Subcritical superprocesses conditioned on non-extinction

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Superprocesses

- Superprocesses are a class of measure-valued Markov processes satisfying the branching property.
- Typically, Superprocess $(X_t)_{t\geq 0}$ takes values in $M_F(E) = \{$ finite measures on $E\}$ where E is a "good" topological space.
- $X_t(A)$ represent the population in the measurable set $A \subset E$ at the time $t \geq 0$.
- The branching property says that

convolution.

$$\mathrm{Law}(X_t \mid X_0 = \mu + \nu) = \mathrm{Law}(X_t \mid X_0 = \mu) * \mathrm{Law}(X_t \mid X_0 = \nu).$$

- Typically, there are three cases to consider:

  Supercritical $\rightsquigarrow$ The population tends to grow exponentially.

  Critical $\rightsquigarrow$ The border case.

  Subcritical $\rightsquigarrow$ The population tends to shrink exponentially.

- This talk is about the asymptotic behavior of a class of subcritical superprocesses

Galton-Watson Processes

- Galton-Watson process $(Z_n)_{n\geq 0}$ is a $\mathbb{Z}_+$-valued time-homogeneous Markov process satisfying the branching property

$$\mathrm{Law}(Z_n \mid Z_0 = i+j) = \mathrm{Law}(Z_n \mid Z_0 = i) * \mathrm{Law}(Z_n \mid Z_0 = j).$$

- Assume that $Z_0 = 1$. We say (Z_n) is supercritical, critical, or subcritical if

$\mathbb{E} Z_1 > 1$, $\mathbb{E} Z_1 = 1$, or $\mathbb{E} Z_1 < 1$, respectively.

- Further assume that (Z_n) is subcritical and non-trivial $\mathbb{E} Z_1 > 0$.
- It is known that the extinction probability

$$\mathbb{P}(\exists n, Z_n = 0) = \lim_{n\to+\infty} \mathbb{P}(Z_n = 0) = 1.$$

Q: What is the asymptotic behavior of

$$\mathrm{Law}(Z_n \mid Z_n > 0) \quad \text{and} \quad \mathrm{Law}(Z_n \mid Z_{n+m} > 0) \ ?$$

Yaglom's limit theorem

Yaglom (1947), Heathcot, Seneta & Vere-Jones (1967), Joffe (1967), Athreya & Ney (1972)

Define $q_{n,m} = \mathrm{Law}(Z_n \mid Z_{n+m} > 0)$, then

$\forall m \in \mathbb{Z}_+$, $\exists$ a probability measure $q_{\infty,m}$ on $\mathbb{N}$, s.t. $q_{n,m} \xrightarrow{\text{weakly}} q_{\infty,m}$ as $n \to +\infty$; and

$\forall n \in \mathbb{Z}_+$, $\exists$ a probability measure $q_{n,\infty}$ on $\mathbb{N}$, s.t. $q_{n,m} \xrightarrow{\text{weakly}} q_{n,\infty}$ as $m \to +\infty$.

Heathcot, Seneta & Vere-Jones' LlogL theorem

Heathcot, Seneta & Vere-Jones (1967), Pakes (1999)

$$\lim_{n\to\infty} \frac{\mathbb{P}(Z_n > 0)}{\mathbb{E}[Z_1]^n} > 0. \iff \mathbb{E}[Z_1 \log^+ Z_1] < +\infty.$$

$\iff$ $q_{\infty,m}$ converges weakly when $m \to +\infty$.

$\iff$ $q_{n,\infty}$ converges weakly when $n \to +\infty$.

Pakes' double limit theorem

Further more, if one of the above condition holds, then

$$\operatorname*{w-lim}_{m\to+\infty} q_{\infty,m} = \operatorname*{w-lim}_{n\to+\infty} q_{n,\infty}.$$

Q: Does results similar to

1) Yaglom's limit theorem,

2) Heathcot, Seneta and Vere-Jones' LlogL theorem, and

3) Pakes' double limit theorem

hold for subcritical superprocesses?

The short answer:

Yes, if the subcritical superprocess is

intrinsically ultracontractive and non-persistent.

Let me explain ...

Definition of superprocesses

- Let E be a Polish space (a seperate completely metrizable space) and

  $M_F(E) = \{$ finite Borel measures on $E \}$ equipped with the weak topology.

- To define a superprocess we need two parameters:

  1. The branching mechanism is

  $$\psi(x,z) := -\beta(x) z + \sigma(x)^2 z^2 + \int_{(0,+\infty)} \left(e^{-zu} - 1 + zu\right) \pi(x,du), \qquad x \in E,\ z \geq 0$$

  where β, σ are bounded measurable function on E,

  and $(u \wedge u^2)\pi(x,du)$ is a finite kernel from E to $(0,+\infty)$.

