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# ON THE COMING DOWN FROM INFINITY OF COALESCING BROWNIAN MOTIONS

BY CLAYTON BARNES[1,a], LEONID MYTNIK[1,b] AND ZHENYAO SUN[2,c]

[1]*The Faculty of Data and Decision Sciences, Technion—Israel Institute of Technology,* [a]*clayleroy2@gmail.com,* [b]*leonid@ie.technion.ac.il*

[2]*School of Mathematics and Statistics, Beijing Institute of Technology,* [c]*zhenyao.sun@gmail.com*

Consider a system of Brownian particles on the real line where each pair of particles coalesces at a certain rate according to their intersection local time. Assume that there are infinitely many initial particles in the system. We give a necessary and sufficient condition for the number of particles to come down from infinity. We also identify the rate of this coming down from infinity for different initial configurations.

## 1. Introduction.

1.1. *Motivation and background.* Coming down from infinity is a property observed for certain stochastic population dynamics models, where the population begins with infinitely many members, and where members undergo competitive interactions that inevitably bring the entire population to a finite size in positive time. Kingman's coalescent, perhaps one of the simplest such examples, is a continuous time Markov process taking values in the set of partitions of $\mathbb{N}$. If one thinks of blocks in a partition of $\mathbb{N}$ as particles, then Kingman's coalescent is a particle system where each pair of particles coalesce into one particle with unit rate, independently of other pairs. If initially there are infinitely many particles, the coming down from infinity property says that almost surely, after any positive amount of time, there are only finitely many particles in the system. In 1999, Aldous showed that Kingman's coalescent comes down from infinity [1]. There are two goals when addressing the coming down from infinity for a given stochastic process. The first is to show whether, and under what conditions, the coming down from infinity property holds. The second, and perhaps more central, goal is then to find the rate with which the number of particles approaches infinity as time goes to zero.

Let $N_t$ be the number of particles at time $t > 0$ in Kingman's coalescent with infinitely many initial particles. It is known in [1] that $N_t$ behaves like $v(t) := 2/t$ when $t \downarrow 0$. More precisely, it is shown in [5] that $N_t/v(t)$ converges to 1 as $t \downarrow 0$, almost surely, and in $L^p$ for any $p \geq 1$. Here the rate function $v(t)$ arises as the solution to the nonlinear ordinary differential equation (ODE) with the singular initial condition,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\frac{1}{2}v(t)^2, & t > 0, \\ v(0) = \infty. \end{cases} \tag{1.1}$$

For more references on coalescent theory, we refer our readers to [6] and the references therein.

The phenomenon of coming down from infinity is also considered in [2] and [23] for spatial Kingman's coalescent, extending the notion of Kingman's coalescent to the discrete

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spatial setting. In this model each particle undergoes continuous-time simple random walk on a connected graph with uniformly bounded degree, and each pair of particles located at the same vertex coalesce into one particle with unit rate. It is proved in [2] and [23] that the spatial Kingman's coalescent comes down from infinity if and only if the underlying graph is finite.

In 1988, Shiga [28] proposed a model which is naturally considered an analogy of Kingman's coalescent in the continuum spatial setting. In this particle system, particles move as independent Brownian motions on $\mathbb{R}$, and each pair of the particles coalesce into one particle with rate $1/2$ according to their intersection local time. We will refer to this model as *(slowly) coalescing Brownian motions*. Coalescing Brownian motions has drawn much attention due to its connection to the stochastic heat equation with Wright–Fisher white noise [3, 4, 7–15, 26, 28].

It is perhaps natural to ask the following question:

(1.2) *Does coalescing Brownian motions come down from infinity? If it does, what is the rate of this coming down from infinity?*

Hobson and Tribe [15] consider this question for coalescing Brownian motions on the unit circle $\mathbb{S}_1$, with initial particles sampled according to a Poisson point measure with intensity $n$ times the uniform measure of $\mathbb{S}_1$. Denote by $\hat{N}_t^{(n)}$ the total number of particles in the system at time $t \geq 0$. They proved that, as $n \uparrow \infty$, $\hat{N}_t^{(n)}$ has a finite weak limit $\hat{N}_t$ for every strictly positive $t$. They also showed that $\hat{N}_t / v(t)$ converges to 1 in probability as $t \downarrow 0$ with the rate function $v(t) = 2/t$. However, their proofs rely on the compactness of the circle $\mathbb{S}_1$ that does not extend to coalescing Brownian motions on the real line $\mathbb{R}$. In this paper, we will give an answer to (1.2) in a more general setting.

1.2. *Definition and main results.* Let $\mathcal{N}$ be the space of locally finite atomic measures on $\mathbb{R}$ equipped with the vague topology. In this paper the coalescing Brownian motions (with possibly infinite many initial particles) will be defined as $\mathcal{N}$-valued càdlàg Markov processes on $(0, \infty)$ whose entrance laws and transition probabilities will be specified later. To motivate that formal definition; however, we want to first construct an essentially equivalent particle system which describes the trajectory of each particles. This construction, due to Tribe [30], Section 2, assign integer labels to the particles so that when particles labeled $i$ and $j$ coalesce, the remaining particle is labeled $\min\{i, j\}$. This labeling rule has the convenient consequence that the lifetime of a particle labeled $i$ is determined by the trajectories of the particles whose labels are no larger than $i$. It is worth noting that how we label the particle does not effect their spatial movements due to the strong Markov property of the Brownian motions.

Let $I_0$ be the collection of labels of the initial particles: if there are finitely many initial particles, then $I_0 = \{i : 1 \leq i \leq n\}$ for some $n \in \mathbb{N} \cup \{0\}$; otherwise, if there are infinitely many initial particles, then $I_0 = \mathbb{N}$. Denote by $x_i \in \mathbb{R}$ the location of the initial particle labeled by $i$. If $I_0 = \varnothing$, then $(x_i)_{i \in I_0}$ is the empty list, and nothing needs to be constructed. Otherwise, if $I_0 \neq \varnothing$, the construction is formulated as follows:

(1.3) Let $\{(B_t^{(i)})_{t \geq 0} : i \in I_0\}$ be a list of independent Brownian motions on $\mathbb{R}$, defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, such that $B_0^{(i)} = x_i$ for each $i \in I_0$.

(1.4) Let $\{\mathbf{e}^{(i)} : i \in I_0\}$ be a family of i.i.d. exponential random variables with mean 2, defined on $(\Omega, \mathscr{F}, \mathbb{P})$, independent of the Brownian motions in (1.3).

(1.5) For each $i, j \in I_0$ with $i > j$, denote by $(L_t^{(i,j)})_{t \geq 0}$ the local time at zero of the process $(B_t^{(i)} - B_t^{(j)})_{t \geq 0}$.

(1.6) Define $\zeta_1 := \infty$, and inductively for each $i \in I_0$ with $i > 1$,

$$\zeta_i := \inf\left\{ t \geq 0 : \sum_{j=1}^{i-1} L_{t \wedge \zeta_j}^{(i,j)} \geq \mathbf{e}^{(i)} \right\}.$$

We call the stopping time $\zeta_i$ the lifetime of the $i$th particle, and we define

$$X_t^{(i)} := \begin{cases} B_t^{(i)} & \text{if } t \in [0, \zeta_i), \\ \dagger & \text{otherwise,} \end{cases}$$

for every $i \in I_0$. Here $\dagger$ is called the cemetery state and is not contained in $\mathbb{R}$. Now, we say the set of random variables $\mathbf{X} := \{X_t^{(i)} : t \geq 0, i \in I_0\}$ is a *coalescing Brownian particle system* with initial configuration $(x_i)_{i \in I_0}$. The space of all the possible initial configurations is denoted by

$$\mathcal{X} := \{\varnothing\} \cup \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \mathbb{R}^n.$$

After we have constructed the trajectory of each particles, it is natural to consider the counting measure formed by the locations of the particles at a fixed time. Denote by $I_t := \{i \in I_0 : t \in [0, \zeta_i)\}$ the set of the labels of the particles alive at time $t \geq 0$. For any open set $A \subset \mathbb{R}$, let $\mathscr{B}(A)$ be Borel $\sigma$-algebra on $A$. For every $U \in \mathscr{B}(\mathbb{R})$ and $t \geq 0$, define a $\mathbb{N} \cup \{0, \infty\}$-valued random variable

$$Z_t(U) := \sum_{i \in I_t} \mathbf{1}_U\big(X_t^{(i)}\big), \tag{1.6}$$

which is the number of living particles contained in $U$ at time $t$. Let us also define a filtration $(\mathscr{F}_t)_{t \geq 0}$ so that

$$\mathscr{F}_t := \sigma\big(Z_s(U) : s \leq t, U \in \mathscr{B}(\mathbb{R})\big), \quad t \geq 0. \tag{1.7}$$

At this point, however, it is not clear whether $(Z_t)_{t>0}$ is an $\mathcal{N}$-valued process. Before we show that this is indeed the case, let us define

$$\mathcal{T}_{\mathrm{a}} := \big\{(\Lambda, \mu) : \Lambda \text{ is a closed subset of } \mathbb{R}, \mu \text{ is an atomic Radon measure on } \mathbb{R} \setminus \Lambda\big\},$$

and introduce a map $\Psi$ from $\mathcal{X}$ to $\mathcal{T}_{\mathrm{a}}$ so that $(\Lambda, \mu) = \Psi((x_i)_{i \in I_0})$, provided

$$\Lambda = \Big\{y \in \mathbb{R} : \sum_{i \in I_0} \mathbf{1}_{(y-r, y+r)}(x_i) = \infty, \forall r > 0\Big\}, \tag{1.8}$$

$$\mu = \sum_{i \in I_0} \mathbf{1}_{\mathbb{R} \setminus \Lambda}(x_i) \delta_{x_i}. \tag{1.9}$$

We call $(\Lambda, \mu) = \Psi((x_i)_{i \in I_0})$ the *initial trace* of the coalescing Brownian particle system with initial configuration $(x_i)_{i \in I_0}$. Note that when there are infinitely many initial particles, $\Lambda$ is the set of the limiting points of the sequence $(x_i)_{i \in \mathbb{N}}$. Also, note that $\mu$ is a Radon measure on $\mathbb{R} \setminus \Lambda$, but not necessarily a Radon measure on $\mathbb{R}$. It is clear that $\mathcal{T}_{\mathrm{a}}$ is the collection of all the possible initial traces of the coalescing Brownian particle systems, in the sense that $\Psi : \mathcal{X} \to \mathcal{T}_{\mathrm{a}}$ is a surjection. In what follows, we denote by $\mathrm{supp}(\Lambda, \mu) := \Lambda \cup \mathrm{supp}(\mu)$ the support of a given $(\Lambda, \mu) \in \mathcal{T}_{\mathrm{a}}$. Recall that a set $S \subset \mathbb{R}$ is said to be bounded if $\mathrm{diam}(S) := \inf\{K : |x - y| < K, \forall x, y \in S\} < \infty$. We are now ready to state our first result.

THEOREM 1.1. *Let $(\Lambda, \mu) = \Psi((x_i)_{i \in I_0})$, and let $U \subset \mathbb{R}$ be an arbitrary open interval.*

(i) *If $U \cap \mathrm{supp}(\Lambda, \mu)$ is unbounded, then $\mathbb{P}(Z_t(U) = \infty, \forall t \geq 0) = 1$.*
(ii) *If $U \cap \mathrm{supp}(\Lambda, \mu)$ is bounded, then $\mathbb{P}(Z_t(U) < \infty, \forall t > 0) = 1$.*

If $U$ is bounded, then Theorem 1.1 (ii) shows that $Z_t(U)$ is finite for all time $t>0$, almost surely. This implies that $(Z_t)_{t>0}$ is an $\mathcal{N}$-valued process. We will verify that $(Z_t)_{t>0}$ is a càdlàg Markov process and characterize its law through its duality with Wright–Fisher stochastic partial differential equation (Wright–Fisher SPDE),

$$\partial_t u_{t,x} = \frac{1}{2}\partial_x^2 u_{t,x} + \sqrt{u_{t,x}(1-u_{t,x})}\dot{W}_{t,x}, \quad t>0, x\in\mathbb{R}. \tag{1.10}$$

Let us briefly review some results about (1.10) here. Denote by $\mathcal{C}_{[0,1]}$ the collection of $[0,1]$-valued continuous functions on $\mathbb{R}$, equipped with the topology of uniform convergence on compact sets. Let $f$ be an arbitrary $[0,1]$-valued Borel measurable function on $\mathbb{R}$. According to [29], there exists a filtered probability space $(\mathbf{\Omega}, \mathscr{G}, (\mathscr{G}_t), \mathbf{P}_f)$, and on this space, an adapted $\mathcal{C}_{[0,1]}$-valued continuous process $(u_{t,\cdot})_{t>0}$ and a space-time white noise $W$, satisfying the mild form of the Wright–Fisher SPDE (1.10) with the initial condition $u_0=f$. Namely, for every $(t,x)\in(0,\infty)\times\mathbb{R}$,

$$u_{t,x} = \int G_{t,x-y} f_y \,\mathrm{d}y + \iint_0^t G_{t-s,x-y}\sqrt{u_{s,y}(1-u_{s,y})}W(\mathrm{d}s\,\mathrm{d}y) \quad \text{a.s.} \tag{1.11}$$

Here $G$ is the heat kernel, given by $G_{t,x} := e^{-x^2/(2t)}/\sqrt{2\pi t}$, for every $(t,x)\in(0,\infty)\times\mathbb{R}$, and the second term on the right-hand side of (1.11) is given by Walsh's stochastic integral driven by a space-time white noise [31]. The law of the $\mathcal{C}_{[0,1]}$-valued continuous process $(u_{t,\cdot})_{t>0}$ is uniquely determined by the initial value $f$ [28], Theorem 5.1(2). The duality relation between the coalescing Brownian particle systems and the Wright–Fisher SPDEs is given by Shiga [28]: Suppose that there are only finitely many initial particles for the coalescing Brownian particle systems $\mathbf{X}$, that is, $I_0=\{0,\dots,n\}$ for some $n\in\mathbb{N}$, then

$$\mathbb{E}\Big[\prod_{i\in I_t}\big(1-f(X_t^{(i)})\big)\Big] = \mathbf{E}_f\Big[\prod_{i=1}^n\big(1-u_t(x_i)\big)\Big], \quad t\ge 0. \tag{1.12}$$

Let us now give the formal definition of the coalescing Brownian motions. For any $(\Lambda,\mu)\in\mathcal{T}_{\mathrm{a}}$, we say that a process $(Y_t)_{t>0}$, living in a filtered probability space with filtration $(\mathscr{F}_t^Y)_{t\ge 0}$ and probability $\mathbb{P}_{(\Lambda,\mu)}$, is a *coalescing Brownian motions process* with initial trace $(\Lambda,\mu)$, if the following statements hold:

(1.13) $(Y_t)_{t>0}$ is an $\mathcal{N}$-valued càdlàg Markov process.

