^*}$ is a monotone concave functional on $\mathcal{B}(E, [0, \infty])$ with $\mathscr{L}_{\mathbf{Q}_r^*}(\infty \mathbf{1}_E) = \infty$ and that*

$$1 - e^{-\mathscr{L}_{\mathbf{Q}_r^*} V_s f} = e^{sr}(1 - e^{-\mathscr{L}_{\mathbf{Q}_r^*} f}), \quad s \geq 0,\ f \in \mathcal{B}(E, [0, \infty]).$$

Proposition 2.7. *Let G be the unique functional on $\mathcal{B}(E, [0, \infty])$ given by Proposition 2.3. Let $r \in [\lambda, 0)$. If G_r is a monotone concave functional on $\mathcal{B}(E, [0, \infty])$ with $G_r(\infty \mathbf{1}_E) = \infty$ and that*

$$1 - e^{-G_r V_s f} = e^{sr}(1 - e^{-G_r f}), \quad s \geq 0,\ f \in \mathcal{B}(E, [0, \infty]),$$

then $1 - e^{-G_r f} = (1 - e^{-Gf})^{r/\lambda}$ for any $f \in \mathcal{B}(E, [0, \infty])$.

Proof of Theorem 1.2. The non-existence of QSD for X with mass decay rate $r < \lambda$ is due to Proposition 2.6(1). The existence of QSD for X with mass decay rate $r \in [\lambda, 0)$ is due to Proposition 2.5. The uniqueness of QSD for X with mass decay rate $r \in [\lambda, 0)$ is due to Propositions 2.6, 2.7 and [21, Theorem 1.17]. □

3. Proofs of Propositions 2.1–2.4

3.1. *Proof of Proposition 2.1*

Define a function ψ_0 by

$$\psi_0(x, z) = \psi(x, z) + \beta(x) z, \quad x \in E, z \in [0, \infty),$$

and an operator $\Psi_0 : \mathcal{B}(E, [0, \infty]) \to \mathcal{B}(E, [0, \infty])$ by

$$\Psi_0 f(x) = \lim_{n \to \infty} \psi_0(x, f(x) \wedge n), \quad f \in \mathcal{B}(E, [0, \infty]), x \in E.$$

Then it follows from [21, Theorem 2.23] and monotonicity that

$$V_s f + \int_0^s P_{s-u}^{\beta} \Psi_0 V_u f \, du = P_s^{\beta} f, \quad f \in \mathcal{B}(E, [0, \infty]), s \geq 0. \tag{3.1}$$

The following fact will be used repeatedly:

$$\{V_t f : t > T,\ f \in \mathcal{B}(E, [0, \infty])\} \subset L_1^+(\nu). \tag{3.2}$$

To see this, note from (2.1), (2.4) and (H2) that, for all $t > T$ and $f \in \mathcal{B}(E, [0, \infty])$, $\nu(V_t f) \leq \nu(v_t) = -\log \mathbb{P}_\nu(\|X_t\| = 0) < \infty$.

Proof of Proposition 2.1. Note that for all $s > 0$ and $\epsilon > 0$,

$$\begin{aligned} V_{s+\epsilon+T} f(x) &\overset{(2.2)}{=} V_s V_{T+\epsilon} f(x) \leq P_s^{\beta} V_{T+\epsilon} f(x) \quad \text{by (3.1)}, \\ &\overset{(H1),\ (3.2)}{=} e^{\lambda s} \phi(x) \nu(V_{T+\epsilon} f)(1 + H_{s,x,V_{T+\epsilon} f}) \\ &\leq e^{\lambda s} \phi(x) \nu(v_{T+\epsilon})(1 + \sup_{x \in E, g \in L_1^+(\nu)} |H_{s,x,g}|), \end{aligned} \tag{3.3}$$

where in the last inequality we used the fact that $\nu(V_t f) \leq \nu(v_t) = -\log \mathbb{P}_\nu(\|X_t\| = 0) < \infty$ for all $f \in \mathcal{B}(E, [0, \infty])$ and $t > T$. From this and the fact that $\lambda < 0$, we immediately get the desired result. □

3.2. Proof of Proposition 2.2

Another fact that will be used repeatedly is the following:

$$\begin{aligned}&\text{For any } f \in \mathcal{B}(E,[0,\infty]),\ \nu(f)=0 \text{ implies } \nu(V_t f)=0 \text{ for all } t\geq 0;\\ &\text{and } \nu(f)>0 \text{ implies } \nu(V_t f)>0 \text{ for all } t\geq 0.\end{aligned} \tag{3.4}$$

To see this, note by (1.2) that $\mathbb{P}_\nu[X_t(f)] = \nu(P_t^\beta f) = e^{\lambda t}\nu(f)$. If $\nu(f)=0$, then $X_t(f)=0$, $\mathbb{P}_\nu$-a.s., therefore $\nu(V_t f) = -\log \mathbb{P}_\nu[e^{-X_t(f)}]=0$. If $\nu(f)>0$, then under $\mathbb{P}_\nu$, $X_t(f)$ is a random variable with positive mean. Therefore, $\nu(V_t f) = -\log \mathbb{P}_\nu[e^{-X_t(f)}]>0$.

Combining (3.4) with (3.3) we get that

$$\text{for all } t>T,\ x\in E \text{ and } f\in\mathcal{B}(E,[0,\infty]) \text{ with } \nu(f)=0, \text{ we have } V_t f(x)=0. \tag{3.5}$$

Note from (H1) and (3.2) that for all $s>0, t>T, x\in E$ and $f\in\mathcal{B}(E,[0,\infty])$, we have

$$P_s^\beta V_t f(x) = e^{\lambda s}\phi(x)\nu(V_t f)(1+H_{s,x,V_t f})<\infty. \tag{3.6}$$

In the proof of Proposition 2.2 we will use the following three lemmas whose proofs are postponed later.

Lemma 3.1. *For all $s>0,\ t>T,\ x\in E$ and $f\in\mathcal{B}(E,[0,\infty])$, we have $P_s^\beta V_t f(x) = \phi(x)\nu(V_{t+s}f)(1+C^7_{s,t,x,f})$ for some real $C^7_{s,t,x,f}$ with*

$$\lim_{s\to\infty}\overline{\lim_{t\to\infty}}\sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])}|C^7_{s,t,x,f}|=0.$$

For $f\in\mathcal{B}(E,[0,\infty])$ and $0<\epsilon<s<\infty$, we define

$$I_{s,\epsilon}f = \int_0^{s-\epsilon} P^\beta_{s-u}\Psi_0 V_u f\, du, \qquad J_{s,\epsilon}f=\int_{s-\epsilon}^s P^\beta_{s-u}\Psi_0 V_u f\, du.$$

Lemma 3.2. *For all $t>T,\ 0<\epsilon<s<\infty,\ x\in E$ and $f\in\mathcal{B}(E,[0,\infty])$ with $\nu(f)>0$, we have $I_{s,\epsilon}V_t f(x)=\phi(x)\nu(V_{s+t}f)C^8_{t,\epsilon,s,x,f}$ for some non-negative $C^8_{t,\epsilon,s,x,f}$ with*

$$\lim_{t\to\infty}\sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])}C^8_{t,\epsilon,s,x,f}=0.$$

Lemma 3.3. *For all $t>T,\ 0<\epsilon<s<\infty,\ x\in E$ and $f\in\mathcal{B}(E,[0,\infty])$ with $\nu(f)>0$, we have $J_{s,\epsilon}V_t f(x)=\phi(x)\nu(V_{s+t}f)C^9_{t,\epsilon,s,x,f}$ for some non-negative $C^9_{t,\epsilon,s,x,f}$ with*

$$\lim_{\epsilon\to 0}\overline{\lim_{t+s\to\infty}}\sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])}C^9_{t,\epsilon,s,x,f}=0.$$

Proof of Proposition 2.2. Thanks to (3.4) and (3.5), we only need to consider the case that $\nu(f)>0$. In this case, by Lemmas 3.1–3.3. we have for any $s>0$ and $\epsilon\in(0,s)$,

$$\begin{aligned}V_{t+s}f(x) &\overset{(2.2)}{=} V_sV_tf(x) \overset{(3.1),(3.6)}{=} P_s^\beta V_t f(x) - \int_0^s P^\beta_{s-u}\Psi_0 V_u V_t f(x)du\\ &= P_s^\beta V_t f(x) - I_{s,\epsilon}V_t f(x) - J_{s,\epsilon}V_t f(x)\\ &= \phi(x)\nu(V_{t+s}f)\left(1+C^7_{s,t,x,f}-C^8_{t,\epsilon,s,x,f}-C^9_{t,\epsilon,s,x,f}\right).\end{aligned} \tag{3.7}$$

On the other hand, we have

$$V_t f(x) = \phi(x)\nu(V_t f)(1 + C^{10}_{t,x,f}) \quad \text{for some real } C^{10}_{t,x,f}. \tag{3.8}$$

Combining (3.7) and (3.8), we have for all $s > 0$ and $\epsilon \in (0, s)$,

$$C^{10}_{t+s,x,f} = C^7_{s,t,x,f} - C^8_{t,\epsilon,s,x,f} - C^9_{t,\epsilon,s,x,f}.$$

Using this and the fact that

$$\lim_{\epsilon\to 0} \overline{\lim_{s\to\infty}}\ \overline{\lim_{t\to\infty}} \sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])} |C^7_{s,t,x,f} - C^8_{t,\epsilon,s,x,f} - C^9_{t,\epsilon,s,x,f}| = 0,$$

it is easy to check that $\overline{\lim}_{s\to\infty} \overline{\lim}_{t\to\infty} \sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])} |C^{10}_{t+s,x,f}| = 0$. This implies

$$\lim_{t\to\infty} \sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])} |C^{10}_{t,x,f}| = 0. \quad \square$$

Now we prove the three lemmas above.