(1.14) For any $t>0$ and $[0,1]$-valued Borel function $f$ on $\mathbb{R}$,

$$\mathbb{E}_{(\Lambda,\mu)}\big[\exp\{\langle\log(1-f),Y_t\rangle\}\big] = \mathbf{E}_f\big[\mathbf{1}_{\{u_{t,x}=1,\forall x\in\Lambda\}}\exp\{\langle\log(1-u_t),\mu\rangle\}\big].$$

(1.15) For any $t>s>0$ and $[0,1]$-valued Borel function $f$ on $\mathbb{R}$,

$$\mathbb{E}_{(\Lambda,\mu)}\big[\exp\{\langle\log(1-f),Y_t\rangle\}|\mathscr{F}_s^Y\big] = \Theta_{t-s}^f(Y_s),$$

where $\Theta_{t-s}^f(\kappa) := \mathbf{E}_f[\exp\{\langle\log(1-u_{t-s}),\kappa\rangle\}]$ for $\kappa\in\mathcal{N}$.

Here we denote by $\mathbb{E}_{(\Lambda,\mu)}$ and $\mathbf{E}_f$ the expectations for the probabilities $\mathbb{P}_{(\Lambda,\mu)}$ and $\mathbf{P}_f$, respectively. Notice that (1.14) and (1.15) characterize the entrance laws and the transition probabilities of the coalescing Brownian motions through the Laplace transform. In particular, if such process $(Y_t)_{t>0}$ exists, then its law is uniquely determined by its initial trace $(\Lambda,\mu)$.

Our next result is to show the existence of such processes.

THEOREM 1.2.

(i) *Let $(x_i)_{i\in I_0}\in\mathcal{X}$, $(\Lambda,\mu)=\Psi((x_i)_{i\in I_0})$ and let $\{Z_t(U): t\ge 0, U\in\mathscr{B}(\mathbb{R})\}$ be random variables in a filtered probability space $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P})$ constructed through* (1.3)–(1.7). *Then the process $(Z_t)_{t>0}$ is a coalescing Brownian motions process with initial trace $(\Lambda,\mu)$.*

(ii) *For any* $(\Lambda,\mu)\in\mathcal{T}_a$, *there exists a coalescing Brownian motions process with initial trace* $(\Lambda,\mu)$.

REMARK 1.3. Since $\Psi:\mathcal{X}\to\mathcal{T}_a$ is a surjection, for any $(\Lambda,\mu)\in\mathcal{T}_a$, there exist $I_0$ and $(x_i)_{i\in I_0}\in\mathcal{X}$ such that $(\Lambda,\mu)=\Psi((x_i)_{i\in I_0})$. Thus, part (ii) of Theorem 1.2 is an immediate corollary of part (i) of that theorem.

To present our result on the coming down from infinity of the coalescing Brownian motions, we will discuss briefly the one-dimensional nonlinear partial differential equation

$$\partial_t v_{t,x}=\frac{1}{2}\partial_x^2 v_{t,x}-\frac{1}{2}v_{t,x}^2,\quad t>0,x\in\mathbb{R}.$$

(See [22], and [24] for a more comprehensive treatment.) Denote by $\mathcal{C}^{1,2}((0,\infty)\times\mathbb{R})$ the collection of functions $(h_{t,x})_{t>0,x\in\mathbb{R}}$, which is continuously differentiable in $t$ and twice continuously differentiable in $x$. For every open $U\subset\mathbb{R}$, denote by $\mathcal{C}_{\mathrm{c}}(U)$ the collection of continuous function, whose support is a compact subset of $U$. According to [22], Theorem 4, for any closed set $A\subset\mathbb{R}$ and nonnegative Radon measure $\nu$ on $A^{\mathrm{c}}$, there exists a unique nonnegative $v^{(A,\nu)}\in\mathcal{C}^{1,2}((0,\infty)\times\mathbb{R})$ such that

$$\begin{cases}\partial_t v_{t,x}^{(A,\nu)}=\frac{1}{2}\partial_x^2 v_{t,x}^{(A,\nu)}-\frac{1}{2}\big(v_{t,x}^{(A,\nu)}\big)^2,\quad (t,x)\in(0,\infty)\times\mathbb{R};\\ \Big\{y\in\mathbb{R}:\forall r>0,\lim_{t\downarrow 0}\int_{y-r}^{y+r}v_{t,x}^{(A,\nu)}\,\mathrm{d}x=\infty\Big\}=A;\\ \lim_{t\downarrow 0}\int\phi_x v_{t,x}^{(A,\nu)}\,\mathrm{d}x=\int\phi_x\nu(\mathrm{d}x),\quad \phi\in\mathcal{C}_{\mathrm{c}}(A^{\mathrm{c}}).\end{cases}\tag{1.16}$$

Notice that the PDE (1.16) is a spatial analogy of the ODE (1.1). The pair $(A,\nu)$ is known as the *initial trace* of the solution $v^{(A,\nu)}$; see also [24] for more details.

The property of coming down from infinity of the coalescing Brownian motions with initial trace $(\Lambda,\mu)\in\mathcal{T}_{\mathrm{a}}$ is closely related to the solution $v^{(\Lambda,\mu)}$ of PDE (1.16). We observe this in our next result.

THEOREM 1.4. *Let* $(\Lambda,\mu)\in\mathcal{T}_{\mathrm{a}}$ *be arbitrary. Suppose that* $(Y_t)_{t>0}$ *is a coalescing Brownian motions process with initial trace* $(\Lambda,\mu)$ *living in a filtered probability space with probability* $\mathbb{P}_{(\Lambda,\mu)}$. *Then the following two statements hold for arbitrary open interval* $U\subset\mathbb{R}$:

(i) *If* $U\cap\operatorname{supp}(\Lambda,\mu)$ *is unbounded, then* $\mathbb{P}_{(\Lambda,\mu)}(Y_t(U)=\infty,\forall t>0)=1$.
(ii) *If* $U\cap\operatorname{supp}(\Lambda,\mu)$ *is bounded, then* $\mathbb{P}_{(\Lambda,\mu)}(Y_t(U)<\infty,\forall t>0)=1$.

*Moreover, in the latter case when* $U\cap\operatorname{supp}(\Lambda,\mu)$ *is bounded, the following three statements hold*:

(iii) $\mathbb{E}_{(\Lambda,\mu)}[Y_t(U)]<\infty$ *for every* $t>0$.
(iv) *If* $\overline{U}\cap\Lambda=\varnothing$, *then* $\limsup_{t\downarrow 0}\mathbb{E}_{(\Lambda,\mu)}[Y_t(U)]<\infty$.
(v) *If* $\overline{U}\cap\Lambda\neq\varnothing$, *then as* $t\downarrow 0$, $\mathbb{E}_{(\Lambda,\mu)}[Y_t(U)]\to\infty$ *and*

$$\Big(\int_U v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x\Big)^{-1}Y_t(U)\longrightarrow 1\quad \text{in } L^1 \text{ w.r.t. } \mathbb{P}_{(\Lambda,\mu)}.$$

*Here* $\overline{U}$ *is the closure of* $U$.

By taking $U = \mathbb{R}$, the above theorem says that the total population in a coalescing Brownian motions process is finite for all positive time, provided its initial trace $(\Lambda, \mu)$ is compactly supported. Conversely, if the initial trace is not compactly supported, then the total population remains infinite. This answers the first part of the question (1.2). Furthermore, in the case when the initial trace is compactly supported and $\Lambda \neq \varnothing$, the total population $(Y_t(\mathbb{R}))_{t>0}$ comes down from infinity, and its small times behavior is described "approximately" by $\int_{\mathbb{R}} v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x$. This answers the second part of the question (1.2). Let us also mention that even if the initial trace is not compactly supported, Theorem 1.4(ii) implies that the population is *locally* bounded at all positive times.

1.3. *Examples of rate functions.* Theorem 1.4 says that the total population comes down from infinity if and only if the initial trace $(\Lambda, \mu)$ is compactly supported and $\Lambda \neq \varnothing$. In this case the rate in which the total population comes down from infinity is given by the rate function $t \mapsto \|v_{t,\cdot}^{(\Lambda,\mu)}\|_{L^1(\mathbb{R})}$. It turns out that this rate function is related to the fractal structure of the set $\Lambda$. In this section we will give some examples where this rate can be explicitly calculated.

To present our result, we want to introduce the concept of Minkowski dimension. Let $\lambda$ denote Lebesgue measure on $\mathbb{R}$, and let $\#A$ denote the cardinality of a given set $A$. Define the Minkowski sum

$$A + \tilde{A} := \{a + \tilde{a} : a \in A, \tilde{a} \in \tilde{A}\}$$

for any subsets $A$ and $\tilde{A}$ of $\mathbb{R}$. Define

$$rA := \{ra : a \in A\}$$

for any $r \in \mathbb{R}$ and $A \subset \mathbb{R}$. Denote by $B^{\mathrm{o}} := (-1, 1)$ the unit open ball centered at the origin. For every $A \subset \mathbb{R}$, if there exists a (unique) $\delta \in [0, 1]$ such that

$$1 - (\log r)^{-1} \log \lambda\big(A + rB^{\mathrm{o}}\big) \to \delta$$

as $r \downarrow 0$, then we say $A$ has Minkowski dimension $\delta$. For many examples and applications on the Minkowski dimension, see [20].

The proof of the following Proposition is postponed in Section 4.

PROPOSITION 1.5. *Let $A$ be a compact subset of $\mathbb{R}$*:

1. *If $A$ has positive finite cardinality, then $\sqrt{t}\|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})}$ converges to $C_1\#A$ as $t \downarrow 0$, where $C_1 := \|v_{1,\cdot}^{(\{0\},\mathbf{0})}\|_{L^1(\mathbb{R})} \in (0, \infty)$.*
2. *If $A$ has positive Lebesgue measure, then $t\|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})}$ converges to $2\lambda(A)$ as $t \downarrow 0$.*
3. *If $A$ has Minkowski dimension $\delta \in (0, 1)$, then $(\log t)^{-1} \log \|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})}$ converges to $-(1 + \delta)/2$ as $t \downarrow 0$.*

EXAMPLE 1.6 (Infinitely many initial particles at one point). For instance, assume that there are infinitely many initial particles which are all located at the origin, that is, $x_i = 0$ for every $i \in \mathbb{N}$. In this case $\Lambda$ is the singleton $\{0\}$, and $\mu$ is the null measure $\mathbf{0}$ on $\mathbb{R}\backslash\{0\}$. Now, Theorem 1.4(v) and Proposition 1.5 says that that $\sqrt{t}Y_t(\mathbb{R})$ converges to $C_1$ in $L^1$ as $t \downarrow 0$.

REMARK. In Example 1.6 the total population $Y_t(\mathbb{R})$ is comparable to $C_1/\sqrt{t}$, which decreases even faster than the rate function $2/t$ of Kingman's coalescent. One might find this surprising since one might expect the spatial movement of the particles, if not slowing down the coalescing, should not speed it up. However, this is only an illusion, due to the different clocks used by the two models for their coalescing mechanisms—one uses the ordinary clock, while the other uses the local time. For a better comparison, one might choose to observe $(Y_t(\mathbb{R}))_{t\geq 0}$ according to the clock of the local time. Recall that, if $(L_t)_{t\geq 0}$ is the local time of a Brownian motion at zero, then $\mathbb{E}[L_t]=\sqrt{2t/\pi}$. Denote by $\tilde{N}_l$ the total population $Z_t(\mathbb{R})$ when the expected local time $\mathbb{E}[L_t]$ is at the level $l$. Now, from Example 1.6 we have that $\tilde{N}_l$ is of order $1/l$ for small $l$, which behaves similar to Kingman's coalescent.

EXAMPLE 1.7 ($\Lambda$ with positive Lebesgue measure). Suppose that $\Lambda$ is compact and has positive finite Lebesgue measure, and $\mu$ is the null measure supported on $\Lambda^c$. Then by Proposition 1.5(2), we have $tY_t(\mathbb{R})\xrightarrow{L^1} 2\lambda(\Lambda)$ as $t\downarrow 0$.

EXAMPLE 1.8 ($\Lambda$ with known Minkowski dimension). Suppose that $\Lambda$ is compact and has Minkowski dimension $\delta\in(0,1)$. Suppose also that $\mu$ is the null measure supported on $\Lambda^c$. Then by Proposition 1.5(3), we have $(\log t)^{-1}\log Y_t(\mathbb{R})$ converges to $-(1+\delta)/2$ in probability as $t\downarrow 0$.

1.4. *Proof strategy.* Instead of comparing coalescing Brownian motions with the PDE (1.16) directly, the idea is to consider their dual counterpart. There are two duality relations used in this paper. One is the moment duality (1.12), given by Shiga [28], between coalescing Brownian motions and the Wright–Fisher SPDE (1.10). The other is the classical duality established between the PDE (1.16) and the super-Brownian motion as well as the Brownian snake (see [22]). Brownian snake was introduced by Le Gall [21] to study the genealogical structure of the super-Brownian motion, whose density, in the one-dimensional case satisfies the following SPDE (see [19, 27]):

$$\partial_t\tilde{u}_{t,x}=\frac{1}{2}\partial_x^2\tilde{u}_{t,x}+\sqrt{\tilde{u}_{t,x}}\dot{W}_{t,x},\quad t>0,x\in\mathbb{R}. \tag{1.17}$$

The duality between the PDE (1.16) and the super-Brownian motion can be expressed as follows: for any nonnegative $\tilde{u}$ solving (1.17) with a deterministic initial value $\tilde{u}_{0,\cdot}$ and any locally bounded nonnegative function $f$ on $\mathbb{R}$, satisfying some mild conditions, one has

$$E\left[\exp\left(-\int\tilde{u}_{t,y}f(y)\,\mathrm{d}y\right)\right]=\exp\left(-\int\tilde{u}_{0,y}v_{t,y}^{(0,\mu)}\,\mathrm{d}y\right)\quad\forall t\geq 0,$$

where $\mu(dy)=f(y)\,\mathrm{d}y$. Observe that the SPDE (1.17) is quite similar to (1.11), in the sense that their noise coefficient are very close to each other for small values of $u$. In particular, one expects that the distribution of the nonnegative random fields $(u_{t,x})$ and $(\tilde{u}_{t,x})$ stay close to each other on finite time intervals if they share the same small initial conditions $u_0=\tilde{u}_0\ll 1$.