Proof of Lemma 3.1. Integrating both sides of (3.1) with respect to ν and replacing f by $V_t f$, we get that for all $t, s \geq 0$ and $f \in \mathcal{B}(E, [0, \infty])$,

$$e^{-\lambda(t+s)}\nu(V_{t+s} f) + \int_0^s e^{-\lambda(t+u)}\nu(\Psi_0 V_{t+u} f)du = e^{-\lambda t}\nu(V_t f). \tag{3.9}$$

As a consequence of (3.9), we can get that for all $t > T$, $s \geq 0$ and $f \in \mathcal{B}(E, [0, \infty])$ with $\nu(f) > 0$,

$$\frac{\nu(V_{t+s} f)}{\nu(V_t f)} = \exp\left\{\lambda s - \int_t^{t+s} \frac{\nu(\Psi_0 V_u f)}{\nu(V_u f)}\, du\right\}. \tag{3.10}$$

In fact, first observe from (3.2) and (3.4) that both sides of (3.9) are finite and positive if $t > T$ and $\nu(f) > 0$. Therefore the function $H : u \mapsto e^{-\lambda u}\nu(V_u f)$ is absolutely continuous on (T, ∞) and

$$dH(u) = -e^{-\lambda u}\nu(\Psi_0 V_u f)du, \quad u \in (T, \infty),$$

which implies that

$$d \log H(u) = -\frac{\nu(\Psi_0 V_u f)}{\nu(V_u f)}du, \quad u \in (T, \infty).$$

Now an elementary integration argument gives (3.10).

Define an operator Ψ_0' on $\mathcal{B}(E, [0, \infty])$ by

$$\Psi_0' f(x) = \lim_{n\to\infty} \frac{\partial\psi_0}{\partial z}(x, n \wedge f(x)), \quad x \in E, f \in \mathcal{B}(E, [0, \infty]).$$

We first claim that for all $t > T, x \in E$ and $f \in \mathcal{B}(E, [0, \infty])$,

$$\overline{\lim_{t\to\infty}} \sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])} \Psi_0' V_t f(x) < \infty. \tag{3.11}$$

In fact, since

$$\frac{\partial\psi_0}{\partial z}(x, z) = 2\sigma(x)^2 z + \int_0^\infty (1 - e^{-rz})r\pi(x, dr), \quad x \in E, z \geq 0, \tag{3.12}$$

we have,

$$\begin{aligned}\Psi_0' V_t f(x) &\le 2\sigma(x)^2 V_t f(x) + V_t f(x) \int_0^1 r^2 \pi(x, dr) + \int_1^\infty r\pi(x, dr) \\ &\overset{\text{Proposition 2.1}}{=} C^3_{t,x,f}\phi(x)\left(2\sigma(x)^2 + \int_0^1 r^2\pi(x, dr)\right) + \int_1^\infty r\pi(x, dr).\end{aligned}$$

Since ϕ, σ are bounded, and $(r \wedge r^2)\pi(x, du)$ is a bounded kernel, (3.11) follows easily.

We next claim that for all $t > T$ and $f \in \mathcal{B}(E, [0, \infty])$,

$$\lim_{t\to\infty} \sup_{f\in\mathcal{B}(E,[0,\infty])} \nu(\Psi_0' V_t f) = 0. \tag{3.13}$$

In fact, it follows from (3.12) that, for any fixed $x \in E$, $z \mapsto \frac{\partial\psi_0}{\partial z}(x, z)$ is a non-negative, non-decreasing and continuous function on $[0, \infty)$ with $\frac{\partial\psi_0}{\partial z}(\cdot, 0) \equiv 0$. Therefore for any $x \in E$, we have

$$\lim_{t\to\infty} \Psi_0' v_t(x) = \lim_{t\to\infty} \frac{\partial\psi_0}{\partial z}(x, v_t(x)) \overset{\text{Proposition 2.1}}{=} 0.$$

Using this, (3.11) and the bounded convergence theorem, we easily get $\lim_{t\to\infty} \nu(\Psi_0' v_t) = 0$. The claim follows immediately from the monotonicity of $\Psi_0' V_t f$ in $f \in \mathcal{B}(E, [0, \infty])$.

Here is another claim that will be used below:

For all $t > T, x \in E$ and $f \in \mathcal{B}(E, [0, \infty])$, it holds that

$$V_t f(x) = \phi(x)\nu(V_t f)C^{11}_{t,x,f} \tag{3.14}$$

for some non-negative $C^{11}_{t,x,f}$ with $\overline{\lim}_{t\to\infty} \sup_{x\in E,\, f\in\mathcal{B}(E,[0,\infty])} C^{11}_{t,x,f} < \infty$.

To see this, first note that (3.14) is trivial when $\nu(f) = 0$ thanks to (3.4) and (3.5). Therefore, we only need to consider the case that $\nu(f) > 0$. In this case, it follows from the elementary fact

$$\psi_0(x, z) \le z\frac{\partial\psi_0}{\partial z}(x, z), \quad x \in E, z \ge 0, \tag{3.15}$$

that

$$\nu(\Psi_0 V_t f) \le \nu((V_t f) \cdot (\Psi_0' V_t f)) \le \nu(V_t f) \sup_{y\in E} \Psi_0' V_t f(y).$$

From (3.2) we get that $\nu(V_t f) < \infty$. Thus from (3.11) for $t > T$ and $f \in \mathcal{B}(E, [0, \infty])$,

$$\nu(\Psi_0 V_t f) = \nu(V_t f)C^{12}_{t,f} \tag{3.16}$$

for some non-negative $C^{12}_{t,f}$ with $\overline{\lim}_{t\to\infty} \sup_{f\in\mathcal{B}(E,[0,\infty])} C^{12}_{t,f} < \infty$. Therefore, for any $s \ge 0$,

$$\begin{aligned}\frac{\nu(V_{t+s} f)}{\nu(V_t f)} &\overset{(3.10)}{=} \exp\left\{\lambda s - \int_t^{t+s} \frac{\nu(\Psi_0 V_u f)}{\nu(V_u f)} du\right\} \\ &\overset{(3.16)}{=} \exp\left\{\lambda s - \int_t^{t+s} C^{12}_{u,f}\, du\right\}.\end{aligned} \tag{3.17}$$

Now note that for any $\epsilon \in (0, t-T)$,

$$
\begin{aligned}
V_t f(x) &\overset{(2.1)}{=} V_\epsilon V_{t-\epsilon} f \le P^\beta_\epsilon V_{t-\epsilon} f(x) \quad \text{by (3.1)},\\
&\overset{(H1)}{=} \phi(x)\nu(V_{t-\epsilon} f)e^{\lambda\epsilon}(1 + H_{\epsilon,x,V_{t-\epsilon} f})\\
&\overset{(3.17)}{=} \phi(x)\nu(V_t f)\exp\left\{\int_{t-\epsilon}^t C^{12}_{u,f}\, du\right\}(1 + H_{\epsilon,x,V_{t-\epsilon} f}).
\end{aligned}
\tag{3.18}
$$

According to (3.2) and (H1) we have

$$
\overline{\lim_{t\to\infty}} \sup_{x\in E, f\in\mathcal{B}(E,[0,\infty])} |H_{\epsilon,x,V_{t-\epsilon} f}| < \infty, \quad \epsilon > 0.
$$

From this, (3.18) and the fact that $\overline{\lim}_{u\to\infty} \sup_{f\in\mathcal{B}(E,[0,\infty])} C^{12}_{u,f} < \infty$, (3.14) follows immediately.

We now use (3.11), (3.13) and (3.14) to give the asymptotic ratio of $\nu(\Psi_0 V_t f)$ and $\nu(V_t f)$. Note that we already obtained some result for this ratio in (3.16). We claim that the following stronger assertion is valid:

$$
\lim_{t\to\infty} \sup_{f\in\mathcal{B}(E,[0,\infty])} C^{12}_{t,f} = 0, \quad f \in \mathcal{B}(E, [0,\infty]).
\tag{3.19}
$$

To see this, we observe that

$$
\begin{aligned}
&\nu(\Psi_0 V_t f) \le \nu((V_t f)\cdot(\Psi_0' V_t f)), \quad \text{by (3.15)},\\
&\le \nu(\Psi_0' V_t f)\sup_{x\in E} V_t f(x) \overset{(3.14)}{=} \nu(\Psi_0' V_t f)\cdot \nu(V_t f)\sup_{x\in E}(\phi(x)C^{11}_{t,x,f}).
\end{aligned}
$$

Since ϕ is bounded, (3.19) follows from (3.13) and (3.14).

Using (3.19), we can get the following asymptotic ratio of $\nu(V_{t+s} f)$ and $\nu(V_t f)$:

For all $t > T$, $s \ge 0$ and $f \in \mathcal{B}(E, [0,\infty])$, we have

$$\nu(V_{t+s} f) = \nu(V_t f)\exp\{\lambda s(1 + C^{13}_{t,s,f})\}$$

for some real $C^{13}_{t,s,f}$ with $\lim_{t\to\infty} \sup_{s\ge 0, f\in\mathcal{B}(E,[0,\infty])} |C^{13}_{t,s,f}| = 0$. In particular, for all $f \in \mathcal{B}(E, [0,\infty])$ with $\nu(f) > 0$ and $s \ge 0$, we have $\displaystyle\lim_{t\to\infty} \frac{\nu(V_{t+s} f)}{\nu(V_t f)} = e^{\lambda s}$. (3.20)

To see this, thanks to (3.4), we only need to consider the case $\nu(f) > 0$. In this case, it holds that

$$
\frac{\nu(V_{t+s} f)}{\nu(V_t f)} \overset{(3.17)}{=} \exp\left\{\lambda s - \int_t^{t+s} C^{12}_{u,f}\, du\right\} =: \exp\{\lambda s(1 + C^{13}_{t,s,f})\}.
$$

Noticing that $C^{13}_{t,s,f} = -\frac{1}{\lambda s}\int_t^{t+s} C^{12}_{u,f}\, du$ and by (3.19) that $\lim_{u\to\infty} \sup_{f\in\mathcal{B}(E,[0,\infty])} C^{12}_{u,f} = 0$, so we have $\lim_{t\to\infty} \sup_{s>0, f\in\mathcal{B}(E,[0,\infty])} |C^{13}_{t,s,f}| = 0$.

We are now ready to prove the conclusion of Lemma 3.1. Again we only need to consider the case $\nu(f) > 0$ thanks to (3.4) and (3.5). In this case, by (3.2) and (3.4), we have $0 < \nu(V_t f) < \infty$. Therefore, we have

$$
\begin{aligned}
P^\beta_s V_t f(x) &\overset{(H1)}{=} e^{\lambda s}\phi(x)\nu(V_t f)(1 + H_{s,x,V_t f})\\
&\overset{(3.20)}{=} \phi(x)\nu(V_{t+s} f)\exp\{-\lambda s C^{13}_{t,s,f}\}(1 + H_{s,x,V_t f}).
\end{aligned}
$$

From (H1) and (3.2), we know that $\lim_{s\to\infty}\sup_{x\in E,t>T,f\in\mathcal{B}(E,[0,\infty])}|H_{s,x,V_t f}|=0$. From (3.20), we know that $\sup_{s\geq 0}\lim_{t\to\infty}\sup_{f\in\mathcal{B}(E,[0,\infty])}|sC^{13}_{t,s,f}|=0$. Therefore, we have

$$\lim_{s\to\infty}\overline{\lim_{t\to\infty}}\sup_{x\in E,f\in\mathcal{B}(E,[0,\infty])}|\exp\{-\lambda sC^{13}_{t,s,f}\}(1+H_{s,x,V_t f})-1|=0.$$

Combining the two displays above we get the conclusion of Lemma 3.1. □

Proof of Lemma 3.2. For all $u\geq 0$, we have

$$\nu(P^\beta_u\Psi_0 V_t f)=e^{\lambda u}\nu(\Psi_0 V_t f)<\infty, \tag{3.21}$$

where the inequality follows from (3.16) and (3.2). Therefore, we have

$$\begin{aligned}
I_{s,\epsilon}V_t f(x)&=\int_0^{s-\epsilon}P^\beta_{s-u}\Psi_0 V_{t+u}f(x)du=\int_0^{s-\epsilon}P^\beta_\epsilon(P^\beta_{s-\epsilon-u}\Psi_0 V_{t+u}f)(x)du\\
&\overset{\text{(H1)}}{=}\int_0^{s-\epsilon}e^{\lambda\epsilon}\phi(x)\nu(P^\beta_{s-\epsilon-u}\Psi_0 V_{t+u}f)\left(1+H_{\epsilon,x,P^\beta_{s-\epsilon-u}\Psi_0 V_{t+u}f}\right)du\\
&\overset{(3.21)}{=}e^{(t+s)\lambda}\int_0^{s-\epsilon}\phi(x)e^{-\lambda(t+u)}\nu(\Psi_0 V_{t+u}f)\left(1+H_{\epsilon,x,P^\beta_{s-\epsilon-u}\Psi_0 V_{t+u}f}\right)du\\
&\leq\phi(x)\Big(1+\sup_{g\in L_1^+(\nu)}|H_{\epsilon,x,g}|\Big)e^{(t+s)\lambda}\int_0^s e^{-\lambda(t+u)}\nu(\Psi_0 V_{t+u}f)du\quad\text{by (3.21)}\\
&\overset{(3.9)}{=}\phi(x)\left(1+\sup_{g\in L_1^+(\nu)}|H_{\epsilon,x,g}|\right)e^{(t+s)\lambda}\left(e^{-\lambda t}\nu(V_t f)-e^{-\lambda(t+s)}\nu(V_{t+s}f)\right)\\
&\overset{(3.2),\,(3.4)}{=}\phi(x)\Big(1+\sup_{g\in L_1^+(\nu)}|H_{\epsilon,x,g}|\Big)\nu(V_{t+s}f)\Big(\frac{e^{s\lambda}\nu(V_t f)}{\nu(V_{t+s}f)}-1\Big)\\
&\overset{(3.20)}{=}\phi(x)\Big(1+\sup_{g\in L_1^+(\nu)}|H_{\epsilon,x,g}|\Big)\nu(V_{t+s}f)(\exp\{-\lambda sC^{13}_{t,s,f}\}-1).
\end{aligned}$$

It is easy to check that

$$\lim_{t\to\infty}\sup_{x\in E,f\in\mathcal{B}(E,[0,\infty])}\left|\Big(1+\sup_{g\in L_1^+(\nu)}|H_{\epsilon,x,g}|\Big)(\exp\{-\lambda sC^{13}_{t,s,f}\}-1)\right|=0.$$

The desired result then follows. □

Proof of Lemma 3.3. It follows from (3.15) that for all $t>T$, $x\in E$ and $f\in\mathcal{B}(E,[0,\infty])$,

$$\Psi_0 V_t f(x)\leq V_t f(x)\cdot\Psi_0' V_t f(x).$$

Now by (3.11) we have

$$\Psi_0 V_t f(x)=V_t f(x)C^{14}_{t,x,f} \tag{3.22}$$

for some non-negative $C^{14}_{t,x,f}$ with $\overline{\lim}_{t\to\infty}\sup_{x\in E,f\in\mathcal{B}(E,[0,\infty])}C^{14}_{t,x,f}<\infty$.

Recall the quantity $C^{13}_{t,s,f}$ given in (3.20). Now we claim that for all $u\geq 0$, $t>T$, $x\in E$ and $f\in\mathcal{B}(E,[0,\infty])$,

$$P^\beta_u\Psi_0 V_t f(x)=\phi(x)\nu(V_{t+u}f)\exp\{-\lambda uC^{13}_{t,u,f}\}C^{15}_{t,u,x,f} \tag{3.23}$$

for some non-negative $C^{15}_{t,u,x,f}$ with $\overline{\lim}_{t\to\infty}\sup_{u\geq 0,x\in E,f\in\mathcal{B}(E,[0,\infty])} C^{15}_{t,u,x,f} < \infty$.

$$
\begin{aligned}
&P^{\beta}_u \Psi_0 V_t f(x) = \int_E \Psi_0 V_t f(y) P^{\beta}_u(x,dy) \overset{(3.22)}{=} \int_E V_t f(y) C^{14}_{t,y,f} P^{\beta}_u(x,dy)\\
&\overset{(3.14)}{=} \int_E \phi(y)\nu(V_t f) C^{11}_{t,y,f} C^{14}_{t,y,f} P^{\beta}_u(x,dy)\\
&\overset{(3.20)}{=} \int_E \phi(y)\nu(V_{t+u} f)\exp\{-\lambda u(1+C^{13}_{t,u,f})\} C^{11}_{t,y,f} C^{14}_{t,y,f} P^{\beta}_u(x,dy)\\
&\leq \nu(V_{t+u} f)\exp\{-\lambda u(1+C^{13}_{t,u,f})\}\Big(\sup_{z\in E} C^{11}_{t,z,f} C^{14}_{t,z,f}\Big)\int_E \phi(y) P^{\beta}_u(x,dy)\\
&= \nu(V_{t+u} f)\exp\{-\lambda u(1+C^{13}_{t,u,f})\}\Big(\sup_{z\in E} C^{11}_{t,z,f} C^{14}_{t,z,f}\Big) e^{\lambda u}\phi(x).
\end{aligned}
$$

Now (3.23) follows from the fact that $\overline{\lim}_{t\to\infty}\Big(\sup_{z\in E,f\in\mathcal{B}(E,[0,\infty])} C^{11}_{t,z,f} C^{14}_{t,z,f}\Big) < \infty$.

Note that (3.23) gives the asymptotic behavior of $P^{\beta}_u \Psi_0 V_t f(x)$. We want to reformulate it into the asymptotic behavior of $P^{\beta}_u \Psi_0 V_{t-u} f(x)$. To do this, we use the following elementary facts: for any real function h on $[0,\infty)^2$,

$$
\begin{aligned}
&\overline{\lim_{t\to\infty}}\sup_{u\geq 0}|h(t,u)| < \infty \implies \sup_{\epsilon>0}\overline{\lim_{t\to\infty}}\sup_{u\in(0,\epsilon)}|h(t-u,u)| < \infty; \qquad (3.24)\\
&\lim_{t\to\infty}\sup_{u\geq 0}|h(t,u)| = 0 \implies \sup_{\epsilon>0}\lim_{t\to\infty}\sup_{u\in(0,\epsilon)} u\cdot|h(t-u,u)| = 0.
\end{aligned}
$$

Observe that for all $u>0$, $t>T+u$ and $f\in\mathcal{B}(E,[0,\infty])$,

$$
P^{\beta}_u \Psi_0 V_{t-u} f(x) \overset{(3.23)}{=} \phi(x)\nu(V_t f)\exp\{-\lambda u C^{13}_{t-u,u,f}\} C^{15}_{t-u,u,x,f}.
$$

From (3.24), we know that

$$
\sup_{\epsilon>0}\overline{\lim_{t\to\infty}}\sup_{u\in(0,\epsilon),x\in E,f\in\mathcal{B}(E,[0,\infty])} C^{15}_{t-u,u,x,f} < \infty
$$

and that

$$
\sup_{\epsilon>0}\lim_{t\to\infty}\sup_{u\in(0,\epsilon),f\in\mathcal{B}(E,[0,\infty])} u C^{13}_{t-u,u,f} = 0.
$$

Thus,

$$
P^{\beta}_u \Psi_0 V_{t-u} f(x) = \phi(x)\nu(V_t f) C^{16}_{t,u,f,x} \qquad (3.25)
$$

for some non-negative $C^{16}_{t,u,f,x}$ with $\sup_{\epsilon>0}\overline{\lim}_{t\to\infty}\sup_{u\in(0,\epsilon),x\in E,f\in\mathcal{B}(E,[0,\infty])} C^{16}_{t,u,f,x} < \infty$.