In the actual proof, we build a connection between Wright–Fisher SPDE (1.10) and the PDE (1.16) by searching for the Doob–Meyer decomposition of the process

$$s\mapsto\exp\left(-\int u_{s,y}v_{t-s,y}^{(\Lambda,\mu)}\,\mathrm{d}y\right),\quad s\in[0,t).$$

This idea was already explored by Tribe [30], where he established the compact support property of the Wright–Fisher SPDE (1.10). The challenge here is that it is not clear how to apply Itô's formula directly to the above process up to time $t$, because $v_{t-s,y}$ approaches a singular value when $s\to t$ and $y\in\Lambda$. So we will use a sequence of so-called nonsingular initial traces $\{(\varnothing,\mu^n):n\in\mathbb{N}\}$ to approximate the singular initial trace $(\Lambda,\mu)$ (trace with $\Lambda\neq\varnothing$), following the techniques developed in [24].

1.5. *Paper outline.* The rest of the paper is organized as follows. In Section 2 we give the proof of Theorems 1.1 and 1.2. In Section 3 we give the proof of Theorem 1.4. In Section 4 we give the proof of Proposition 1.5.

## 2. Proof of Theorems 1.1 and 1.2.

2.1. *Initial traces.* Let us first review some concepts and notations from [24]. Denote by $\mathcal{T}$ the collection of pairs $(A,\nu)$, where $A$ is a closed subset of $\mathbb{R}$ and $\nu$ is a nonnegative Radon measure on $A^{\mathrm{c}}$. Then $\mathcal{T}$ is the space of all the possible initial traces for the PDE (1.16). For any $(A,\nu)$ and $(\tilde{A},\tilde{\nu})$ in $\mathcal{T}$, we say $(A,\nu)\preceq(\tilde{A},\tilde{\nu})$ if $A\subset\tilde{A}$ and $\nu(B)\leq\tilde{\nu}(B)$ for every $B\in\mathscr{B}(\tilde{A}^{\mathrm{c}})$. Note that $\preceq$ is a partial order on $\mathcal{T}$. Denote by $\mathscr{M}_{\mathrm{reg}}^{+}$ the space of outer regular (not necessary locally bounded) Borel measures on $\mathbb{R}$. Define a map $\eta:(A,\nu)\mapsto\eta^{(A,\nu)}$ from $\mathcal{T}$ to $\mathscr{M}_{\mathrm{reg}}^{+}$ so that, for every $B\in\mathscr{B}(\mathbb{R})$,

$$\eta^{(A,\nu)}(B)=\begin{cases}\infty & B\cap A\neq\varnothing,\\ \nu(B) & B\cap A=\varnothing.\end{cases}\tag{2.1}$$

For any $\eta\in\mathscr{M}_{\mathrm{reg}}^{+}$, denote by $\mathcal{S}_\eta$ the set of singular points of $\eta$, that is,

$$\mathcal{S}_\eta:=\{x\in\mathbb{R}:\eta(U)=\infty\text{ for every open neighborhood }U\text{ of }x\},$$

and by $\mathcal{R}_\eta:=\mathcal{S}_\eta^c$ the set of regular points of $\eta$. For an $\mathscr{M}_{\mathrm{reg}}^{+}$-sequence $(\eta_n)_{n\in\mathbb{N}}$, we say it converges m-weakly to an $\eta\in\mathscr{M}_{\mathrm{reg}}^{+}$ if the following two conditions hold:

(2.2) If $U$ is an open subset of $\mathbb{R}$ with $\eta(U)=\infty$ then $\lim_{n\to\infty}\eta_n(U)=\infty$.

(2.3) For every compact $K\subset\mathcal{R}_\eta$, the sequence $(\eta_n(K))_{n\in\mathbb{N}}$ is eventually bounded and $\lim_{n\to\infty}\int\phi\,\mathrm{d}\eta_n=\int\phi\,\mathrm{d}\eta$ for every $\phi\in\mathcal{C}_{\mathrm{c}}(\mathcal{R}_\eta)$.

Let us now list some properties of the solutions to (1.16), given their initial traces. The following two statements can be verified from [24], Proposition 3.10 and [22], Theorem 4:

(2.4) For every $(A,\nu),(\tilde{A},\tilde{\nu})\in\mathcal{T}$, if $(A,\nu)\preceq(\tilde{A},\tilde{\nu})$, then $v_{t,x}^{(A,\nu)}\leq v_{t,x}^{(\tilde{A},\tilde{\nu})}$ for every $t>0$ and $x\in\mathbb{R}$.

(2.5) For any $\mathcal{T}$-sequence $((A_n,\nu_n))_{n\in\mathbb{N}}$ and $(A,\nu)\in\mathcal{T}$, if $\eta^{(A_n,\nu_n)}$ converges m-weakly to $\eta^{(A,\nu)}$, then $v_{t,x}^{(A_n,\nu_n)}$ converges to $v_{t,x}^{(A,\nu)}$ for every $t>0$ and $x\in\mathbb{R}$.

The following three statements can be verified directly from the uniqueness of the solution to (1.16):

(2.6) $v_{t,x}^{(\mathbb{R},\mathbf{0})}=2/t$ for every $t>0$ and $x\in\mathbb{R}$.

(2.7) $v_{t,x}^{(A,\mathbf{0})}=v_{t,x+z}^{(A+\{z\},\mathbf{0})}$ for every closed $A\subset\mathbb{R}$, $t>0$, $x\in\mathbb{R}$, and $z\in\mathbb{R}$.

(2.8) $v_{t,x}^{(A,\mathbf{0})}=v_{t,-x}^{(-A,\mathbf{0})}$ for every closed $A\subset\mathbb{R}$, $t>0$ and $x\in\mathbb{R}$.

According to [16], Proposition 3.2 & Lemma 3.4, there exists a constant $C_2\geq2$ such that

$$v_{t,x}^{((-\infty,0],\mathbf{0})}\leq\frac{C_2}{t}\left(1+\frac{x}{\sqrt{t}}\right)e^{-\frac{x^2}{2t}},\qquad t>0,x\geq0.\tag{2.9}$$

Using (2.4), (2.6), (2.7), (2.8), and (2.9), it is straightforwad to verify that

$$v_{t,x}^{([-k,k],\mathbf{0})}\leq\frac{C_2}{t}\left(1+\frac{d(x,[-k,k])}{\sqrt{t}}\right)e^{-\frac{d(x,[-k,k])^2}{2t}},\qquad k\geq0,t>0,x\in\mathbb{R},\tag{2.10}$$

where $d(x,[-k,k]):=\inf\{|x-z|:z\in[-k,k]\}$.

2.2. *Proof of Theorem* 1.1(*i*). For two lists of real numbers $(x_i)_{i\in I_0}\in\mathcal{X}$ and $(\tilde{x}_i)_{i\in\tilde{I}_0}\in\mathcal{X}$, we say $(\tilde{x}_i)_{i\in\tilde{I}_0}$ is a sublist of $(x_i)_{i\in I_0}$ if there is an strictly increasing map $\iota$ from $\tilde{I}_0$ to $I_0$ so that $\tilde{x}_i=x_{\iota(i)}$ for every $i\in\tilde{I}_0$.

LEMMA 2.1. *Let $(\tilde{x}_i)_{i\in\tilde{I}_0}$ be a sublist of $(x_i)_{i\in I_0}$. Suppose that $\tilde{\mathbf{X}} = \{\tilde{X}_t^{(i)} : t \geq 0, i \in \tilde{I}_0\}$ is a coalescing Brownian particle system with initial configuration $(\tilde{x}_i)_{i\in\tilde{I}_0}$. For every $U \in \mathscr{B}(\mathbb{R})$ and $t \geq 0$, denote by $\tilde{Z}_t(U)$ the number of living particles contained in $U$ at time $t$ in the particle system $\tilde{\mathbf{X}}$. Then for every $U \in \mathscr{B}(\mathbb{R})$, the process $(\tilde{Z}_t(U))_{t\geq 0}$ is stochastically dominated by the process $(Z_t(U))_{t\geq 0}$.*

Lemma 2.1 can be verified by a straightforward coupling method. We omit the details.

PROOF OF THEOREM 1.1(i). We construct another particle system that is easily seen to have infinitely many particles in $U$ for all time, and is dominated by that of coalescing Brownian motions. Notice that $\mathrm{supp}(\Lambda, \mu)$ is also the support of the set $\{x_i : i \in I_0\}$. The condition of the theorem implies that $\mathrm{supp}(\Lambda, \mu)$ is unbounded, and therefore there are infinitely many initial particles, that is, $I_0 = \mathbb{N}$. Since $U$ is an interval and $U \cap \mathrm{supp}(\Lambda, \mu)$ is unbounded, there exists a subsequence $(\tilde{x}_i)_{i\in\mathbb{N}}$ of $(x_i)_{i\in\mathbb{N}}$ such that $\{(\tilde{x}_i - 2, \tilde{x}_i + 2) : i \in \mathbb{N}\}$ is a family of disjoint subsets of $U$. Let $\tilde{\mathbf{X}}$ be a coalescing Brownian particle system with initial configuration $(\tilde{x}_i)_{i\in\mathbb{N}}$. Let $\tilde{Z}_t(U)$ be the number of the particles contained in $U$ at time $t \geq 0$ in the particle system $\tilde{\mathbf{X}}$. From Lemma 2.1, $(\tilde{Z}_t(U))_{t\geq 0}$ is stochastically dominated by $(Z_t(U))_{t\geq 0}$. Now, we only have to show that $\tilde{Z}_t(U) = \infty$, for every $t \geq 0$, almost surely.

To do that, we want to construct another Brownian particle system. Let $\{(\hat{B}_t^{(i)})_{t\geq 0} : i \in \mathbb{N}\}$ be a sequence of independent Brownian motions such that $\hat{B}_0^{(i)} = \tilde{x}_i$ for each $i \in \mathbb{N}$. Define $\hat{\zeta}_i := \inf\{t \geq 0 : |\hat{B}_t^{(i)} - \hat{B}_0^{(i)}| \geq 1\}$ and

$$\hat{X}_t^{(i)} = \begin{cases} \hat{B}_t^{(i)} & \text{if } t \in [0, \hat{\zeta}_i), \\ \dagger & \text{otherwise,} \end{cases}$$

for each $i \in \mathbb{N}$. Now, $\hat{\mathbf{X}} := \{\hat{X}_t^{(i)} : t \geq 0, i \in \mathbb{N}\}$ is a system of independent Brownian particles, where each particle is killed when the distance it travels reaches 1, that is, the $i$th particle is killed at time $\hat{\zeta}_i$. Denote by $\hat{Z}_t(U)$ the number of the particles contained in $U$ at time $t \geq 0$ in the particle system $\hat{\mathbf{X}}$. Since the initial particles in $\hat{\mathbf{X}}$ are located away from each other with distance at least 4, each particle in $\hat{\mathbf{X}}$ is killed before it can meet with any other particles. Therefore, $(\hat{Z}_t(U))_{t\geq 0}$ is stochastically dominated by $(\tilde{Z}_t(U))_{t\geq 0}$.

To finish the proof, we only have to show that $\hat{Z}_t(U) = \infty$ for every $t \geq 0$ almost surely. First, observe that $\hat{Z}_t(U)$ is the total population of the system $\hat{\mathbf{X}}$, since for every $i \in \mathbb{N}$, particle $i$ are killed before it can leave $[x_i - 1, x_i + 1] \subset U$. In particular, we have $\hat{Z}_t(U) = \sum_{i\in\mathbb{N}} \mathbf{1}_{\{\hat{\zeta}_i > t\}}$, which is a nonincreasing process in $t \geq 0$. Also, observe that $(\hat{\zeta}_i)_{i\in\mathbb{N}}$ is a family of i.i.d. random variables with $\mathbb{P}(\hat{\zeta}_i > t) > 0$ for every $t \geq 0$. Now, the desired result follows from the second Borel–Cantelli lemma. □

2.3. *Proof of Theorem 1.1(ii).* Recall that $\mathbf{X} = \{X_t^{(i)} : t \geq 0, i \in I_0\}$ is a coalescing Brownian particle system with initial configuration $(x_i)_{i\in I_0}$ and initial trace $(\Lambda, \mu) = \Psi((x_i)_{i\in I_0})$. If there are finitely many initial particles, that is, $\#I_0 < \infty$, then the statement in Theorem 1.1(ii) holds automatically because $Z_t(U) \leq \#I_0$ by definition. Therefore, in the rest of this subsection, we assume that there are infinitely many initial particles, that is, $I_0 = \mathbb{N}$.

Let $\theta$ be a nonnegative increasing continuous function from $[0, 1)$ to $[1, \infty)$ such that $\theta(0) = 1$ and $\theta(\gamma) = -\log(1 - \gamma)/\gamma$ for every $\gamma \in (0, 1)$. Let us state several lemmas that will be proved later in this subsection.

LEMMA 2.2. *Let $U \subset \mathbb{R}$ be an open interval and $\varepsilon \in (0, 1)$. Let $(u_{t,x})_{t\geq 0, x\in\mathbb{R}}$ be the solution to the Wright–Fisher SPDE (1.10) with initial condition $u_0 = \varepsilon \mathbf{1}_U$. Let $F$ be a closed* *interval containing the set* $\{x_i : i \in \mathbb{N}\}$. *Then for any* $t > 0$ *and* $\gamma \in (\varepsilon, 1)$, *we have*

$$\mathbb{E}\big[(1-\varepsilon)^{Z_t(U)}\big] \geq \mathbf{E}_{\varepsilon \mathbf{1}_U}\left[\exp\left(-\theta(\gamma)\sum_{i=1}^{\infty} u_{t,x_i}\right)\right] - \mathbf{P}_{\varepsilon \mathbf{1}_U}\Big(\sup_{s \leq t, y \in F} u_{s,y} > \gamma\Big), \tag{2.11}$$

*and*

$$\mathbb{E}\big[(1-\varepsilon)^{Z_t(U)}\big] \leq \mathbf{E}_{\varepsilon \mathbf{1}_U}\left[\exp\left(-\sum_{i=1}^{\infty} u_{t,x_i}\right)\right]. \tag{2.12}$$

For every $t > 0$, closed interval $F$, and $(A, \nu) \in \mathcal{T}$, define

$$\mathcal{V}_t^{(A,\nu,F)} := \int_0^t \int_{F^c} \big(v_{r,z}^{(A,\nu)}\big)^2 \, \mathrm{d}z \, \mathrm{d}r, \tag{2.13}$$

where $v$ satisfies (1.16).