Finally, we note that

$$
\begin{aligned}
&J_{s,\epsilon} V_t f(x) = \int_{s-\epsilon}^{s} P^{\beta}_{s-u}\Psi_0 V_{t+u} f(x)du = \int_0^{\epsilon} P^{\beta}_u \Psi_0 V_{t+s-u} f(x)du\\
&\overset{(3.25)}{=} \int_0^{\epsilon}\phi(x)\nu(V_{t+s} f) C^{16}_{t+s,u,f,x}\,du \leq \epsilon\phi(x)\nu(V_{t+s} f)\sup_{u\in(0,\epsilon)} C^{16}_{t+s,u,f,x}.
\end{aligned}
$$

It is elementary to see that

$$
\lim_{\epsilon\to 0}\overline{\lim_{t+s\to\infty}}\sup_{x\in E,f\in\mathcal{B}(E,[0,\infty])}\Big(\epsilon\sup_{u\in(0,\epsilon)} C^{16}_{t+s,u,f,x}\Big) = 0.
$$

Combining the two displays above, we get the conclusion of Lemma 3.3. □

3.3. Proof of Proposition 2.3

Recall that for each $t \geq 0$, $\Gamma_t := \mathscr{L}_{X_t;\mathbb{P}_\nu(\cdot|\|X_t\|>0)}$ is the log-Laplace functional for X_t under probability $\mathbb{P}_\nu(\cdot|\|X_t\| > 0)$. For any unbounded increasing positive sequence $\mathbf{t} = (t_n)_{n\in\mathbb{N}}$, define $G^{\mathbf{t}} f = \underline{\lim}_{n\to\infty} \Gamma_{(t_n)} f$.

To prove Proposition 2.3, we first prove two lemmas.

Lemma 3.4. *For any unbounded increasing positive sequence* $\mathbf{t} = (t_n)_{n\in\mathbb{N}}$*,* $G^{\mathbf{t}}$ *is a* $[0,\infty]$*-valued monotone concave functional on* $\mathcal{B}(E,[0,\infty])$ *such that* $G^{\mathbf{t}}(\infty\mathbf{1}_E) = \infty$ *and that*

$$1 - e^{-G^{\mathbf{t}} V_s f} = e^{s\lambda}(1 - e^{-G^{\mathbf{t}} f}), \quad s \geq 0,\ f \in \mathcal{B}(E,[0,\infty]).$$

Proof. Since $(\Gamma_t)_{t\geq 0}$ are $[0,\infty]$-valued functionals, so is $G^{\mathbf{t}}$. Also, from $\Gamma_t(\infty\mathbf{1}_E) = \infty$ for all $t \geq 0$ we have that $G^{\mathbf{t}}(\infty\mathbf{1}_E) = \infty$. We claim that $G^{\mathbf{t}}$ is monotone concave. In fact, for each $f \leq g$ in $\mathcal{B}(E,[0,\infty])$, we have

$$G^{\mathbf{t}} f = \varliminf_{n\to\infty} \Gamma_{(t_n)} f \leq \varliminf_{n\to\infty} \Gamma_{(t_n)} g = G^{\mathbf{t}} g.$$

On the other hand, using Lemma A.2, we have for all $t \geq 0$, $f \in \mathcal{B}(E,[0,\infty])$, $u, v \in [0,\infty)$, $r \in [0,1]$, it holds that

$$\Gamma_t((ru + (1-r)v)f) \geq r\Gamma_t(uf) + (1-r)\Gamma_t(vf).$$

Therefore, for all $f \in \mathcal{B}(E,[0,\infty])$, $u, v \in [0,\infty)$, $r \in [0,1]$, we have

$$\begin{aligned}
&G^{\mathbf{t}}((ru + (1-r)v)f) = \varliminf_{n\to\infty} \Gamma_{(t_n)}((ru + (1-r)v)f)\\
&\geq \varliminf_{n\to\infty} (r\Gamma_{(t_n)}(uf) + (1-r)\Gamma_{(t_n)}(vf))\\
&\geq r(\varliminf_{n\to\infty} \Gamma_{(t_n)}(uf)) + (1-r)(\varliminf_{n\to\infty} \Gamma_{(t_n)}(vf))\\
&= rG^{\mathbf{t}}(uf) + (1-r)G^{\mathbf{t}}(vf).
\end{aligned}$$

Note that for any $t > 0$ and $f \in \mathcal{B}(E,[0,\infty])$, it holds that

$$1 - e^{-\Gamma_t f} = \frac{\mathbb{P}_\nu[1 - e^{-X_t(f)}]}{\mathbb{P}_\nu(\|X_t\| > 0)} = \frac{1 - e^{-\nu(V_t f)}}{1 - e^{-\nu(v_t)}}. \tag{3.26}$$

Fix a function $f \in \mathcal{B}(E,[0,\infty])$. Thanks to (3.4) and (3.26), we only need to consider the case $\nu(f) > 0$. In this case, by (3.4), we have $\nu(V_t f) > 0$ for each $t \geq 0$. Therefore, for any $s, t \geq 0$,

$$\begin{aligned}
&1 - e^{-\Gamma_t V_s f} \overset{(3.26)}{=} \frac{1 - e^{-\nu(V_{t+s} f)}}{1 - e^{-\nu(v_t)}} = \frac{1 - e^{-\nu(V_{t+s} f)}}{1 - e^{-\nu(V_t f)}} \frac{1 - e^{-\nu(V_t f)}}{1 - e^{-\nu(v_t)}}\\
&\overset{(3.26)}{=} \frac{1 - e^{-\nu(V_{t+s} f)}}{1 - e^{-\nu(V_t f)}}(1 - e^{-\Gamma_t f}).
\end{aligned} \tag{3.27}$$

Thus, for any $s \geq 0$,

$$1 - e^{-G^{\mathbf{t}} V_s f} = \lim_{n\to\infty} (1 - e^{-\Gamma_{(t_n)} V_s f}) \overset{(3.27)}{=} \lim_{n\to\infty} \left(\frac{1 - e^{-\nu(V_{t_n+s} f)}}{1 - e^{-\nu(V_{(t_n)} f)}} (1 - e^{-\Gamma_{(t_n)} f}) \right)$$
$$= \left(\lim_{t\to\infty} \frac{1 - e^{-\nu(V_{t+s} f)}}{1 - e^{-\nu(V_t f)}} \right) \cdot \lim_{n\to\infty} (1 - e^{-\Gamma_{(t_n)} f}) = e^{s\lambda} (1 - e^{-G^{\mathbf{t}} f}),$$

where the last equality follows from Proposition 2.1, (3.20), and the fact that $(1 - e^{-x})/x \xrightarrow[x\to 0]{} 1$. $\square$

Lemma 3.5. *Suppose that $r \in [\lambda, 0)$. If G_r is a $[0, \infty]$-valued monotone concave functional on $\mathcal{B}(E, [0, \infty])$ such that $G_r(\infty \mathbf{1}_E) = \infty$ and that*

$$1 - e^{-G_r V_s f} = e^{sr} (1 - e^{-G_r f}), \quad s \geq 0,\ f \in \mathcal{B}(E, [0, \infty]),$$

then for any unbounded increasing positive sequence $\mathbf{t} = (t_n)_{n\in\mathbb{N}}$,

$$1 - e^{-G_r f} = (1 - e^{-G^{\mathbf{t}} f})^{r/\lambda}, \quad f \in \mathcal{B}(E, [0, \infty]).$$

Proof. Let $(Q_t)_{t\geq 0}$ be the family of $[0, \infty)$-valued functionals on $\mathcal{B}(E, [0, \infty])$ given by

$$Q_t g := e^{-rt} (1 - e^{-G_r(g v_t)}).$$

Note that, by (2.5), $v_t(x) > 0$ for all $x \in E$. It follows from Proposition 2.1 that $v_t(x) < \infty$ for all $x \in E$ and all $t > T$. Thus $v_t(\cdot)$ is a $(0, \infty)$-valued function for all $t > T$.

We claim that for any $u \in [0, 1]$, $Q_t(u \mathbf{1}_E)$ is non-increasing in $t \in (0, \infty)$. In particular, we can define the $[0, \infty]$-valued function $q(u) := \lim_{t\to\infty} Q_t(u \mathbf{1}_E)$, $u \in [0, 1]$. In fact, note that $\mathbb{P}_{\delta_x}[e^{-X_s(u v_t)}] = e^{-V_s(u v_t)}$, $x \in E$, $s, t > 0$, $u \geq 0$. Lemma A.2 says that, for all $s, t > 0$ and $x \in E$, $u \mapsto V_s(u v_t)(x)$ is a $[0, \infty]$-valued concave function on $[0, \infty)$. Therefore, for $u \in [0, 1]$, we have

$$V_s(u v_t) \geq u V_s(v_t) + (1 - u) V_s(0 \cdot v_t) = u v_{s+t}, \quad s, t > 0.$$

Using this, we get

$$Q_{t+s}(u \mathbf{1}_E) = e^{-r(t+s)} (1 - e^{-G_r(u v_{t+s})}) \leq e^{-r(t+s)} (1 - e^{-G_r[V_s(u v_t)]})$$
$$= e^{-rt} (1 - e^{-G_r(u v_t)}) = Q_t(u \mathbf{1}_E), \quad s, t > 0, u \in [0, 1].$$

We want to show that $q(u) = u^{r/\lambda}$, $u \in [0, 1]$. In order to do this, we first show that

the function q is non-decreasing and concave on $[0, 1]$ with $q(1) = 1$.
In particular, thanks to Lemma A.1, q is a continuous function on $(0, 1]$. (3.28)

In fact, from $G_r(\infty \mathbf{1}_E) = \infty$ and $V_t(\infty \mathbf{1}_E) = v_t$, we get

$$Q_t(\mathbf{1}_E) = e^{-rt} (1 - e^{-G_r v_t}) = e^{-rt} e^{rt} (1 - e^{-G_r(\infty \mathbf{1}_E)}) = 1, \quad t \geq 0.$$

Therefore $q(1) = 1$. The above argument also says that $G_r v_t < \infty$ for each $t > 0$. Now from the condition that G_r is monotone concave, we have that for all $t > 0$, the map $u \mapsto G_r(u v_t)$ is a non-decreasing and concave $[0, \infty)$-valued function on $[0, 1]$. From Lemma A.3 we get that, for each $t > 0$, $u \mapsto Q_t(u \mathbf{1}_E)$ is a $[0, \infty)$-valued, non-decreasing and concave function on $[0, 1]$. Since the limit of concave functions is concave, we get (3.28) by letting $t \to \infty$.

We now show that

$$q(u) = u^{r/\lambda}, \quad u \in [0, 1]. \tag{3.29}$$

To see this, note that for all $s \geq 0$, $t > T$ and $x \in E$, we have that

$$\begin{aligned}
&e^{\lambda s}(\phi^{-1}v_t)(x) \overset{\text{Proposition 2.2}}{=} e^{\lambda s}\nu(v_t)(1 + C^4_{t,x,\infty\mathbf{1}_E}) \\
&\overset{(3.20)}{=} \nu(v_{t+s})\exp\{-\lambda s C^{13}_{t,s,\infty\mathbf{1}_E}\}(1 + C^4_{t,x,\infty\mathbf{1}_E}) \\
&\overset{\text{Proposition 2.2}}{=} (\phi^{-1}v_{t+s})(x)(1 + C^4_{t+s,x,\infty\mathbf{1}_E})^{-1}\exp\{-\lambda s C^{13}_{t,s,\infty\mathbf{1}_E}\}(1 + C^4_{t,x,\infty\mathbf{1}_E}) \\
&= (\phi^{-1}v_{t+s})(x)(1 + C^{17}_{s,t,x}),
\end{aligned}$$

for some real $C^{17}_{s,t,x}$ with $\lim_{t\to\infty}\sup_{x\in E}|C^{17}_{s,t,x}| = 0$. Thus, we know that for all $s \geq 0$ and $\epsilon > 0$ there exists $T^1_{s,\epsilon} > 0$ such that

$$1 - \epsilon \leq \frac{e^{\lambda s}v_t(x)}{v_{t+s}(x)} \leq 1 + \epsilon, \quad x \in E, t > T^1_{s,\epsilon}. \tag{3.30}$$

From this we get that for all $s \geq 0, \epsilon > 0$, $t \geq T^1_{s,\epsilon}$, and $u \geq 0$,

$$\begin{aligned}
&Q_{t+s}[(1-\epsilon)u\mathbf{1}_E] = e^{-r(t+s)}(1 - e^{-G_r[(1-\epsilon)uv_{t+s}]}) \overset{(3.30)}{\leq} e^{-rt}e^{-rs}(1 - e^{-G_r(ue^{\lambda s}v_t)}) \\
&= e^{-rs}Q_t(ue^{\lambda s}\mathbf{1}_E) \overset{(3.30)}{\leq} e^{-r(t+s)}(1 - e^{-G_r[(1+\epsilon)uv_{t+s}]}) \\
&= Q_{t+s}[(1+\epsilon)u\mathbf{1}_E].
\end{aligned}$$

Letting $t \to \infty$ in the display above, we get that for all $s \geq 0$, $\epsilon > 0$ and u satisfying $0 < (1-\epsilon)u < (1+\epsilon)u < 1$, it holds that

$$q((1-\epsilon)u) \leq e^{-rs}q(ue^{\lambda s}) \leq q((1+\epsilon)u). \tag{3.31}$$

Using (3.28), letting $\epsilon \to 0$ and then $u \uparrow 1$ in (3.31), we get that

$$q(1) = 1 = e^{-rs}q(e^{\lambda s}), \quad s \geq 0.$$

In other words, $q(u) = u^{r/\lambda}$ for $u \in (0, 1]$. Finally noticing that q is non-negative and non-decreasing on $[0, 1]$, we also have $q(0) = 0$.

We are now ready to finish the proof of Lemma 3.5. Fix an unbounded increasing positive sequence $\mathbf{t} = (t_n)_{n\in\mathbb{N}}$ and a function $f \in \mathcal{B}(E, [0, \infty])$, we only need to prove that $1 - G_r f = (1 - G^{\mathbf{t}} f)^{r/\lambda}$.

From the definition of $G^{\mathbf{t}} f$, we can choose a subsequence $\mathbf{t}' = (t'_n)_{n\in\mathbb{N}}$ of $\mathbf{t}$ such that for each $n \in \mathbb{N}$, we have $t'_n > T$ and

$$G^{\mathbf{t}} f = \Gamma_{t'_n} f + C^{18}_n \tag{3.32}$$

for some real C^{18}_n (depending on both f and $\mathbf{t}'$) such that $\lim_{n\to\infty}|C^{18}_n| = 0$.

Therefore, we have for any $n \in \mathbb{N}$,

$$\begin{aligned}
&1 - e^{-G^{\mathbf{t}} f} \overset{(3.32)}{=} 1 - e^{-\Gamma_{t'_n} f - C^{18}_n} = (1 - e^{-\Gamma_{t'_n} f})e^{-C^{18}_n} + (1 - e^{-C^{18}_n}) \\
&\overset{(3.26)}{=} \frac{1 - e^{-\nu(V_{(t'_n)}f)}}{1 - e^{-\nu(v_{(t'_n)})}}e^{-C^{18}_n} + (1 - e^{-C^{18}_n}) \\
&= \frac{\nu(V_{(t'_n)}f)}{\nu(v_{(t'_n)})}(1 + C^{19}_n) + (1 - e^{-C^{18}_n})
\end{aligned}$$

for some real C_n^{19} with $\lim_{n\to\infty}|C_n^{19}| = 0$, by Proposition 2.1 and the fact that $(1-e^{-x})/x \xrightarrow[x\to 0]{} 1$. Thus

$$1 - e^{-G^{\mathbf{t}} f} \overset{\text{Proposition 2.2}}{=} \frac{V_{(t_n')} f(x)}{v_{(t_n')}(x)} \frac{1 + C^4_{t_n', x, \infty \mathbf{1}_E}}{1 + C^4_{t_n', x, f}} (1 + C_n^{19}) + (1 - e^{-C_n^{18}}). \tag{3.33}$$

It is elementary to see that

$$\lim_{n\to\infty} \sup_{x\in E} \left| \frac{1 + C^4_{t_n', x, \infty \mathbf{1}_E}}{1 + C^4_{t_n', x, f}} (1 + C_n^{19}) - 1 \right| = 0.$$

Therefore, for any $\epsilon > 0$, there exists $N_\epsilon > 0$ such that for any $n > N_\epsilon$,

$$\left| \left(\frac{1 + C^4_{t_n', x, \infty \mathbf{1}_E}}{1 + C^4_{t_n', x, f}} (1 + C_n^{19}) \right)^{-1} - 1 \right| < \epsilon; \text{ and } |1 - e^{-C_n^{18}}| < \epsilon. \tag{3.34}$$

Note from (2.1), $0 \le V_t f \le v_t$ for each $t \ge 0$. It is elementary to verify from (3.33) and (3.34) that, for any $\epsilon > 0$, $n > N_\epsilon$ and $x \in E$,

$$(1-\epsilon) \Big((1 - e^{-G^{\mathbf{t}} f} - \epsilon) \vee 0 \Big) \le \frac{V_{(t_n')} f(x)}{v_{(t_n')}(x)} \le (1+\epsilon)(1 - e^{-G^{\mathbf{t}} f} + \epsilon) \wedge 1.$$

Since G_r is a monotone functional, we know that for each $t \ge 0$, Q_t is also a monotone functional. This implies that for any $\epsilon > 0$ and $n > N_\epsilon$,

$$\begin{aligned} &Q_{(t_n')} \Big[(1-\epsilon) \Big((1 - e^{-G^{\mathbf{t}} f} - \epsilon) \vee 0 \Big) \mathbf{1}_E \Big] \le Q_{(t_n')} \left(\frac{V_{(t_n')} f}{v_{(t_n')}} \right) \\ &\le Q_{(t_n')} \Big[\Big((1+\epsilon)(1 - e^{-G^{\mathbf{t}} f} + \epsilon) \wedge 1 \Big) \mathbf{1}_E \Big]. \end{aligned} \tag{3.35}$$