LEMMA 2.3. *Let* $t > 0$:

1. *If* $U$ *is an open interval so that* $U \cap \{x_i : i \in \mathbb{N}\}$ *is bounded, then* $\int_U v_{t,y}^{(\Lambda,\mu)} \, \mathrm{d}y$ *is finite.*
2. *If* $F$ *is a closed interval containing* $\bigcup_{i \in \mathbb{N}}(x_i - 1, x_i + 1)$, *then* $\mathcal{V}_t^{(\Lambda,\mu,F)}$ *is finite.*

LEMMA 2.4. *Let* $U$ *be an open interval. Let* $F$ *be a closed interval containing* $\bigcup_{i \in \mathbb{N}}(x_i - 1, x_i + 1)$. *Let* $t > 0$ *and* $0 \leq \varepsilon \leq \gamma < 1$. *Suppose that* $(u_{t,x})_{t \geq 0, x \in \mathbb{R}}$ *is a solution to the Wright–Fisher SPDE* (1.10) *with initial condition* $u_0 = \varepsilon \mathbf{1}_U$. *Then*

$$\begin{aligned}\mathbf{E}_{\varepsilon \mathbf{1}_U}&\left[\exp\left(-\theta(\gamma)\sum_{i=1}^{\infty} u_{t,x_i}\right)\right] \\ &\geq \exp\left(-\frac{\varepsilon}{1-\gamma}\int_U v_{t,y}^{(\Lambda,\mu)} \, \mathrm{d}y\right) - \frac{\varepsilon}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)} - \mathbf{P}_{\varepsilon \mathbf{1}_U}\Big(\sup_{s \leq t, y \in F} u_{s,y} > \gamma\Big),\end{aligned} \tag{2.14}$$

*and*

$$\mathbf{E}_{\varepsilon \mathbf{1}_U}\left[\exp\left(-\sum_{i=1}^{\infty} u_{t,x_i}\right)\right] \leq \exp\left(-\varepsilon \int_U v_{t,y}^{(\Lambda,\mu)} \, \mathrm{d}y\right). \tag{2.15}$$

LEMMA 2.5. *Let* $U$ *be an open interval, and* $F$ *be a closed interval. Let* $t > 0$ *and* $0 < \gamma < 1$. *Suppose that* $U \cap F$ *is bounded. Then*

$$C_3(U, F, t, \gamma) := \sup_{\varepsilon \in (0, \gamma/2)} \frac{1}{\varepsilon}\mathbf{P}_{\varepsilon \mathbf{1}_U}\Big(\sup_{s \leq t, y \in F} u_{s,y} > \gamma\Big) < \infty.$$

We use the above lemmas to prove the next proposition.

PROPOSITION 2.6. *Suppose that* $F$ *is a closed interval containing* $\bigcup_{i \in \mathbb{N}}(x_i - 1, x_i + 1)$. *Suppose that* $U$ *is an open interval such that* $U \cap F$ *is bounded. Then for any* $t > 0$ *and* $\gamma \in (0, 1)$,

$$\mathbb{E}\big[Z_t(U)\big] \leq \frac{1}{1-\gamma}\left[\int_U v_{t,y}^{(\Lambda,\mu)} \, \mathrm{d}y + \frac{1}{2}\mathcal{V}_t^{(\Lambda,\mu,F)}\right] + 2C_3(U, F, t, \gamma) < \infty. \tag{2.16}$$

PROOF. Let $\varepsilon \in (0, \gamma/2)$ be arbitrary and $(u_{t,x})_{t\geq 0, x\in\mathbb{R}}$ be a solution to the Wright–Fisher SPDE (1.10) with initial condition $u_0 = \varepsilon \mathbf{1}_U$. From the condition of the proposition, it is easy to see that $U \cap \{x_i : i \in \mathbb{N}\}$ is bounded. By Lemmas 2.2, 2.4 and 2.5, we have

$$\mathbb{E}\big[(1-\varepsilon)^{Z_t(U)}\big] \geq \exp\biggl(-\frac{\varepsilon}{1-\gamma}\int_U v_{t,y}^{(\Lambda,\mu)}\,\mathrm{d}y\biggr) - \frac{\varepsilon}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)} - 2\varepsilon C_3(U, F, t, \gamma).$$

We extract the first moment of $Z_t(U)$ by taking the derivative of its moment generating function, that is,

$$\begin{aligned}\mathbb{E}\big[Z_t(U)\big] &= \lim_{\varepsilon\to 0^+}\frac{1-\mathbb{E}((1-\varepsilon)^{Z_t(U)})}{\varepsilon}\\ &\leq \frac{1}{1-\gamma}\biggl[\int_U v_{t,y}^{(\Lambda,\mu)}\,\mathrm{d}y + \frac{1}{2}\mathcal{V}_t^{(\Lambda,\mu,F)}\biggr] + 2C_3(U, F, t, \gamma),\end{aligned}$$

which is finite for every $t > 0$ by Lemmas 2.3 and 2.5. $\square$

REMARK 2.7. Let $U$ be an open interval such that $U \cap \operatorname{supp}(\Lambda, \mu)$ is bounded. Take $F$ to be the smallest closed interval containing $\bigcup_{i\in\mathbb{N}}(x_i - 1, x_i + 1)$. Since $\operatorname{supp}(\Lambda, \mu) = \operatorname{cl}(\{x_i : i \in \mathbb{N}\})$, where $\operatorname{cl}(A)$ denotes the closure of a set $A$, it is clear that $U \cap F$ is bounded. Now, the above proposition implies that, for fixed $t > 0$, $\mathbb{E}[Z_t(U)] < \infty$ and, therefore, $\mathbb{P}(Z_t(U) < \infty) = 1$.

For any open subset $G$ of $[0, \infty) \times \mathbb{R}$, let us define random variable

$$\mathcal{Z}(G) := \sum_{i\in I_0}\mathbf{1}_{\{\exists t>0 \text{ s.t. } (t, X_t^{(i)})\in G\}},$$

which is the number of particles whose time-space trajectory intersects $G$.

PROPOSITION 2.8. *Let $U$ be an open interval such that $U \cap \operatorname{supp}(\Lambda, \mu)$ is bounded. Let $0 < \delta < T < \infty$ be arbitrary. Then $\mathbb{E}[\mathcal{Z}((\delta, T) \times U)] < \infty$.*

PROOF. Let us consider covering the space $\mathbb{R}$ by open intervals $L_j := (j - 1, j + 1)$, $j \in \mathbb{N}$ and show that $\mathbb{E}[Z_\delta(L_j)]$ is uniformly bounded in $j$. In fact, fixing arbitrary $0 < \gamma < 1$ and $j \in \mathbb{N}$, by Proposition 2.6 we have

$$\mathbb{E}\big[Z_\delta(L_j)\big] \leq \frac{1}{1-\gamma}\biggl[\int_{j-1}^{j+1} v_{\delta,y}^{(\Lambda,\mu)}\,\mathrm{d}y + \frac{1}{2}\mathcal{V}_\delta^{(\Lambda,\mu,\mathbb{R})}\biggr] + 2C_3(L_j, \mathbb{R}, \delta, \gamma).$$

Notice that, by (2.4) and (2.6),

$$\int_{j-1}^{j+1} v_{\delta,y}^{(\Lambda,\mu)}\,\mathrm{d}y \leq \int_{j-1}^{j+1} v_{\delta,y}^{(\mathbb{R},\mathbf{0})}\,\mathrm{d}y = \frac{4}{\delta},$$

and that $\mathcal{V}_\delta^{(\Lambda,\mu,\mathbb{R})} = 0$. It is also clear from the definition of $C_3$ in Lemma 2.5 that

$$C_3(L_j, \mathbb{R}, \delta, \gamma) = C_3(L_0, \mathbb{R}, \delta, \gamma).$$

Therefore, we have

$$\mathbb{E}\big[Z_\delta(L_j)\big] \leq \frac{4}{(1-\gamma)\delta} + 2C_3(L_0, \mathbb{R}, \delta, \gamma) =: C_4(\delta, \gamma).$$

Define $\beta_t^{(i)} := B_{\delta+t}^{(i)} - B_\delta^{(i)}$ for every $i \in \mathbb{N}, t \geq 0$. From the Markov property of Brownian motions, $\{(\beta_t^{(i)})_{t\geq 0} : i \in \mathbb{N}\}$ is a family of independent Brownian motions which are independent of $(B_\delta^{(i)})_{i\in\mathbb{N}}$. Therefore,

$$N_j := Z_\delta(L_j) = \#\big\{i \in \mathbb{N} : \zeta_i > \delta, B_\delta^{(i)} \in L_j\big\}, \quad j \in \mathbb{N}$$

is independent of

$$M^{(i)}_{T-\delta} := \sup_{t\in[0,T-\delta]} \left|\beta^{(i)}_t\right|, \quad i \in \mathbb{N}.$$

One can verify the following inequality:

$$\mathcal{Z}\big((\delta,T)\times U\big) \le Z_\delta(U) + \sum_{j\in\mathbb{N}:L_j\not\subset U}\ \sum_{i\in\mathbb{N}:\delta>\zeta_i, B^{(i)}_\delta\in L_j} \mathbf{1}_{\{M^{(i)}_{T-\delta}\ge \operatorname{dist}(L_j,U)\}}.$$

Here $\operatorname{dist}(L_j,U)$ denotes the distance between sets $L_j$ and $U$, which is also the distance a Brownian particle at least needs to travel from $L_j$ to $U$. Taking expectations, we have

$$\mathbb{E}\big[\mathcal{Z}\big((\delta,T)\times U\big)\big] \le \mathbb{E}\big[Z_\delta(U)\big] + \sum_{j\in\mathbb{N}:L_j\not\subset U} \mathbb{E}\Bigg[\sum_{i\in\mathbb{N}:\delta>\zeta_i, B^{(i)}_\delta\in L_j} \mathbf{1}_{\{M^{(i)}_{T-\delta}\ge \operatorname{dist}(L_j,U)\}}\Bigg].$$

It follows from Proposition 2.6 and Remark 2.7 that the first expectation on the right-hand side is finite. Now, from the independence of $(N_j)_{j\in\mathbb{N}}$ with the collection of the i.i.d. random variables $(M^{(i)}_{T-\delta})_{i\in\mathbb{N}}$, we can apply Wald's lemma to calculate the second expectation on the right-hand side. Thus,

$$\begin{aligned}\mathbb{E}\big[\mathcal{Z}\big((\delta,T)\times U\big)\big] &\le \mathbb{E}\big[Z_\delta(U)\big] + \sum_{j\in\mathbb{N}:L_j\not\subset U} \mathbb{E}[N_j]\cdot\mathbb{P}\big(M^{(0)}_{T-\delta}\ge \operatorname{dist}(L_j,U)\big)\\ &\le \mathbb{E}\big[Z_\delta(U)\big] + C_4(\delta,\gamma)\sum_{j\in\mathbb{N}:L_j\not\subset U} C_5(T-\delta)\exp\biggl(-\frac{\operatorname{dist}(L_j,U)^2}{2(T-\delta)}\biggr) < \infty,\end{aligned}$$

as desired. Here $C_5(T-\delta)$ is a finite constant only depending on $T-\delta$. $\square$

PROOF OF THEOREM 1.1(ii). Clearly, $Z_t(U) \le \mathcal{Z}((\delta,T)\times U)$, for every $t\in(\delta,T)$, almost surely. Thus, it suffices to show that $\mathbb{P}(\mathcal{Z}((\delta,T)\times U)<\infty)=1$ for arbitrarily fixed $0<\delta<T$. Now, the desired result follows from Proposition 2.8. $\square$

Let us now give the proofs of Lemmas 2.2–2.5.

PROOF OF LEMMA 2.2. Noticing that, for every $n\in\mathbb{N}$, $\mathbf{X}^{(n)} := \{X^{(i)}_t : t\ge 0, i = 1,\dots,n\}$ is a coalescing Brownian particle system with initial configuration $(x_i)^n_{i=1}$. Denote by $I^{(n)}_t$ the random set of the indexes of the particles alive at a given time $t\ge 0$ in the particle system $\mathbf{X}^{(n)}$. From Shiga's duality [28], Theorem 5.2,

$$\mathbb{E}\Bigg[\prod_{i\in I^{(n)}_t}\big(1-\varepsilon\mathbf{1}_U\big(X^{(i)}_t\big)\big)\Bigg] = \mathbf{E}_{\varepsilon\mathbf{1}_U}\Bigg[\prod_{i=1}^n(1-u_{t,x_i})\Bigg], \quad n\in\mathbb{N}.$$

Taking $n\uparrow\infty$, we get by monotone convergence theorem that

$$\mathbb{E}\Bigg[\prod_{i\in I_t}\big(1-\varepsilon\mathbf{1}_U\big(X^{(i)}_t\big)\big)\Bigg] = \mathbf{E}_{\varepsilon\mathbf{1}_U}\Bigg[\prod_{i=1}^\infty(1-u_{t,x_i})\Bigg].$$

We can rewrite the above as

$$\mathbb{E}\big[(1-\varepsilon)^{Z_t(U)}\big] = \mathbf{E}_{\varepsilon\mathbf{1}_U}\Bigg[\exp\Bigg(\sum_{i=1}^\infty \log(1-u_{t,x_i})\Bigg)\Bigg].$$

On one hand, from the fact that $\log(1-z)\le -z$ for every $z\in[0,1]$, we have (2.12) holds, as desired. On the other hand, from the fact that $z\mapsto\theta(z) = -\log(1-z)/z$ is a positive increasing function on $(0,1)$, we can verify that $-\log(1-z) \le \theta(\gamma)z$ for every $z \in [0,\gamma]$. Note that almost surely on the event $\{\sup_{s\le t, y\in F} u_{s,y} \le \gamma\}$, for every $i \in \mathbb{N}$, we have $u_{t,x_i} \in [0,\gamma]$ and, therefore, $-\log(1-u_{t,x_i}) \le \theta(\gamma)u_{t,x_i}$. So we can verify that

$$\begin{aligned}\mathbb{E}\big[(1-\varepsilon)^{Z_t(U)}\big] &\ge \mathbf{E}_{\varepsilon \mathbf{1}_U}\left[\exp\left(-\theta(\gamma)\sum_{i=1}^{\infty} u_{t,x_i}\right)\mathbf{1}_{\{\sup_{s\le t, y\in F} u_{s,y}\le\gamma\}}\right] \\ &\ge \mathbf{E}_{\varepsilon \mathbf{1}_U}\left[\exp\left(-\theta(\gamma)\sum_{i=1}^{\infty} u_{t,x_i}\right)\right] - \mathbf{P}_{\varepsilon \mathbf{1}_U}\Big(\sup_{s\le t, y\in F} u_{s,y} > \gamma\Big),\end{aligned}$$

as desired. □

PROOF OF LEMMA 2.3. Notice that, for the statement (2), we only have to show that $\mathcal{V}_t^{(\Lambda,\mu,F)}$ is finite when $F$ is the *smallest* closed interval containing $\bigcup_{i\in\mathbb{N}}(x_i-1, x_i+1)$. So let us denote by $F$ this smallest closed interval, and by $\tilde{F}$ the smallest closed interval containing $\{x_i : i \in \mathbb{N}\}$. From the condition that $U \cap \{x_i : i \in \mathbb{N}\}$ is bounded, it is easy to see that $U \cap \tilde{F}$ is bounded. It is also clear that $(\Lambda,\mu) \preceq (\tilde{F}, \mathbf{0})$. There are four cases to consider, for which we derive both statements (1) and (2):

*Case (1), $\tilde{F} = \mathbb{R}$*: In this case $U$ must be a bounded interval, say $(\alpha,\beta)$. From (2.6) we know that $\int_U v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x \le 2(\beta-\alpha)/t$ is finite. It is also obvious that $\mathcal{V}_t^{(\Lambda,\mu,\mathbb{R})} = 0$ is finite.

*Case (2), $\tilde{F} = (-\infty, b]$ for some $b \in \mathbb{R}$*: In this case $F = (-\infty, b+1]$, and $U$ must be the subset of $(\alpha,\infty)$ for some $\alpha \in \mathbb{R}$. From (2.4) and (2.7), we have

$$\int_U v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x \le \int_\alpha^\infty \big(v_{t,x}^{(\mathbb{R},\mathbf{0})}\mathbf{1}_{x<b} + v_{t,x-b}^{((-\infty,0],\mathbf{0})}\mathbf{1}_{x\ge b}\big)\,\mathrm{d}x.$$

One can verify that the integral on the right-hand side is finite using (2.6) and (2.9). One can also verify that

$$\mathcal{V}_t^{(\Lambda,\mu,F)} \le \int_0^t \int_{b+1}^\infty \big(v_{r,z-b}^{((-\infty,0],\mathbf{0})}\big)^2\,\mathrm{d}z\,\mathrm{d}r,$$

where the right-hand side is finite by (2.9).

*Case (3), $\tilde{F} = [a,\infty)$ for some $a \in \mathbb{R}$*: This is similar to Case 2, thanks to the spatial symmetry of the PDE (1.16).

*Case (4), $\tilde{F} = [a,b]$ for some $-\infty < a < b < \infty$*: Now, $F = [a-1, b+1]$. From (2.4) and (2.7), we have

$$\int_U v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x \le \int v_{t,x-\frac{a+b}{2}}^{([-\frac{b-a}{2},\frac{b-a}{2}],\mathbf{0})}\,\mathrm{d}x.$$

One can verify that the integral on the right-hand side is finite using (2.10). One can also verify that

$$\mathcal{V}_t^{(\Lambda,\mu,F)} \le \int_0^t \left(\int_{-\infty}^{a-1} + \int_{b+1}^\infty\right)\big(v_{r,z-\frac{a+b}{2}}^{([-\frac{b-a}{2},\frac{b-a}{2}],\mathbf{0})}\big)^2\,\mathrm{d}z\,\mathrm{d}r,$$

where the right-hand side is finite by (2.10). □

PROOF OF LEMMA 2.4. Let $n \in \mathbb{N}$ be arbitrary. Define a stopping time,

$$\tau := \inf\Big\{s \ge 0 : \sup_{y\in F} u_{s,y} > \gamma\Big\},$$

and a process,

$$M_s^{(\gamma,n)} := \begin{cases} \dfrac{1}{1-\gamma}\displaystyle\int u_{s,y} v_{t-s,y}^{(\varnothing,\mu^{(\gamma,n)})}\,\mathrm{d}y & s\in[0,t), \\ \dfrac{1}{1-\gamma}\displaystyle\int u_{t,y}\mu^{(\gamma,n)}(\mathrm{d}y) & s=t, \end{cases}$$

where $\mu^{(\gamma,n)}(\mathrm{d}y)=(1-\gamma)\theta(\gamma)\sum_{i=1}^n \delta_{x_i}$. One can show that $\{M_s^{(\gamma,n)} : s\in[0,t]\}$ is a continuous semimartingale. In fact, using the stochastic Fubini's theorem for space-time white noise (c.f. [17], Lemma 2.4), it is standard to verify that almost surely, for every $s\in[0,t]$,

$$\begin{aligned} &M_s^{(\gamma,n)} - M_0^{(\gamma,n)} \\ &\quad = \frac{1}{1-\gamma}\biggl(\frac12\iint_0^s \bigl(v_{t-r,z}^{(\varnothing,\mu^{(\gamma,n)})}\bigr)^2 u_{r,z}\,\mathrm{d}r\,\mathrm{d}z + \iint_0^s v_{t-r,z}^{(\varnothing,\mu^{(\gamma,n)})}\sqrt{u_{r,z}(1-u_{r,z})}W(\mathrm{d}r\,\mathrm{d}z)\biggr). \end{aligned}$$

By Ito's formula we can verify that, almost surely for every $s\in[0,t]$,

$$\begin{aligned} &e^{-M_s^{(\gamma,n)}} - e^{-M_0^{(\gamma,n)}} \\ &\quad = -\frac{1}{2(1-\gamma)^2}\iint_0^s e^{-M_r^{(\gamma,n)}}\bigl(v_{t-r,z}^{(\varnothing,\mu^{(\gamma,n)})}\bigr)^2\bigl(u_{r,z}^2-\gamma u_{r,z}\bigr)\,\mathrm{d}r\,\mathrm{d}z \\ &\qquad - \frac{1}{1-\gamma}\iint_0^s e^{-M_r^{(\gamma,n)}} v_{t-r,z}^{(\varnothing,\mu^{(\gamma,n)})}\sqrt{u_{r,z}(1-u_{r,z})}W(\mathrm{d}r\,\mathrm{d}z). \end{aligned} \tag{2.17}$$

To show (2.14), let us take the expectation on (2.17), while setting $s=\tau\wedge t$. Notice that

$$\mathbf{E}_{\varepsilon\mathbf{1}_U}\biggl[\iint_0^{\tau\wedge t} e^{-M_r^{(\gamma,n)}} v_{t-r,z}^{(\varnothing,\mu^{(\gamma,n)})}\sqrt{u_{r,z}(1-u_{r,z})}W(\mathrm{d}r\,\mathrm{d}z)\biggr]=0$$

and

$$\begin{aligned} &\iint_0^{\tau\wedge t} e^{-M_r^{(\gamma,n)}}\bigl(v_{t-r,z}^{(\varnothing,\mu^{(\gamma,n)})}\bigr)^2\bigl(u_{r,z}^2-\gamma u_{r,z}\bigr)\,\mathrm{d}r\,\mathrm{d}z \\ &\quad \le (1-\gamma)\int_0^t \mathrm{d}r\int_{F^{\mathrm{c}}}\bigl(v_{t-r,z}^{(\varnothing,\mu^{(\gamma,n)})}\bigr)^2 u_{r,z}\,\mathrm{d}z \quad \text{a.s.} \end{aligned}$$

The last inequality follows by the observation that $u_{r,z}^2-\gamma u_{r,z}\le 0$ for $r\le\tau$ and $z\in F$. By the above and noticing that from (1.11) that $\mathbf{E}_{\varepsilon\mathbf{1}_U}[u_{r,z}]\le\varepsilon$, for every $r\ge 0$, $z\in\mathbb{R}$, we get

$$\mathbb{E}\bigl[e^{-M_{\tau\wedge t}^{(\gamma,n)}}\bigr]-e^{-M_0^{(\gamma,n)}} \ge -\frac{\varepsilon}{2(1-\gamma)}\int_0^t\mathrm{d}r\int_{F^{\mathrm{c}}}\bigl(v_{r,z}^{(\varnothing,\mu^{(\gamma,n)})}\bigr)^2\,\mathrm{d}z.$$

Now, we have

$$\begin{aligned} \mathbb{E}\bigl[e^{-M_t^{(\gamma,n)}}\bigr]+\mathbb{P}(\tau\le t) &\ge \mathbb{E}\bigl[e^{-M_{\tau\wedge t}^{(\gamma,n)}};\tau>t\bigr]+\mathbb{E}\bigl[e^{-M_{\tau\wedge t}^{(\gamma,n)}};\tau\le t\bigr] \\ &\ge e^{-M_0^{(\gamma,n)}}-\frac{\varepsilon}{2(1-\gamma)}\int_0^t\mathrm{d}r\int_{F^{\mathrm{c}}}\bigl(v_{r,z}^{(\varnothing,\mu^{(\gamma,n)})}\bigr)^2\,\mathrm{d}z. \end{aligned} \tag{2.18}$$

We want to take the limit in (2.18), as $n\to\infty$. Notice that the $\mathcal{T}$-sequence $((\varnothing,\mu^{(\gamma,n)}))_{n\in\mathbb{N}}$ is increasing with respect to the partial order $\preceq$. Recall the map $\eta$, given by (2.1). Define $\mu^{(\gamma)} := \eta^{(\Lambda,(1-\gamma)\theta(\gamma)\mu)}$. From (1.8) and (1.9), it is straightforward to verify that $(\mu^{(\gamma,n)})_{n\in\mathbb{N}}$ converges to $\mu^{(\gamma)}$ m-weakly as $n\to\infty$. Therefore, from (2.4) and (2.5), the increasing sequence $(v_{r,z}^{(\varnothing,\mu^{(\gamma,n)})})_{n\in\mathbb{N}}$ converges to $v_{r,z}^{(\Lambda,(1-\gamma)\theta(\gamma)\mu)}$ for every $r>0$ and $z\in\mathbb{R}$. From (2.4) and the fact that $(1-\gamma)\theta(\gamma)\le 1$, we have $v_{r,z}^{(\Lambda,(1-\gamma)\theta(\gamma)\mu)}\le v_{r,z}^{(\Lambda,\mu)}$ for every $r>0$ and

$z \in \mathbb{R}$. Taking the limit in (2.18) as $n \uparrow \infty$, we can now verify from the monotone convergence theorem that

$$
\begin{aligned}
&M_t^{(\gamma,n)} \longrightarrow \theta(\gamma)\sum_{i=1}^{\infty} u_{t,x_i},\\
&M_0^{(\gamma,n)} \longrightarrow \frac{\varepsilon}{1-\gamma}\int_U v_{t,y}^{(\Lambda,(1-\gamma)\theta(\gamma)\mu)}\,\mathrm{d}y \le \frac{\varepsilon}{1-\gamma}\int_U v_{t,y}^{(\Lambda,\mu)}\,\mathrm{d}y,\\
&\int_0^t \mathrm{d}r\int_{F^{\mathrm{c}}} \bigl(v_{r,z}^{(\varnothing,\mu^{(\gamma,n)})}\bigr)^2\,\mathrm{d}z \longrightarrow \mathcal{V}_t^{(\Lambda,(1-\gamma)\theta(\gamma)\mu,F)} \le \mathcal{V}_t^{(\Lambda,\mu,F)},
\end{aligned}
\tag{2.19}
$$

which implies the desired (2.14).

To show (2.15), let us take the expectation on (2.17) while replacing $s$ by $t$ and the arbitrary $\gamma \in [0,1)$ by 0. This gives us

$$
\mathbb{E}\bigl[e^{-M_t^{(0,n)}}\bigr] \le e^{-M_0^{(0,n)}}. \tag{2.20}
$$

Taking the arbitrary $n \uparrow \infty$, we can get the desired (2.15) from (2.19) and (2.20). □

PROOF OF LEMMA 2.5. Take $\varepsilon < \gamma/2$, and assume that $(u_{t,x})_{t\ge 0, x\in\mathbb{R}}$ solves the Wright–Fisher SPDE (1.10) with initial condition $u_{0,\cdot} = \varepsilon\mathbf{1}_U$. From (1.11) we have that

$$
u_{s,y} \le \varepsilon + N_s(y), \quad s \ge 0,\ y \in \mathbb{R}, \tag{2.21}
$$

where

$$
N_s(y) := \iint_0^s G_{s-r,y-z}\sqrt{u_{r,z}(1-u_{r,z})}\,W(\mathrm{d}r\,\mathrm{d}z).
$$

Since $\varepsilon < \gamma/2$, we see from (2.21) that $u_{s,y}$ is less than $\gamma$ if $|N_s(y)|$ is less than $\gamma/2$. Hence, it suffices to show that there exists a constant $C_6(U,F,t,\gamma) > 0$, independent of $\varepsilon \in (0,\gamma/2)$, such that

$$
\mathbf{P}_{\varepsilon\mathbf{1}_U}\Bigl(\sup_{s\le t, y\in F}|N_s(y)| > \frac{\gamma}{2}\Bigr) \le \varepsilon C_6(U,F,t,\gamma). \tag{2.22}
$$

Since $F$ is a closed interval, there are four cases:

1. $F = [a,\infty)$ for some $a \in \mathbb{R}$.
2. $F = (-\infty, b]$ for some $b \in \mathbb{R}$.
3. $F = \mathbb{R}$, and
4. $F = [a,b]$ for some $-\infty < a \le b < \infty$.

*Case (1)*: Since $U$ is an interval and $U \cap F$ is bounded, we have $U \subset (-\infty,\beta)$ for some $\beta \in \mathbb{R}$. Now, we can apply [30], Lemma 3.1, and get that there exists a constant $C_7 > 0$, independent of $U, F, t, \gamma$, and $\varepsilon$ such that

$$
\begin{aligned}
&\mathbf{P}_{\varepsilon\mathbf{1}_U}\Bigl(\sup_{s\le t, y\in[a,\infty)}|N_s(y)| > \frac{\gamma}{2}\Bigr)\\
&\quad\le C_7\Bigl(\frac{\gamma}{2}\Bigr)^{-20}(t\vee t^{22})\int \varepsilon\mathbf{1}_U(x)\,\mathrm{d}x\int G_{t,x-z}\mathbf{1}_{[a,\infty)}(z)\,\mathrm{d}z\\
&\quad\le \varepsilon\frac{C_7}{\sqrt{2\pi t}}\Bigl(\frac{\gamma}{2}\Bigr)^{-20}(t\vee t^{22})\int_{-\infty}^{\beta}\mathrm{d}x\int_a^{\infty} G_{t,x-z}\,\mathrm{d}z < \infty,
\end{aligned}
\tag{2.23}
$$

which implies the desired result (2.22) for this case.

*Case (2)*: This case is similar to Case (1) due to the spatial symmetry of the SPDE (1.10).

*Case (3)*: Since $U \cap F$ is bounded, we have $U = (\alpha, \beta)$ for some $-\infty < \alpha \le \beta < \infty$. In this case by [30], Lemma 3.1, (2.23) still holds for arbitrary $a \in \mathbb{R}$. Taking $a \downarrow -\infty$, we get from monotone convergence theorem that

$$\mathbf{P}_{\varepsilon \mathbf{1}_U}\Big(\sup_{s \le t, y \in \mathbb{R}} |N_s(y)| > \frac{\gamma}{2}\Big) \le C_7 \varepsilon \Big(\frac{\gamma}{2}\Big)^{-20} (t \vee t^{22})(\beta - \alpha).$$

Therefore, the desired result (2.22) also holds for Case (3).