Note from the definition of $(Q_t)_{t\ge 0}$ and G_r, we always have for $t > T$ that

$$Q_t \left(\frac{V_t f}{v_t} \right) = e^{-rt} (1 - e^{-G_r V_t f}) = 1 - e^{-G_r f}.$$

Therefore, taking $n \to \infty$ in (3.35), and using (3.29) we get that

$$\Big((1-\epsilon) \Big((1 - e^{-G^{\mathbf{t}} f} - \epsilon) \vee 0 \Big) \Big)^{r/\lambda} \le 1 - e^{-G_r f} \le \Big((1+\epsilon)(1 - e^{-G^{\mathbf{t}} f} + \epsilon) \wedge 1 \Big)^{r/\lambda}.$$

Taking $\epsilon \to 0$, we get the desired result. □

Proof of Proposition 2.3. Combining Lemmas 3.4 and 3.5 (taking $r = \lambda$) with a sub-sub-sequence type argument, we can easily get the conclusion of Proposition 2.3. □

3.4. *Proof of Proposition 2.4*

Proof of Proposition 2.4. We first consider the case that $g = 0$ ν-almost surely. From (3.1) and (H1), we have

$$V_1 g_n(x) \le P_1^\beta g_n(x) \le C^{20} \phi(x) \nu(g_n), \quad n \in \mathbb{N}, x \in E, \tag{3.36}$$

where $C^{20} := \sup_{x\in E, f\in L^1_+(\nu)} e^\lambda (1 + |H_{1,x,f}|)$. By the bounded convergence theorem, we have

$$\lim_{n\to\infty} \nu(g_n) = \nu(g) = 0. \tag{3.37}$$

On the other hand, from (3.9), we know that $t \mapsto e^{-\lambda t}\nu(v_t)$ is a non-increasing $(0,\infty)$-valued continuous function on (T,∞). Since $\lambda < 0$, we have

$$t \mapsto \nu(v_t) \text{ is a strictly decreasing } (0,\infty)\text{-valued continuous function on } (T,\infty). \tag{3.38}$$

By Proposition 2.1, we have

$$\lim_{t\to\infty} \nu(v_t) = 0. \tag{3.39}$$

Using (3.37), (3.38) and (3.39) we can see that there exist $n_0 > 0$ and a sequence $\{t_n : n > n_0\}$ of positive numbers such that

$$\lim_{n\to\infty} t_n = \infty \tag{3.40}$$

and that, for any $n > n_0$,

$$2C^{20}\nu(g_n) \le \nu(v_{t_n}). \tag{3.41}$$

It follows from Proposition 2.2 that there exists $n_1 > n_0$ such that for all $n > n_1$ and $x \in E$,

$$\nu(v_{t_n}) \le 2\phi(x)^{-1} v_{t_n}(x). \tag{3.42}$$

Now, for any $n > n_1$ and $x \in E$, we have

$$\begin{aligned} V_1 g_n(x) &\overset{(3.36)}{\le} C^{20}\phi(x)\nu(g_n) \overset{(3.41)}{\le} \frac{1}{2}\phi(x)\nu(v_{t_n}) \\ &\overset{(3.42)}{\le} v_{t_n}(x). \end{aligned} \tag{3.43}$$

Therefore, for any $n > n_1$,

$$1 - e^{-Gg_n} \overset{(2.6)}{=} e^{-\lambda}(1 - e^{-GV_1 g_n}) \le e^{-\lambda}(1 - e^{-Gv_{t_n}}) = e^{-\lambda} e^{\lambda t_n},$$

where in the inequality above we used (3.43) and the monotonicity of G (Proposition 2.3), and in the last equality, we used Proposition 2.3 with $f = \infty \mathbf{1}_E$. Letting $n \to \infty$ in the display above, noticing (3.40) and the fact that $\lambda < 0$, we get the desired result in this case.

We now consider the case that $g_n \downarrow g$ pointwisely where $\nu(g) > 0$. The monotonicity of G (Proposition 2.3) implies that $\lim_{n\to\infty} Gg_n$ exists and is greater than Gg. So we only need to show that $\lim_{n\to\infty} Gg_n \le Gg$. From Proposition 2.2, for any $\epsilon > 0$ there exists $T^2_\epsilon > 0$ such that for any $t \ge T^2_\epsilon$, $x \in E$ and $f \in \mathcal{B}(E,[0,\infty])$,

$$(1-\epsilon)\phi(x)\nu(V_t f) \le V_t f(x) \le (1+\epsilon)\phi(x)\nu(V_t f). \tag{3.44}$$

Therefore, we have for any $\epsilon > 0$, $t \ge T^2_\epsilon$, $x \in E$ and $f, h \in \mathcal{B}(E,[0,\infty])$ with $\nu(h) > 0$ that

$$\begin{aligned} V_t f(x) &\overset{(3.44)}{\ge} (1-\epsilon)\phi(x)\nu(V_t f) \\ &\overset{(3.4)}{=} (1-\epsilon)\phi(x)\frac{\nu(V_t f)}{\nu(V_t h)}\nu(V_t h) \overset{(3.44)}{\ge} \frac{1-\epsilon}{1+\epsilon}\frac{\nu(V_t f)}{\nu(V_t h)} V_t h(x) \\ &\ge \left(\frac{1-\epsilon}{1+\epsilon}\frac{\nu(V_t f)}{\nu(V_t h)} \wedge 1\right) V_t h(x). \end{aligned} \tag{3.45}$$

Since G is a monotone concave functional (Proposition 2.3), we know that for any $f \in \mathcal{B}(E,[0,\infty])$, $u \mapsto 1 - e^{-G(uf)}$ is a concave function on $[0,1]$ (Lemma A.3); and therefore,

$$1 - e^{-G(uf)} \ge u(1 - e^{-Gf}) + (1-u)(1 - e^{-G(0\mathbf{1}_E)}) = u(1 - e^{-Gf}), \quad u \in [0,1]. \tag{3.46}$$

Now we have for any $\epsilon > 0$, $t \geq T_\epsilon^2$, $x \in E$ and $f, h \in \mathcal{B}(E, [0, \infty])$ with $\nu(h) > 0$ that

$$1 - e^{-Gf} \overset{\text{Proposition 2.3}}{=} e^{-\lambda t}(1 - e^{-GV_t f}) \overset{(3.45)}{\geq} e^{-\lambda t}\left(1 - e^{-G\left(\left(\frac{1-\epsilon}{1+\epsilon}\frac{\nu(V_t f)}{\nu(V_t h)} \wedge 1\right)V_t h\right)}\right)$$
$$\overset{(3.46)}{\geq} e^{-\lambda t}\left(\frac{1-\epsilon}{1+\epsilon}\frac{\nu(V_t f)}{\nu(V_t h)} \wedge 1\right)(1 - e^{-GV_t h})$$
$$\overset{\text{Proposition 2.3}}{=} \left(\frac{1-\epsilon}{1+\epsilon}\frac{\nu(V_t f)}{\nu(V_t h)} \wedge 1\right)(1 - e^{-Gh}).$$

Replacing f by g, h by g_n, and then taking $n \to \infty$, noticing that by monotone convergence theorem $\nu(V_t g_n) \xrightarrow[n\to\infty]{} \nu(V_t g)$, we get

$$1 - e^{-Gg} \geq \frac{1-\epsilon}{1+\epsilon} \lim_{n\to\infty}(1 - e^{-Gg_n}),$$

as desired (noticing $\epsilon > 0$ is arbitrary). □

4. Proofs of Propositions 2.5–2.7

4.1. *Proof of Proposition 2.5*

Proof of Proposition 2.5(1). Denote by G the functional given by Proposition 2.3; and by $\mathbf{Q}_\lambda$ the Yaglom limit given by Theorem 1.1. By (2.9), we know that G is the log-Laplace functional of $\mathbf{Q}_\lambda$. Now note that for $t \geq 0$,

$$(\mathbf{Q}_\lambda\mathbb{P})(\|X_t\| > 0) \overset{(2.4)}{=} \int_{\mathcal{M}_f(E)} (1 - e^{-\mu(v_t)})\mathbf{Q}_\lambda(d\mu) \overset{(2.9)}{=} 1 - e^{-Gv_t}$$
$$\overset{\text{Proposition 2.3}}{=} e^{\lambda t}. \tag{4.1}$$

Therefore, we have that for all $f \in \mathcal{B}(E, [0, \infty])$ and $t \geq 0$,

$$(\mathbf{Q}_\lambda\mathbb{P})[1 - e^{-X_t(f)} | \|X_t\| > 0] \overset{(4.1)}{=} e^{-\lambda t}(\mathbf{Q}_\lambda\mathbb{P})[1 - e^{-X_t(f)}]$$
$$\overset{(2.3)}{=} e^{-\lambda t}\int_{\mathcal{M}_f(E)} (1 - e^{-\mu(V_t f)})\mathbf{Q}_\lambda(d\mu) \overset{(2.9)}{=} e^{-\lambda t}(1 - e^{-GV_t f})$$
$$\overset{\text{Proposition 2.3}}{=} 1 - e^{-Gf} \overset{(2.9)}{=} \int_{\mathcal{M}_f(E)} (1 - e^{-\mu(f)})\mathbf{Q}_\lambda(d\mu).$$

According to [21, Theorem 1.17], this says that

$$(\mathbf{Q}_\lambda\mathbb{P})(\cdot | \|X_t\| > 0) = \mathbf{Q}_\lambda(\cdot), \quad t \geq 0.$$

Therefore $\mathbf{Q}_\lambda$ is a QSD of X. From (4.1) and (1.4), its mass decay rate is λ. □

Proof of Proposition 2.5(2). Denote $\gamma = r/\lambda \in (0, 1)$. We first claim that there exists a $\mathbb{Z}_+$-valued random variable $\{Z; P\}$ with probability generating function $P[s^Z] = 1 - (1 - s)^\gamma$, $s \in [0, 1]$. To see this, we set

$$P(Z = n) = \frac{\gamma(1-\gamma)\cdots(n-1-\gamma)}{n!}, \quad n \in \mathbb{Z}_+.$$

Using Newton's binomial theorem (see [30, Exercise 8.22]), we get

$$1-(1-s)^\gamma=\sum_{n=1}^\infty \frac{\gamma(1-\gamma)\cdots(n-1-\gamma)}{n!}s^n,\quad s\in[0,1],$$

thus, such a random variable exists.