*Case (4)*: In this case $F = [a, b]$ is bounded. Let $p > 4$. From [29], Lemma 6.2, there exists a constant $C_8 > 0$ such that, for any $s, s' \ge 0$ and $y, y' \in \mathbb{R}$,

$$\mathcal{K}_{s,y;s',y'} := \iint_0^\infty |G_{s'-r,y'-z} - G_{s-r,y-z}|^2 \, \mathrm{d}r \, \mathrm{d}z \le C_8\big(|y' - y| + \sqrt{|s' - s|}\big),$$

where $G := 0$ on $(-\infty, 0) \times \mathbb{R}$ for convention. From (1.11), Burkholder–Davis–Gundy inequality [18], Theorem 20.12, and Jensen's inequality, there exists a constant $C_9(p) > 0$, depending only on $p$, such that, for every $s, s' \ge 0$ and $y, y' \in \mathbb{R}$,

$$\begin{aligned}
&\mathbf{E}_{\varepsilon \mathbf{1}_U}\big[|N_{s'}(y') - N_s(y)|^{2p}\big] \\
&\quad = \mathbf{E}_{\varepsilon \mathbf{1}_U}\bigg[\bigg|\iint_0^\infty (G_{s'-r,y'-z} - G_{s-r,y-z})\sqrt{u_{r,z}(1 - u_{r,z})}\, W(\mathrm{d}r \, \mathrm{d}z)\bigg|^{2p}\bigg] \\
&\quad \le C_9(p) \mathbf{E}_{\varepsilon \mathbf{1}_U}\bigg[\bigg(\iint_0^\infty |G_{s'-r,y'-z} - G_{s-r,y-z}|^2 u_{r,z} \, \mathrm{d}r \, \mathrm{d}z\bigg)^p\bigg] \\
&\quad = C_9(p) \mathcal{K}^p_{s,y;s',y'} \mathbf{E}_{\varepsilon \mathbf{1}_U}\bigg[\bigg(\frac{1}{\mathcal{K}_{s,y;s',y'}} \iint_0^\infty |G_{s'-r,y'-z} - G_{s-r,y-z}|^2 u_{r,z} \, \mathrm{d}r \, \mathrm{d}z\bigg)^p\bigg] \\
&\quad \le C_9(p) \mathcal{K}^p_{s,y;s',y'} \mathbf{E}_{\varepsilon \mathbf{1}_U}\bigg[\frac{1}{\mathcal{K}_{s,y;s',y'}} \iint_0^\infty |G_{s'-r,y'-z} - G_{s-r,y-z}|^2 u^p_{r,z} \, \mathrm{d}r \, \mathrm{d}z\bigg] \\
&\quad \le C_9(p) C_8^p \varepsilon\big(|y' - y| + \sqrt{|s' - s|}\big)^p.
\end{aligned}$$

Here in the last inequality, we used the fact that $\mathbf{E}_{\varepsilon \mathbf{1}_U}[u^p_{r,z}] \le \mathbf{E}_{\varepsilon \mathbf{1}_U}[u_{r,z}] \le \varepsilon$ for every $r \ge 0$ and $z \in \mathbb{R}$. Now, from Kolmogorov continuity theorem for random fields [31], Corollary 1.2, there exists a constant $C_{10}(t, p, F) > 0$, depending only on $t$, $p$ and the bounded $F$ such that

$$\mathbf{E}_{\varepsilon \mathbf{1}_U}\Big[\sup_{(s',y'),(s,y) \in [0,t] \times F} |N_{s'}(y') - N_s(y)|^{2p}\Big] \le C_{10}(t, p, F)\varepsilon.$$

Finally, for some $y_0 \in F$, by Markov's inequality and the fact that $N_0(y_0) = 0$,

$$\begin{aligned}
\mathbf{P}_{\varepsilon \mathbf{1}_U}\Big(\sup_{(s,y) \in [0,t] \times F} |N_s(y)| \ge \frac{\gamma}{2}\Big) &\le \mathbf{P}_{\varepsilon \mathbf{1}_U}\Big(\sup_{(s,y) \in [0,t] \times F} |N_s(y) - N_0(y_0)| \ge \frac{\gamma}{2}\Big) \\
&\le \frac{\mathbf{E}_{\varepsilon \mathbf{1}_U}[\sup_{(s,y) \in [0,t] \times F} |N_s(y) - N_0(y_0)|^{2p}]}{(\gamma/2)^{2p}} \\
&\le \frac{C_{10}(t, p, F)}{(\gamma/2)^{2p}} \varepsilon.
\end{aligned}$$

The desired result (2.22) in Case (4) now follows. □

2.4. *Proof of Theorem* 1.2. As we have mentioned in Remark 1.3, it is enough to proof Theorem 1.2(i), since Theorem 1.2(ii) is an immediate corollary of it.

PROOF OF THEOREM 1.2(i). From Theorem 1.1 (ii), we already know that $(Z_t)_{t>0}$ is an $\mathcal{N}$-valued process. Let us now verify that it is a càdlàg process. Fix an arbitrary $0 < \delta <$

$T < \infty$ and $l > 0$. From Proposition 2.8 we have $\mathbb{E}[Z((\delta, T) \times (-l, l))] < \infty$. This implies that almost surely there are only finitely many particle whose trajectory intersects with the time-space region $(\delta, T) \times (-l, l)$. Denote by $I_{(\delta,T)\times(-l,l)}$ the labels of those particles, that is,

$$I_{(\delta,T)\times(-l,l)} := \{i \in I_0 : \exists t \in (\delta, T) \text{ s.t. } X_t^{(i)} \in (-l, l)\}.$$

Then in other words, the event $\Omega_{\delta,T,l} := \{I_{(\delta,T)\times(-l,l)} \text{ has finite cardinality}\}$ has probability 1. Notice that if $t \in (\delta, T)$ and $\varphi$ is a continuous testing function supported on $(-l, l)$, then

$$\langle \varphi, Z_t \rangle = \sum_{i \in I_t} \varphi(X_t^{(i)}) = \sum_{i \in I_{(\delta,T)\times(-l,l)}} \varphi(X_t^{(i)})$$

with a convention that $\varphi(\dagger) = 0$. Now, on the event $\Omega_{\delta,T,l}$, for every continuous function $\varphi$ supported on $(-l, l)$, we have $(\langle \varphi, Z_t \rangle)_{\delta<t<T}$ is a càdlàg process because it is the sum of finitely many càdlàg processes. From this it is straightforward to verify that $(Z_t)_{t>0}$ an $\mathcal{N}$-valued càdlàg process.

We still needs to verify that (1.14) and (1.15) hold for the process $(Z_t)_{t>0}$. Let us first consider the case when there are only finitely many initial particles, that is, $\#I_0 < \infty$. Equivalently, speaking in terms of the initial trace, $\Lambda = \varnothing$ and $\mu(\mathbb{R}) < \infty$. In this case (1.14) holds for the process $(Z_t)_{t>0}$ because it degenerates to Shiga's duality formula [28], Theorem 5.2

$$\mathbb{E}\Big[\prod_{i \in I_t} (1 - f(X_t^{(i)}))\Big] = \mathbf{E}_f\Big[\prod_{i \in I_0} (1 - u_{t,x_i})\Big].$$

Similarly, (1.15) also holds for the process $(Z_t)_{t>0}$ and filtration $(\mathscr{F}_t)$ by applying Shiga's duality to conditional expectation.

Let us now verify that (1.14) holds for the process $(Z_t)_{t>0}$ when $\#I_0 = \infty$. Notice that in this case for every $n \in \mathbb{N}$, $\mathbf{X}^{(n)} := \{X_t^{(i)} : t \ge 0, i = 1, \dots, n\}$ is a coalescing Brownian particle system with initial configuration $(x_i)_{i=1}^n$. Denote by $I_t^{(n)}$ the random set of the index of the particles alive at a given time $t > 0$ in the particle system $\mathbf{X}^{(n)}$. Let $f$ be an arbitrary $[0, 1]$-valued Borel measurable function on $\mathbb{R}$. From Shiga's duality [28], Theorem 5.2,

$$\mathbb{E}\Big[\prod_{i \in I_t^{(n)}} (1 - f(X_t^{(i)}))\Big] = \mathbf{E}_f\Big[\prod_{i=1}^{n} (1 - u_{t,x_i})\Big], \quad n \in \mathbb{N}.$$

Taking $n \uparrow \infty$, we get by monotone convergence theorem that

$$\mathbb{E}\Big[\prod_{i \in I_t} (1 - f(X_t^{(i)}))\Big] = \mathbf{E}_f\Big[\prod_{i=1}^{\infty} (1 - u_{t,x_i})\Big]. \tag{2.24}$$

From the fact that $u_{t,x}$ is continuous in $x$, we can verify the following analytical fact:

$$\prod_{i=1}^{\infty} (1 - u_{t,x_i}) = \mathbf{1}_{\{u_{t,x}=0, \forall x \in \Lambda\}} \prod_{i \in \mathbb{N}: x_i \notin \Lambda} (1 - u_{t,x_i}).$$

From this we can rewrite (2.24) into

$$\mathbb{E}[\exp\{\langle \log(1 - f), Z_t \rangle\}] = \mathbf{E}_f[\mathbf{1}_{\{u_{t,x}=1, \forall x \in \Lambda\}} \exp\{\langle \log(1 - u_t), \mu \rangle\}],$$

as desired.

Similarly, we can verify that (1.15) holds for process $(Z_t)_{t>0}$ and filtration $(\mathscr{F}_t)_{t\ge0}$. In fact, by applying Shiga's duality with conditional expectation to the particle system $\mathbf{X}^{(n)}$, we have

$$\mathbb{E}\Big[\prod_{i \in I_t^{(n)}} (1 - f(X_t^{(i)})) \Big| \mathscr{F}_s\Big] = \int_{\mathcal{C}_{[0,1]}} \Big(\prod_{i \in I_s^{(n)}} (1 - g(X_s^{(i)}))\Big) \mathbf{P}_f(u_{t-s} \in \mathrm{d}g), \quad n \in \mathbb{N}.$$

Here $\mathbf{P}_f(u_{t-s} \in \mathrm{d}g)$ is the law of the random function $u_{t-s}$ under the probability $\mathbf{P}_f$. Taking $n \to \infty$, we get

$$\mathbb{E}\Big[\prod_{i \in I_t}(1 - f(X_t^{(i)}))\Big|\mathscr{F}_s\Big] = \int_{\mathcal{C}_{[0,1]}}\Big(\prod_{i \in I_s}(1 - g(X_s^{(i)}))\Big)\mathbf{P}_f(u_{t-s} \in \mathrm{d}g),$$

which can be rewritten as

$$\mathbb{E}\big[\exp\{\langle\log(1-f), Z_t\rangle\}|\mathscr{F}_s\big] = \Theta_{t-s}^f(Z_s).$$

This also implies the Markov property of $(Z_t)_{t>0}$. □

**3. Proof of Theorem 1.4.** From Theorem 1.2 we know that a coalescing Brownian motions process with initial trace $(\Lambda, \mu)$ can be realized by the process $(Z_t)_{t>0}$ constructed through (1.3)–(1.7) by taking a list of real numbers $(x_i)_{i \in I_0} \in \mathcal{X}$ so that $(\Lambda, \mu) = \Psi((x_i)_{i \in I_0})$. Therefore, we only have to prove that the statements (i)–(v) hold for this precise realization $(Z_t)_{t>0}$. First, notice that this is trivial if $\#I_0 < \infty$. In fact, if $\#I_0 < \infty$, then $\Lambda = \varnothing$ and $\mu = \sum_{i=1}^n \delta_{x_i}$ for some finite $n$. In this case statements (ii)–(iv) hold for the process $(Z_t)_{t>0}$ simply because $Z_t(U) \le n$ for any $t \ge 0$ and any open interval $U$, and there is no open interval $U$ satisfying the conditions of statements (i) and (v). Therefore, in the rest of this Section, we will assume that $I_0 = \mathbb{N}$. This also allows us to use the lemmas from Section 2.3.

Let us first list some lemmas whose proofs are postponed at the end of this section.

LEMMA 3.1. *Let $F$ be a closed interval*:

1. *If $U$ is an open interval such that $U \cap F$ is bounded*, *then*

$$\limsup_{t \downarrow 0} C_3(U, F, \gamma, t) < \infty, \qquad \gamma \in (0, 1),$$

*where $C_3(U, F, \gamma, t)$ is given as in Lemma 2.5.*

2. *If $F$ contains $\bigcup_{i \in \mathbb{N}}(x_i - 1, x_i + 1)$, then*

$$\limsup_{t \downarrow 0} \mathcal{V}_t^{(\Lambda, \mu, F)} < \infty.$$

LEMMA 3.2. *Let $U$ be an open interval such that $U \cap \operatorname{supp}(\Lambda, \mu)$ is bounded*:

1. *If $\bar{U} \cap \Lambda = \varnothing$, then* $\limsup_{t \downarrow 0} \int_U v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x < \infty$.
2. *If $\bar{U} \cap \Lambda \neq \varnothing$, then* $\lim_{t \downarrow 0} \int_U v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x = \infty$.

LEMMA 3.3. *Let $U \subset \mathbb{R}$ be an open interval such that $U \cap \operatorname{supp}(\Lambda, \mu)$ is bounded. Suppose that $\bar{U} \cap \Lambda \neq \varnothing$. Then as $t \downarrow 0$,*

$$\Big(\int_U v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x\Big)^{-1}\mathbb{E}\big[Z_t(U)\big] \longrightarrow 1.$$

LEMMA 3.4. *Let $U \subset \mathbb{R}$ be an open interval such that $U \cap \operatorname{supp}(\Lambda, \mu)$ is bounded. Suppose that $\bar{U} \cap \Lambda \neq \varnothing$. Then as $t \downarrow 0$,*

$$\Big(\int_U v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x\Big)^{-1} Z_t(U) \longrightarrow 1 \quad \textit{in probability}.$$

PROOF OF THEOREM 1.4. As mentioned at the beginning of this section, we only have to show that the statements (i)–(v) hold for the process $(Z_t)_{t>0}$ with the assumption that $\#I_0=\infty$. For statements (i) and (ii), this is done in Theorem 1.1. For statements (iii) this is done in Proposition 2.6 and Remark 2.7.

Let us now verify that statement (iv) holds for the process $(Z_t)_{t>0}$. Let $\gamma\in(0,1)$ be arbitrary. Let $F$ be the smallest closed interval containing $\bigcup_{i\in\mathbb{N}}(x_i-1,x_i+1)$. From the condition that $U\cap\operatorname{supp}(\Lambda,\mu)$ is bounded, we immediately get that $U\cap F$ is bounded. From Lemma 3.1 we know that both $\limsup_{t\downarrow 0}\mathcal{V}_t^{(\Lambda,\mu,F)}$ and $\limsup_{t\downarrow 0}C_3(U,F,t,\gamma)$ are finite. With this at hand, the desired result follows from Proposition 2.6 and Lemma 3.2(1).