Now let $\{(Y_n)_{n\in\mathbb{N}};P\}$ be an $\mathcal{M}_f^o(E)$-valued i.i.d. sequence with law of the Yaglom limit $\mathbf{Q}_\lambda$. Let Z and $(Y_n)_{n\in\mathbb{N}}$ be independent of each other. Define the probability $\mathbf{Q}_r$ on $\mathcal{M}_f^o(E)$ as the law of the finite random measure $\sum_{n=1}^Z Y_n$.

In the rest of this proof, we will argue that $\mathbf{Q}_r$ is a QSD of X with mass decay rate r. To do this, we calculate that

$$\begin{aligned} e^{-\mathscr{L}_{\mathbf{Q}_r}f}&=P[e^{-\sum_{n=1}^Z Y_n(f)}]=P\left[P\left[\prod_{n=1}^Z e^{-Y_n(f)}\middle|\sigma(Z)\right]\right]=P\left[e^{-Z\cdot\mathscr{L}_{\mathbf{Q}_\lambda}f}\right]\\ &=1-(1-e^{-\mathscr{L}_{\mathbf{Q}_\lambda}f})^\gamma,\quad f\in\mathcal{B}(E,[0,\infty]). \end{aligned}\tag{4.2}$$

Therefore, for each $t>0$ and $f\in\mathcal{B}(E,[0,\infty])$, we have

$$\begin{aligned}(\mathbf{Q}_r\mathbb{P})\left[1-e^{-X_t(f)}\middle|\|X_t\|>0\right]&=(\mathbf{Q}_r\mathbb{P})(\|X_t\|>0)^{-1}\cdot(\mathbf{Q}_r\mathbb{P})[1-e^{-X_t(f)}]\\ &\overset{(2.3),(2.4)}{=}(1-e^{-\mathscr{L}_{\mathbf{Q}_r}v_t})^{-1}(1-e^{-\mathscr{L}_{\mathbf{Q}_r}V_tf})\overset{(4.2)}{=}(1-e^{-\mathscr{L}_{\mathbf{Q}_\lambda}v_t})^{-\gamma}(1-e^{-\mathscr{L}_{\mathbf{Q}_\lambda}V_tf})^\gamma\\ &\overset{(2.3),(2.4)}{=}(\mathbf{Q}_\lambda\mathbb{P})\left[1-e^{-X_t(f)}\middle|\|X_t\|>0\right]^\gamma\overset{\text{Proposition 2.5 (1)}}{=}(1-e^{-\mathscr{L}_{\mathbf{Q}_\lambda}f})^\gamma\overset{(4.2)}{=}1-e^{-\mathscr{L}_{\mathbf{Q}_r}f}.\end{aligned}$$

This proves that $\mathbf{Q}_r$ is a QSD. To see its mass decay rate is r, we calculate that for each $t\geq 0$,

$$\begin{aligned}(\mathbf{Q}_r\mathbb{P})(\|X_t\|>0)&\overset{(2.4)}{=}1-e^{-\mathscr{L}_{\mathbf{Q}_r}v_t}\\ &\overset{(4.2)}{=}(1-e^{-\mathscr{L}_{\mathbf{Q}_\lambda}v_t})^\gamma\overset{(2.4)}{=}(\mathbf{Q}_\lambda\mathbb{P})(\|X_t>0\|)^\gamma\overset{\text{Proposition 2.5 (1)}}{=}e^{rt}.\quad\square\end{aligned}$$

4.2. *Proof of Proposition 2.6*

Proof of Proposition 2.6(1). First observe that for any $t\geq 0$,

$$e^{rt}=(\mathbf{Q}_r^*\mathbb{P})(\|X_t\|>0)\overset{(2.4)}{=}1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}(v_t)}.\tag{4.3}$$

According to Lemma A.2, for any $t>0$, we know that $u\mapsto\mathscr{L}_{\mathbf{Q}_r^*}(uv_t)$ is a $[0,\infty]$-valued concave function on $[0,\infty)$. According to Lemma A.3, for any $t>0$, we know that $u\mapsto 1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}(uv_t)}$ is a $[0,1]$-valued concave function on $[0,\infty)$. In particular, we have for any $t>0$ and $u\in[0,1]$ that

$$1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}(uv_t)}\geq u(1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}(1\cdot v_t)})+(1-u)(1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}(0\cdot v_t)})=u(1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}(v_t)}).\tag{4.4}$$

Recall that $T^1_{s,\epsilon}$ is the constant given in (3.30). Now for any $s>0,\epsilon>0$ and $t>T^1_{s,\epsilon}$ we have

$$e^{rs}\overset{(4.3)}{=}\frac{1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}v_{t+s}}}{1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}v_t}}\overset{(3.30)}{\geq}\frac{1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}(\frac{e^{\lambda s}}{1+\epsilon}v_t)}}{1-e^{-\mathscr{L}_{\mathbf{Q}_r^*}(v_t)}}\overset{(4.4)}{\geq}\frac{e^{\lambda s}}{1+\epsilon}.$$

Letting $\epsilon\to 0$, we get the desired result. $\square$

Proof of Proposition 2.6(2). From the definition of QSD, we know that $\mathbf{Q}_r^*$ has no concentration on $\{\mathbf{0}\}$. Therefore $\mathscr{L}_{\mathbf{Q}_r^*}(\infty\mathbf{1}_E)=\infty$. According to Lemma A.2, we know that

$\mathscr{L}_{\mathbf{Q}_r^*}$ is a monotone concave functional. Knowing that $\mathbf{Q}_r^*$ is a QSD for X with mass decay rate r, it can be verified that for each $f \in \mathcal{B}(E, [0, \infty])$ and $t \geq 0$,

$$\begin{aligned}
&1 - e^{-\mathscr{L}_{\mathbf{Q}_r^*} f} = (\mathbf{Q}_r^* \mathbb{P}) \left[1 - e^{-X_t(f)} \middle| \|X_t\| > 0\right] \\
&= e^{-rt} (\mathbf{Q}_r^* \mathbb{P})[1 - e^{-X_t(f)}] \overset{(2.3)}{=} e^{-rt} \int_{\mathcal{M}_f(E)} (1 - e^{-\mu(V_t f)}) \mathbf{Q}_r^*(d\mu) \\
&= e^{-rt}(1 - e^{-\mathscr{L}_{\mathbf{Q}_r^*} V_t f}). \quad \square
\end{aligned}$$

4.3. *Proof of Proposition 2.7*

Proof of Proposition 2.7. This is now obvious from Lemma 3.5 and the fact that $Gf = \lim_{t \to \infty} \Gamma_t f$ for $f \in \mathcal{B}(E, [0, \infty])$ (Proposition 2.3). $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We thank Zenghu Li and Leonid Mytnik for helpful conversations. We also thank the two referees for helpful comments on the first version of this paper.

Appendix

A.1. *Extended values*

In this paper, we often work with the extended non-negative real number system $[0, \infty]$ which consists of the non-negative real line $[0, \infty)$ and an extra point ∞. We consider $[0, \infty]$ as the one point compactification of $[0, \infty)$; and therefore, it is a compact Hausdorff space. We also make the following conventions that

- $x + \infty = \infty$ for each $x \in [0, \infty]$;
- $x \cdot \infty = \infty$ for each $x \in (0, \infty]$;
- $\frac{1}{\infty} = 0$; $\frac{1}{0} = \infty$; $e^{-\infty} = 0$; $-\log 0 = \infty$.

Note that $\infty \cdot 0$ has no meaning, but we use the convention that $\infty \cdot 0 = 0$ when we are dealing with indicator functions. For example, we may write expression like

$$h(x) = g(x) \cdot \mathbf{1}_A(x) + \infty \cdot \mathbf{1}_{E \setminus A}(x), \quad x \in E,$$

as a shorthand of

$$x = \begin{cases} g(x) & \text{if } x \in A, \\ \infty & \text{if } x \in E \setminus A. \end{cases}$$

A.2. Concave functionals

We say an $\mathbb{R}$-valued (or $[0,\infty]$-valued) function f on a convex subset D of $\mathbb{R}$ is concave iff

$$f(rx+(1-r)y) \geq rf(x)+(1-r)f(y), \quad x, y \in D, r \in [0,1].$$

The following lemmas about concave functions are elementary, we refer our readers to [5, Chapter 6] for more details.