Finally, from Lemmas 3.2 (2), 3.3, 3.4, and [18], Theorem 5.12, we can verify that the statement (v) holds for the process $(Z_t)_{t>0}$. □

Let us now give the proofs of Lemmas 3.1–3.4.

PROOF OF LEMMA 3.1. From the definition of $(\mathcal{V}_t^{(\Lambda,\mu,F)})_{t>0}$ and $(C_3(U,F,t,\gamma))_{t>0}$ (see (2.13) and Lemma 2.5, respectively), we know that they are nondecreasing in $t>0$. Now, the desired results follow from Lemma 2.5 and Lemma 2.3(2). □

PROOF OF LEMMA 3.2(1). Since $U$ is an open interval, there are four cases to consider:

*Case (1), $U=\mathbb{R}$*: This case won't happen, because from the condition of the lemma we have $\operatorname{supp}(\Lambda,\mu)$ is bounded and $\Lambda=\varnothing$. Now, $\mu$ is a locally finite measure on $\mathbb{R}$ with compact support, and in particular, $\#I_0=\mu(\mathbb{R})<\infty$. This contradicts the assumption we made at the beginning of this section.

*Case (2), $U=(\alpha,\beta)$ for $-\infty<\alpha\le\beta<\infty$*: From the condition that $\bar U\cap\Lambda=\varnothing$, we get that the closed interval $\bar U=[\alpha,\beta]$ is contained in the open set $\Lambda^c$. Therefore, there exists a small $\delta>0$ such that $[\alpha-\delta,\beta+\delta]$ is also contained in $\Lambda^c$. It can be verified that

$$Z_0([\alpha-\delta,\beta+\delta])=\#(\{x_i:i\in\mathbb{N}\}\cap[\alpha-\delta,\beta+\delta])<\infty,$$

since if it is infinite, then there exists a limiting accumulation point of $\{x_i:i\in\mathbb{N}\}$ in $\Lambda^c$, which contradicts (1.8). Let us take a continuous $[0,1]$-valued function $\phi$ supported on $[\alpha-\delta,\beta+\delta]$ such that $\phi=1$ on $[\alpha,\beta]$. Now, as $t\downarrow 0$, from (1.16) we have

$$\int_\alpha^\beta v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x\le\int_{\Lambda^c}v_{t,x}^{(\Lambda,\mu)}\phi(x)\,\mathrm{d}x\longrightarrow\sum_{i\in\mathbb{N}}\phi(x_i)\le Z_0([\alpha-\delta,\beta+\delta])<\infty,$$

as desired for this case.

*Case (3), $U=(\alpha,\infty)$ for some $\alpha\in\mathbb{R}$*: From the condition that $U\cap\operatorname{supp}(\Lambda,\mu)$ is bounded, one can get that there exists $\beta\in U$ big enough such that $(\beta-1,\infty)\cap\operatorname{supp}(\Lambda,\mu)=\varnothing$. It is then clear that $(\Lambda,\mu)\preceq((-\infty,\beta-1],\mathbf{0})$. Therefore, from (2.4), (2.7), and (2.9), we have

$$\begin{aligned}\int_\beta^\infty v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x&\le\int_\beta^\infty v_{t,x}^{((-\infty,\beta-1],\mathbf{0})}\,\mathrm{d}x=\int_1^\infty v_{t,x}^{((-\infty,0],\mathbf{0})}\,\mathrm{d}x\\&\le\frac{C_2}{\sqrt t}\int_{t^{-1/2}}^\infty(1+z)e^{-\frac12 z^2}\,\mathrm{d}z\le\frac{C_2}{\sqrt t}\int_{t^{-1/2}}^\infty\Big(\frac{z}{t^{-1/2}}+z\Big)e^{-\frac12 z^2}\,\mathrm{d}z\\&=C_2\Big(1+\frac{1}{\sqrt t}\Big)e^{-\frac{1}{2t}}\longrightarrow 0,\quad t\downarrow 0.\end{aligned}\tag{3.1}$$

Also, note that, by taking $\tilde U:=(\alpha,\beta)$, we have $\tilde U\cap\operatorname{supp}(\Lambda,\mu)$ is bounded and $\tilde U\cap\Lambda=\varnothing$. So from what we proved in Case 2, we know that $\limsup_{t\downarrow 0}\int_\alpha^\beta v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x<\infty$. Combining this with (3.1), we get the desired result for this case.

*Case (4), $U = (-\infty, \beta)$ for some $\beta \in \mathbb{R}$*: This is the same as Case 3, thanks to the spatial symmetry. □

PROOF OF LEMMA 3.2(2). Since $\overline{U} \cap \Lambda \neq \varnothing$, there exists $\tilde{x} \in \overline{U} \cap \Lambda$. By shifting $(\Lambda, \mu)$ and $U$ together, we can assume without loss of generality that $\tilde{x} = 0$. Now, since $U$ is an open interval whose closure contains 0, we know that there exists $\delta > 0$ such that either $(-\delta, 0) \subset U$ or $(0, \delta) \subset U$ holds. Define set $-U := \{-y \in \mathbb{R} : y \in U\}$, it is then clear that $(-\delta, 0) \cup (0, \delta) \subset U \cup (-U)$. Now, from (1.16), (2.4), (2.8), and the fact that $(\{0\}, \mathbf{0}) \preceq (\Lambda, \mu)$, we have, as $t \downarrow 0$,

$$\begin{aligned}\int_U v_{t,x}^{(\Lambda,\mu)} \, \mathrm{d}x &\geq \int_U v_{t,x}^{(\{0\},\mathbf{0})} \, \mathrm{d}x = \frac{1}{2}\left(\int_U v_{t,x}^{(\{0\},\mathbf{0})} \, \mathrm{d}x + \int_{-U} v_{t,x}^{(\{0\},\mathbf{0})} \, \mathrm{d}x\right) \\ &\geq \frac{1}{2} \int_{U\cup(-U)} v_{t,x}^{(\{0\},\mathbf{0})} \, \mathrm{d}x \geq \frac{1}{2} \int_{-\delta}^{\delta} v_{t,x}^{(\{0\},\mathbf{0})} \, \mathrm{d}x \longrightarrow +\infty,\end{aligned}$$

as desired. □

PROOF OF LEMMA 3.3. Let $F$ be the smallest closed interval containing $\bigcup_{i\in\mathbb{N}}(x_i - 1, x_i + 1)$. Let $0 < \gamma < 1$ be arbitrary. From the condition that $U \cap \operatorname{supp}(\Lambda, \mu)$ is bounded, we know that $U \cap F$ is also bounded. From Proposition 2.6 we know (2.16) holds for any time $t > 0$. From Lemma 3.1 we know that both $\limsup_{t\downarrow 0} \mathcal{V}_t^{(\Lambda,\mu,F)}$ and $\limsup_{t\downarrow 0} C_3(U, F, t, \gamma)$ are finite. These and Lemma 3.2 (2) easily imply that

$$\limsup_{t\downarrow 0}\left(\int_U v_{t,y}^{(\Lambda,\mu)} \, \mathrm{d}y\right)^{-1} \mathbb{E}[Z_t(U)] \leq \frac{1}{1-\gamma}.$$

Since $\gamma \in (0, 1)$ is arbitrary, we can replace the right-hand side of the above inequality by 1. On the other hand, by Lemmas 2.2 (see (2.12)) and 2.4 (see (2.15)), for any $t > 0$, we have

$$\begin{aligned}\mathbb{E}[Z_t(U)] &= \lim_{\varepsilon\downarrow 0} \frac{1 - \mathbb{E}((1-\varepsilon)^{Z_t(U)})}{\varepsilon} \\ &\geq \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\left(1 - \exp\left(-\varepsilon \int_U v_{t,y}^{(\Lambda,\mu)} \, \mathrm{d}y\right)\right) = \int_U v_{t,y}^{(\Lambda,\mu)} \, \mathrm{d}y.\end{aligned}$$

This implies that

$$\liminf_{t\downarrow 0}\left(\int_U v_{t,y}^{(\Lambda,\mu)} \, \mathrm{d}y\right)^{-1} \mathbb{E}[Z_t(U)] \geq 1.$$

Thus, the desired result follows. □

PROOF OF LEMMA 3.4. Let $\vartheta > 0$ be arbitrary. Define

$$\varepsilon(U, \vartheta, t) := 1 - \exp\left(-\left(\int_U v_{t,x}^{(\Lambda,\mu)} \, \mathrm{d}x\right)^{-1} \vartheta\right), \quad t > 0.$$

From Lemma 3.2(2), it is easy to see that

$$\varepsilon(U, \vartheta, t) \int_U v_{t,y}^{(\Lambda,\mu)} \, \mathrm{d}y \longrightarrow \vartheta \quad \text{as } t \downarrow 0.$$

Also, notice that

$$\mathbb{E}\left[\exp\left(-\vartheta\left(\int_U v_{t,x}^{(\Lambda,\mu)} \, \mathrm{d}x\right)^{-1} Z_t(U)\right)\right] = \mathbb{E}\left[(1 - \varepsilon(U, \vartheta, t))^{Z_t(U)}\right], \quad t > 0. \tag{3.2}$$

By Lemmas 2.2 (see (2.12)) and 2.4 (see (2.15)), we have

$$\mathbb{E}\big[\big(1-\varepsilon(U,\vartheta,t)\big)^{Z_t(U)}\big]\le\exp\biggl(-\varepsilon(U,\vartheta,t)\int_U v_{t,y}^{(\Lambda,\mu)}\,\mathrm{d}y\biggr),\qquad t>0,$$

which implies that

$$\limsup_{t\downarrow 0}\mathbb{E}\big[\big(1-\varepsilon(U,\vartheta,t)\big)^{Z_t(U)}\big]\le e^{-\vartheta}. \tag{3.3}$$

Let us now take $F$ to be the smallest closed interval containing $\bigcup_{i\in\mathbb{N}}(x_i-1,x_i+1)$ and take an arbitrary $\gamma\in(0,1)$. From Lemma 3.2 (2) we have

$$\varepsilon(U,\vartheta,t)\to 0\quad\text{as } t\downarrow 0, \tag{3.4}$$

and, therefore, there exists $t_0(U,\vartheta,\gamma)$ such that $2\varepsilon(U,\vartheta,t)<\gamma$ for every $0<t\le t_0(U,\vartheta,\gamma)$. Now, by Lemmas 2.2 (see (2.11)) and 2.4 (see (2.14)), for every $0<t\le t_0(U,\vartheta,\gamma)$, we have

$$\begin{aligned}
&\mathbb{E}\big[\big(1-\varepsilon(U,\vartheta,t)\big)^{Z_t(U)}\big]\\
&\quad\ge\exp\biggl(-\frac{\varepsilon(U,\vartheta,t)}{1-\gamma}\int_U v_{t,y}^{(\Lambda,\mu)}\,\mathrm{d}y\biggr)\\
&\qquad-\frac{\varepsilon(U,\vartheta,t)}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)}-2\mathbf{P}_{\varepsilon(U,\vartheta,t)\mathbf{1}_U}\Bigl(\sup_{s\le t,y\in F}u_{s,y}>\gamma\Bigr).
\end{aligned}$$

Noticing that, by (3.4) and Lemma 3.1,

$$\frac{\varepsilon(U,\vartheta,t)}{2(1-\gamma)}\mathcal{V}_t^{(\Lambda,\mu,F)}\longrightarrow 0\quad\text{as } t\downarrow 0,$$

and that by Lemma 2.5,

$$\mathbf{P}_{\varepsilon(U,\vartheta,t)\mathbf{1}_U}\Bigl(\sup_{s\le t,y\in F}u_{s,y}>\gamma\Bigr)\le\varepsilon(U,\vartheta,t)C_3(U,F,t,\gamma)\longrightarrow 0\quad\text{as } t\downarrow 0,$$

we can verify

$$\liminf_{t\downarrow 0}\mathbb{E}\big[\big(1-\varepsilon(U,\vartheta,t)\big)^{Z_t(U)}\big]\ge e^{-\frac{\vartheta}{1-\gamma}}.$$

Since $\gamma\in(0,1)$ is arbitrary, we can replace the right-hand side of the above inequliaty by $e^{-\vartheta}$. From this, (3.2), and (3.3), we have

$$\lim_{t\downarrow 0}\mathbb{E}\biggl[\exp\biggl(-\vartheta\biggl(\int_U v_{t,x}^{(\Lambda,\mu)}\,\mathrm{d}x\biggr)^{-1}Z_t(U)\biggr)\biggr]=e^{-\vartheta}.$$

Since $\vartheta>0$ is arbitrary, we are done. $\square$

**4. Behavior of the rate function.** For the proof of Proposition 1.5, we will use Le Gall's probabilistic representation [22], Theorem 4, for the solutions of the PDE (1.16). In the lemma below, we give a weak version of [22], Theorem 4, avoiding the technical details of the Brownian snake. We include its proof later in this section for the sake of completeness. Denote by $\mathbb{K}$ the collection of compact subsets of $\mathbb{R}$ equipped with the Hausdorff metric. Denote by $\mathcal{K}$ the identity map on $\mathbb{K}$. Notice that $\mathcal{K}$ can be considered as a $\mathbb{K}$-valued random variable on the measurable space $(\mathbb{K},\mathscr{B}(\mathbb{K}),\mathsf{P})$, where $\mathsf{P}$ is a probability measure that will be specified in the following lemma. Define $A-\tilde{A}:=\{a-\tilde{a}:a\in A,\tilde{a}\in\tilde{A}\}$ for any subsets $A$ and $\tilde{A}$ of $\mathbb{R}$.

LEMMA 4.1. *There exists a unique probability measure* $\mathsf{P}$ *on* $\mathbb{K}$ *such that*

$$\mathsf{P}(\mathcal{K} \cap A \neq \varnothing) = v_{2,0}^{(A,\mathbf{0})}, \quad A \in \mathbb{K}.$$

*Furthermore, the following statements hold*:

1. $v_{t,x}^{(A,\mathbf{0})} = 2t^{-1}\mathsf{P}(x \in (A - \sqrt{t/2}\mathcal{K}))$ *for every* $t > 0$, $x \in \mathbb{R}$, *and compact* $A \subset \mathbb{R}$.
2. *Extending the probability space* $(\mathbb{K}, \mathscr{B}(\mathbb{K}), \mathsf{P})$ *if necessary, there exist real valued random variables* $Y$, $\tilde{Y}$, *and strictly positive random variables* $R$, $\tilde{R}$, *such that* $\mathsf{E}[\tilde{R}] < \infty$ *and that* $\mathsf{P}((Y - R, Y + R) \subset -\sqrt{1/2}\mathcal{K} \subset (\tilde{Y} - \tilde{R}, \tilde{Y} + \tilde{R})) = 1$.

Here $\mathsf{E}$ is the corresponding expectation of the probability $\mathsf{P}$.

PROOF OF PROPOSITION 1.5. Recall that $\lambda$ is the Lebesgue measure on $\mathbb{R}$. According to Lemma 4.1(1) and Fubini's theorem, we have

$$\|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})} = 2t^{-1}\mathsf{E}[\lambda(A - \sqrt{t/2}\mathcal{K})], \quad t > 0. \tag{4.1}$$

Let random variable $Y$, $\tilde{Y}$, $R$, and $\tilde{R}$ be given as in Lemma 4.1(2):

(1) Suppose that $A$ has positive finite cardinality. Notice that in this case $\lambda(A - \sqrt{t/2}\mathcal{K})/\sqrt{t}$ converges to $\#A \cdot \lambda(\sqrt{1/2}\mathcal{K})$ almost surely, as $t \downarrow 0$. Also, notice that the family of random variables $\{\lambda(A - \sqrt{t/2}\mathcal{K})/\sqrt{t} : t > 0\}$ is dominated by the integrable random variable $2\#A \cdot \tilde{R}$. Therefore, from the dominated convergence theorem

$$\sqrt{t}\|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})} = 2\mathsf{E}\big[\lambda(A - \sqrt{t/2}\mathcal{K})/\sqrt{t}\big] \to 2\mathsf{E}\big[\lambda(\sqrt{1/2}\mathcal{K})\big] \cdot \#A, \quad t \downarrow 0.$$

From (4.1) we have that $C_1 = \|v_{1,\cdot}^{(\{0\},\mathbf{0})}\|_{L^1(\mathbb{R})} = 2\mathsf{E}[\lambda(\sqrt{1/2}\mathcal{K})]$. The desired result in this case now follows:

(2) Suppose that $A$ has positive finite Lebesgue measure. Notice that, in this case,

$$t\|v_{t,\cdot}^{(A,0)}\|_{L^1(\mathbb{R})} = 2\mathsf{E}\big[\lambda(A - \sqrt{t/2}\mathcal{K})\big] \geq 2\lambda(A), \quad t > 0.$$

On the other hand, since $A$ is compact, we can verify from the monotone convergence theorem that

$$t\|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})} = 2\mathsf{E}\big[\lambda(A - \sqrt{t/2}\mathcal{K})\big] \leq 2\mathsf{E}\big[\lambda\big(A + \sqrt{t}\tilde{R}B^o\big)\big] \to 2\lambda(A), \quad t \downarrow 0,$$

where $B^o := (-1, 1)$ is the centered open unit ball in $\mathbb{R}$. So we have $t\|v_{t,\cdot}^{(A,0)}\|_{L^1(\mathbb{R})} \to 2\lambda(A)$, as $t \downarrow 0$.

(3) Suppose that $A$ has Minkowski dimension $\delta \in (0, 1)$. Define $C_{11}(A) := \inf\{n \geq 0 : A \subset [-n, n]\}$. Let $\overline{\delta} \in (\delta, 1)$ be arbitrary. Noticing that

$$\frac{\lambda(A + rB^o)}{r^{1-\overline{\delta}}} = \exp\left((\log r)\left(\frac{\log \lambda(A + rB^o)}{\log r} - (1 - \overline{\delta})\right)\right) \to 0, \quad r \downarrow 0, \tag{4.2}$$

so there exists $C_{12}(A, \overline{\delta}) > 0$ such that $\lambda(A + rB^o)/r^{1-\overline{\delta}} \leq 1$ for every $r \in (0, C_{12}(A, \overline{\delta}))$. Also, notice that

$$\frac{\lambda(A + rB^o)}{r^{1-\overline{\delta}}} \leq \frac{2(C_{11}(A) + r)}{r^{1-\overline{\delta}}} \leq \frac{2C_{11}(A)}{C_{12}(A, \overline{\delta})^{1-\overline{\delta}}} + 2r^{\overline{\delta}}, \quad r \geq C_{12}(A, \overline{\delta}).$$

Therefore, there exists $C_{13}(A, \overline{\delta}) > 0$ such that $\lambda(A + rB^o)/r^{1-\overline{\delta}} \leq C_{13}(A, \overline{\delta}) + 2r^{\overline{\delta}}$ for every $r > 0$. Now, we can verify that the family of random variables $\{\lambda(A + \sqrt{t}\tilde{R}B^o)/t^{(1-\overline{\delta})/2} : t \in$

$(0, 1]\}$ is dominated by the integrable random variable $C_{13}(A, \overline{\delta})\tilde{R}^{1-\overline{\delta}} + 2\tilde{R}$. From (4.1), Proposition 4.1(2), (4.2), and the dominated convergence theorem, we have

$$t^{\frac{1+\overline{\delta}}{2}}\|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})} = 2\mathsf{E}\left[\frac{\lambda(A-\sqrt{t/2}\mathcal{K})}{(\sqrt{t})^{1-\overline{\delta}}}\right] \le 2\mathsf{E}\left[\frac{\lambda(A+\sqrt{t}\tilde{R}B^o)}{(\sqrt{t}\tilde{R})^{1-\overline{\delta}}}\tilde{R}^{1-\overline{\delta}}\right] \to 0, \quad t\downarrow 0.$$

From this and the fact that $\overline{\delta} \in (\delta, 1)$ is arbitrary, we can verify that

$$\limsup_{t\downarrow 0} \frac{\log\|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})}}{\log(1/t)} \le \frac{1+\delta}{2}.$$

On the other hand, let $\underline{\delta} \in (0, \delta)$ be arbitrary. Notice that

$$\frac{\lambda(A+rB^o)}{r^{1-\underline{\delta}}} = \exp\left((\log r)\left(\frac{\log\lambda(A+rB^o)}{\log r} - (1-\underline{\delta})\right)\right) \to \infty, \quad r\downarrow 0.$$

So from (4.1), Lemma 4.1(2), and Fatou's lemma,

$$\begin{aligned}\liminf_{t\downarrow 0} t^{\frac{1+\underline{\delta}}{2}}\|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})} &= \liminf_{t\downarrow 0} 2\mathsf{E}\left[\frac{\lambda(A-\sqrt{t/2}\mathcal{K})}{(\sqrt{t})^{1-\underline{\delta}}}\right] \\ &\ge 2\mathsf{E}\left[\liminf_{t\downarrow 0}\frac{\lambda(A+\sqrt{t}RB^o)}{(\sqrt{t}R)^{1-\underline{\delta}}}R^{1-\underline{\delta}}\right] = \infty.\end{aligned}$$

From this and the fact that $\underline{\delta} \in (0, \delta)$ is arbitrary, we can verify that

$$\liminf_{t\downarrow 0} \frac{\log\|v_{t,\cdot}^{(A,\mathbf{0})}\|_{L^1(\mathbb{R})}}{\log(1/t)} \ge \frac{1+\delta}{2}.$$

The desired result in this case now follows. $\square$

PROOF OF LEMMA 4.1. Denote by $\mathbb{W}$ the space of pairs $(\zeta, w)$, where $\zeta \in [0,\infty)$ and $w = (w_t)_{t\ge 0}$ is an $\mathbb{R}$-valued continuous path such that $w_t = w_{t\wedge\zeta}$ for every $t \ge 0$. The space $\mathbb{W}$ is equipped with the metric

$$d_{\mathbb{W}}\big((\zeta, w), (\zeta', w')\big) = |\zeta-\zeta'| + \sup_{t\ge 0}|w_t - w_t'|, \quad (\zeta, w), (\zeta, w') \in \mathbb{W}.$$

Denote by $\mathcal{C}([0,\infty), \mathbb{W})$ the space of $\mathbb{W}$-valued continuous path $(\zeta_s, (w_{s,t})_{t\ge 0})_{s\ge 0}$ equipped with the topology of local uniform convergence. According to [22], Theorem 4, there exists a $\sigma$-finite measure $\mathbb{N}_0$ on $\mathcal{C}([0,\infty), \mathbb{W})$, known as the excursion measure of the Brownian snake initiated at position 0, such that $v_{r,0}^{(A,\mathbf{0})} = 2\mathbb{N}_0(\mathscr{S}_r \cap A \ne \varnothing)$ for every closed $A \subset \mathbb{R}$ and $r > 0$. Here for each $r > 0$, $\mathscr{S}_r : (\zeta_s, (w_{s,t})_{t\ge 0})_{s\ge 0} \mapsto \{w_{s,r} : s \ge 0, \zeta_s \ge r\}$ is a measurable map from $\mathcal{C}([0,\infty), \mathbb{W})$ to $\mathbb{K}$. From (2.6) we have $2\mathbb{N}_0(\mathscr{S}_2 \ne \varnothing) = v_{2,0}^{(\mathbb{R},\mathbf{0})} = 1$. This allows us to define a probability measure $\tilde{\mathbb{N}}_0$ on $\mathcal{C}([0,\infty), \mathbb{W})$ so that $\mathrm{d}\tilde{\mathbb{N}}_0 = 2\mathbf{1}_{\{\mathscr{S}_2\ne\varnothing\}}\,\mathrm{d}\mathbb{N}_0$. Now, we have

$$v_{2,0}^{(A,\mathbf{0})} = \tilde{\mathbb{N}}_0(\mathscr{S}_2 \cap A \ne \varnothing) \tag{4.3}$$

for every closed $A \subset \mathbb{R}$. This gives us the existence of the desired probability $\mathsf{P}$, which is the law of $\mathscr{S}_2$ under $\tilde{\mathbb{N}}_0$. The uniqueness of $\mathsf{P}$ follows from [25], Theorem 1.13:

(1) One can verify directly from the uniqueness of the solution to the PDE (1.16) that $v_{t,x}^{(A,\mathbf{0})} = \alpha^2 v_{\alpha^2 t,\alpha x}^{(\alpha A,\mathbf{0})}$ for every $\alpha > 0$, $t > 0$, $x \in \mathbb{R}$, and closed $A \subset \mathbb{R}$. Now, we can verify for every $t > 0$, $x \in \mathbb{R}$ and closed $A \subset \mathbb{R}$ that

$$\begin{aligned}v_{t,x}^{(A,\mathbf{0})} &= v_{t,0}^{(A-\{x\},\mathbf{0})} = 2t^{-1}v_{2,0}^{(\sqrt{2/t}(A-\{x\}),\mathbf{0})} \\ &= 2t^{-1}\mathsf{P}\big(\mathcal{K}\cap\big(\sqrt{2/t}(A-\{x\})\big)\ne\varnothing\big) = 2t^{-1}\mathsf{P}\big(x \in (A-\sqrt{t/2}\mathcal{K})\big).\end{aligned}$$

(2) According to [22], Theorem 4, there exists a nonnegative continuous random field $(\tilde{u}_{t,x})_{t>0,x\in\mathbb{R}}$, known as the density of the super-Brownian motion constructed from the Brownian snake, such that

$$v_{2,0}^{(\varnothing,\theta\nu)} = 2\mathbb{N}_0\left[1 - \exp\left(-\frac{\theta}{2}\int \tilde{u}_{2,y}\nu(dy)\right)\right]$$

for every $\theta > 0$ and nonnegative finite Radon measure $\nu$ on $\mathbb{R}$. Furthermore, from how it is constructed, one can verify that $\mathbb{N}_0$-almost everywhere, $(\tilde{u}_{2,y})_{y\in\mathbb{R}}$ is supported on $\mathscr{S}_2$. Fixing a nonnegative finite Radon measure $\nu$ with $\operatorname{supp}(\nu) = \mathbb{R}$, taking $\theta \uparrow \infty$, we get from above and (2.5) that

$$\begin{aligned} 1 = v_{2,0}^{(\mathbb{R},\mathbf{0})} &= 2\mathbb{N}_0\left(\int \tilde{u}_{2,y}\nu(dy) > 0\right) = 2\mathbb{N}_0(\exists y \in \mathbb{R}, \tilde{u}_{2,y} > 0) \\ &= \tilde{\mathbb{N}}_0(\exists y \in \mathbb{R}, \tilde{u}_{2,y} > 0) = \tilde{\mathbb{N}}_0\left(\bigcup_{n\in\mathbb{N}}\{\tilde{u}_{2,y} > 0, \forall y \in (q_n, q'_n)\}\right), \end{aligned}$$

where $\{(q_n, q'_n) : n \in \mathbb{N}\}$ is a sequential arrangement of $\{(q, q') \in \mathbb{Q}^2 : q < q'\}$. This allows us to define a random variable $N := \inf\{n \in \mathbb{N} : \tilde{u}_{2,y} > 0, \forall y \in (q_n, q'_n)\}$, which is finite almost surely under $\tilde{\mathbb{N}}_0$. We then define an $\mathbb{R}$-valued random variable $Y$ and a $(0,\infty)$-valued random variable $R$ so that $(Y - R, Y + R) = -\sqrt{1/2}(q_N, q'_N)$. Now, $\tilde{\mathbb{N}}_0$-almost surely, $(Y - R, Y + R) \subset -\sqrt{1/2}\operatorname{supp}(\tilde{u}_{2,\cdot}) \subset -\sqrt{1/2}\mathscr{S}_2$.

On the other hand, from (4.3), (2.8), (2.7), and (2.9), we can verify that

$$\begin{aligned} \tilde{\mathbb{N}}_0(0 \vee \sup \mathscr{S}_2 \geq n) &= \tilde{\mathbb{N}}_0(\mathscr{S}_2 \cap [n,\infty) \neq \varnothing) \\ &= v_{2,0}^{([n,\infty),\mathbf{0})} = v_{2,0}^{((-\infty,-n],\mathbf{0})} = v_{2,n}^{((-\infty,0],\mathbf{0})} \\ &\leq \frac{1}{2}C_2\left(1 + \frac{n}{\sqrt{2}}\right)e^{-\frac{n^2}{4}}, \quad n > 0. \end{aligned}$$

So we have $\tilde{\mathbb{N}}_0[|0 \vee \sup \mathscr{S}_2|] < \infty$. Similarly, $\tilde{\mathbb{N}}_0[|0 \wedge \inf \mathscr{S}_2|] < -\infty$. We can then define a real valued random variable $\tilde{Y}$ and a $(0,\infty)$-valued random variable $\tilde{R}$ so that the interval $(\tilde{Y} - \tilde{R}, \tilde{Y} + \tilde{R}) = -\sqrt{1/2}(0 \wedge \inf \mathscr{S}_2 - 1, 0 \vee \sup \mathscr{S}_2 + 1)$. Notice that $\tilde{\mathbb{N}}_0$-almost surely, $-\sqrt{1/2}\mathscr{S}_2 \subset (\tilde{Y} - \tilde{R}, \tilde{Y} + \tilde{R})$. We are done. □

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