Lemma A.1. *If f is a non-decreasing $\mathbb{R}$-valued concave function on $(a,b]$ where $a<b$ in $\mathbb{R}$, then f is continuous on $(a,b]$.*

Lemma A.2. *Suppose that $\{Z; P\}$ is a $[0,\infty]$-valued random variable. Define $L(u) := -\log P[e^{-uZ}]$ with $u \in [0,\infty)$, then L is a $[0,\infty]$-valued concave function on $[0,\infty)$.*

Lemma A.3. *Suppose that g is a concave function on some convex subset D of $\mathbb{R}$, then so is $q := 1-e^{-g}$.*

A.3. Continuity theorem for the Laplace functional of random measures

In this subsection, we discuss the continuity theorem for finite random measures on Polish space. The following result is not new. We included it here for the sake of completeness. Let E be a Polish space. Denote by $\mathcal{M}_f(E)$ the collection of all the finite Borel measures on E equipped with the topology of weak convergence. According to [16, Lemma 4.5], $\mathcal{M}_f(E)$ is a Polish space.

Lemma A.4. *Let $(\mathbf{P}_n)_{n\in\mathbb{N}}$ be a sequence of probabilities on $\mathcal{M}_f(E)$. Suppose that (1) for each $f \in \mathcal{B}_b(E,[0,\infty))$, the limit $Lf := \lim_{n\to\infty} \mathscr{L}_{\mathbf{P}_n} f$ exists; and (2) for each $f_n \downarrow f$ pointwisely in $\mathcal{B}_b(E,[0,\infty))$, $Lf_n \downarrow Lf$. Then there exists a unique probability $\mathbf{Q}$ on $\mathcal{M}_f(E)$ such that $(\mathbf{P}_n)_{n\in\mathbb{N}}$ converges weakly to $\mathbf{Q}$ and $\mathscr{L}_{\mathbf{Q}} = L$ on $\mathcal{B}_b(E,[0,\infty))$.*

Proof. We say a $[0,\infty)$-valued functional Γ on $\mathcal{B}_b(E,[0,\infty))$ is positive definite if

$$\sum_{i,j=1}^{n} a_i a_j \Gamma(f_i + f_j) \geq 0$$

for any $\mathbb{R}$-valued list $(a_k)_{k=1}^n$ and $\mathcal{B}_b(E,[0,\infty))$-valued list $(f_k)_{k=1}^n$. It is proved in [4, Theorem 3.3.3] that for any $n \in \mathbb{N}$, $f \mapsto e^{-\mathscr{L}_{\mathbf{P}_n} f}$ is positive definite on $\mathcal{B}_b(E,[0,\infty))$. Therefore, $f \mapsto e^{-Lf}$ is positive definite. Now from [9, Corollary (A.6)] and the condition (2), we know that there exists a sub-probability $\mathbf{Q}$ on $\mathcal{M}_f(E)$ such that

$$\int_{\mathcal{M}_f(E)} e^{-\mu(f)} \mathbf{Q}(d\mu) = e^{-Lf}, \quad f \in \mathcal{B}_b(E,[0,\infty)). \tag{A.1}$$

Taking $f = 0\cdot\mathbf{1}_E$ in condition (1) we get that $L(0\cdot\mathbf{1}_E) = 0$. This says that $\mathbf{Q}$ is a probability on $\mathcal{M}_f(E)$. Now condition (1) and [21, Theorem 1.8] imply that $(\mathbf{P}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{Q}$ weakly. Finally, (A.1) implies that $\mathscr{L}_{\mathbf{Q}} = L$ on $\mathcal{B}_b(E,[0,\infty))$. $\square$

References

[1] S. Asmussen, H. Hering, Branching Processes, in: Progress in Probability and Statistics, vol. 3, Birkhäuser Boston, Inc., Boston, MA, 1983, MR-0701538.

[2] K.B. Athreya, P.E. Ney, Branching processes, in: Die Grundlehren der mathematischen Wissenschaften, vol. Band 196, Springer-Verlag, New York-Heidelberg, 1972, MR-0373040.

[3] N. Champagnat, S. Rœlly, Limit theorems for conditioned multitype Dawson–Watanabe processes and Feller diffusions, Electron. J. Probab. 13 (25) (2008) 777–810, MR-2399296.

[4] D. Dawson, Infinitely divisible random measures and superprocesses, in: Stochastic Analysis and Related Topics (Silivri, 1990), in: Progr. Probab., vol. 31, Birkhäuser Boston, Boston, MA, 1992, pp. 1–129, MR-1203373.

[5] R.M. Dudley, Real analysis and probability, in: Revised Reprint of the 1989 Original, in: Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, Cambridge, 2002, MR-1932358.

[6] A.M. Etheridge, D.R.E. Williams, A decomposition of the $(1+\beta)$-superprocess conditioned on survival, Proc. Roy. Soc. Edinburgh Sect. A 133 (4) (2003) 829–847, MR-2006204.

[7] S. Evans, The entrance space of a measure-valued Markov branching process conditioned on nonextinction, Canad. Math. Bull. 35 (1) (1992) 70–74, MR-1157466.

[8] S. Evans, E. Perkins, Measure-valued Markov branching processes conditioned on nonextinction, Israel J. Math. 71 (3) (1990) 329–337, MR-0995575.

[9] P. Fitzsimmons, Construction and regularity of measure-valued Markov branching processes, Israel J. Math. 64 (3) (1988) 337–361, (1989). MR-0995575.

[10] R. Heathcote, E. Seneta, D. Vere-Jones, A refinement of two theorems in the theory of branching processes, Theory Probab. Appl. 12 (1967) 297–301, MR-0217889.

[11] F. Hoppe, Stationary measures for multitype branching processes, J. Appl. Probab. 12 (1975) 219–227, MR-0373043.

[12] F. Hoppe, E. Seneta, Analytical methods for discrete branching processes, in: Branching Processes (Conf. Saint Hippolyte, Que. 1976), in: Adv. Probab. Related Topics, vol. 5, Dekker, New York, 1978, pp. 219–261, MR-0517536.

[13] A. Joffe, On the Galton–Watson branching process with mean less than one, Ann. Math. Stat. 38 (1967) 264–266, MR-0205337.

[14] A. Joffe, F. Spitzer, On multitype branching processes with $\rho \leq 1$, J. Math. Anal. Appl. 19 (1967) 409–430, MR-0212895.

[15] O. Kallenberg, Foundations of modern probability, second ed., in: Probability and its Applications (New York), Springer-Verlag, New York, 2002, MR-1876169.

[16] O. Kallenberg, Random measures, theory and applications, in: Probability Theory and Stochastic Modelling, vol. 77, Springer, Cham, 2017, MR-3642325.

[17] P. Kim, R. Song, Intrinsic ultracontractivity of non-symmetric diffusion semigroups in bounded domains, Tohoku Math. J. (2) 60 (4) (2008) 527–547, MR-2487824.

[18] C. Labbé, Quasi-stationary distributions associated with explosive CSBP, Electron. Commun. Probab. 18 (57) (2013) 13, MR-3084568.

[19] A. Lambert, Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct, Electron. J. Probab. 12 (14) (2007) 420–446, MR-2299923.

[20] Z. Li, Asymptotic behavior of continuous time and state branching processes, J. Aust. Math. Soc. A 68 (2000) 68–84, MR-1727226.

[21] Z. Li, Measure-valued branching Markov processes, in: Probability and its Applications (New York), Springer, Heidelberg, 2011, MR-2760602.

[22] R. Liu, Y.-X. Ren, Some properties of superprocesses conditioned on non-extinction, Sci. China A 52 (4) (2009) 771–784, MR-2504975.

[23] R. Lyons, R. Pemantle, Y. Peres, Conceptual proofs of Llogl criteria for mean behavior of branching processes, Ann. Probab. 23 (3) (1995) 1125–1138, MR-1349164.

[24] P. Maillard, The λ-invariant measures of subcritical Bienaymé-Galton–Watson processes, Bernoulli 24 (1) (2018) 297–315, MR-3706758.

[25] S. Méléard, D. Villemonais, Quasi-stationary distributions and population processes, Probab. Surv. 9 (2012) 340–410, MR-2994898.

[26] Y.-X. Ren, R. Song, Z. Sun, Limit theorems for a class of critical superprocesses with stable branching, Stochastic Process. Appl. 130 (2020) 4358–4391, http://dx.doi.org/10.1016/j.spa.2020.01.001.

[27] Y.-X. Ren, R. Song, Z. Sun, Spine decompositions and limit theorems for a class of critical superprocesses, Acta Appl. Math. 165 (2020) 91–131, http://dx.doi.org/10.1007/s10440-019-00243-7.

[28] Y.-X. Ren, R. Song, R. Zhang, Limit theorems for some critical superprocesses, Illinois J. Math. 59 (1) (2015) 235–276, MR-3459635.

[29] Y.-X. Ren, R. Song, R. Zhang, Central limit theorems for supercritical branching nonsymmetric Markov processes, Ann. Probab. 45 (1) (2017) 564–623, MR-3601657.

[30] W. Rudin, Principles of mathematical analysis, third ed., in: International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976, MR-0385023.

[31] H.H. Schaefer, Banach lattices and positive operators, in: Die Grundlehren der mathematischen Wissenschaften, vol. Band 215, Springer-Verlag, New York-Heidelberg, 1974, MR-0423039.

[32] E. Seneta, D. Vere-Jones, On the asymptotic behaviour of subcritical branching processes with continuous state space, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 10 (1968) 212–225, MR-0239667.

[33] L. Serlet, The occupation measure of super-Brownian motion conditioned to nonextinction, J. Theoret. Probab. 9 (3) (1996) 561–578, MR-1400587.

[34] A.M. Yaglom, Certain limit theorems of the theory of branching processes, Dokl. Acad. Nauk. SSSR 56 (1947) 795–798, MR-0022045.