  2. The spatial movement $(\xi_t)_{t \leq \zeta}$ is an E-valued Borel right Markov process

  with (possibly sub-Markovian) semigroup $(P_t)_{t \geq 0}$ and lifetime $\zeta \in (0,+\infty]$

Definition of superprocesses

- $bp\mathcal{B}(E) := \{$ bounded, nonnegative Borel measurable functions on $E\}$.
- The cumulant semigroup $(V_t)_{t\geq 0}$ on $bp\mathcal{B}(E)$ is defined s.t.

  $\forall f \in bp\mathcal{B}(E)$, $(V_t f)_{t\geq 0}$ is the unique nonnegative, locally bounded solution

  to the integral equation

$$V_t f(x) + \int_0^t ds \int_E \psi(y, V_{t-s}f(y))\, P_s(x, dy) = P_t f(x), \quad t\geq 0, \ x\in E.$$

- A (ψ, ξ)-superprocess $(X_t)_{t\geq 0}$ is defined as a $M_F(E)$-valued Markov process s.t.

$$\mathbb{E}\exp\left(-\langle X_t, f\rangle\right) = \exp\left(-\langle X_0, V_t f\rangle\right), \quad \forall t\geq 0, \ f\in bp\mathcal{B}(E).$$

- The existence of such processes are given by Watanabe (1968), Dynkin (1993)...

Mean behavior

- The mean semigroup $(T_t)_{t\geq 0}$ on $b\mathcal{B}(E)$ is defined s.t.

$$\langle \mu, T_t f\rangle = \mathbb{E}_\mu \langle X_t, f\rangle, \qquad t\geq 0,\ \mu\in M_F(E),\ f\in b\mathcal{B}(E).$$

- We assume that

$\exists$ an eigenvalue $\lambda\in\mathbb{R}$, an eigenfunction $\phi\in bp\mathcal{B}(E)$, an eigenmeasure $\nu\in M_F(E)$ s.t. $\mathrm{supp}(\phi)=\mathrm{supp}(\nu)=E$ and $T_t\phi = e^{\lambda t}\phi$, $\nu T_t = e^{\lambda t}\nu$ for every $t\geq 0$.

- We normalize ϕ and ν so that $\langle \nu, 1_E\rangle = \langle \nu, \phi\rangle = 1$.

- We say the superprocess is supercritical, critical, or subcritical if

$$\lambda>0, \qquad \lambda=0, \qquad \text{or} \quad \lambda<0, \qquad \text{respectively.}$$

- We assume that $(X_t)_{t\geq 0}$ is subcritical.

Intrinsic ultracontractivity

- For each $f \in p\mathcal{B}(E)$ with $\nu(f) < +\infty$, define $H_t f \in \mathcal{B}(E)$ s.t.

$$T_t f = e^{\lambda t} \langle \nu, f \rangle \phi \cdot (1 + H_t f)$$

- We assume that the mean semigroup $(T_t)_{t\geq 0}$ is intrinsically ultracontractive, that is

$$\underbrace{\sup_f \|H_t f\|_\infty < +\infty, \ \forall t \geq 0}_{T_t(x,dy) \text{ "} \lesssim \text{" } e^{\lambda t}\phi(x)\,\nu(dy).} \quad \text{and} \quad \underbrace{\lim_{t\to+\infty} \sup_f \|H_t f\|_\infty = 0.}_{T_t(x,dy) \text{ "} \sim \text{" } e^{\lambda t}\phi(x)\,\nu(dy).}$$

Non-persistent

- We assume that the superprocess $(X_t)_{t\geq 0}$ initiated at ν is non-persistent, that is

$$\mathbb{P}_\nu(X_t = 0) > 0, \quad \forall t > 0.$$

Yaglom's limit

Liu, Ren, Song, S. (2021), Liu, Ren, Song, S. (2022+)

Define $Q^{\mu}_{t,r}(\cdot) := \mathbb{P}_{\mu}(X_t \in \cdot \mid X_{t+r} \neq 0)$, $t, r \geq 0$, $\mu \in M_F(E) \setminus \{0\}$.

$\forall r \geq 0$, $\exists$ a probability measure $Q_{\infty,r}$ on $M_F(E)$, s.t. $\forall \mu \in M_F(E) \setminus \{0\}$

$$Q^{\mu}_{t,r} \xrightarrow{\text{weakly}} Q_{\infty,r} \quad \text{as } t \to +\infty.$$

$\forall t \geq 0$, $\forall \mu \in M_F(E) \setminus \{0\}$, $\exists$ a probability measure $Q^{\mu}_{t,\infty}$ on $M_F(E)$ s.t.

$$Q^{\mu}_{t,r} \xrightarrow{\text{strongly}} Q^{\mu}_{t,\infty} \quad \text{as } r \to +\infty,$$

$\langle Q^{\mu}_{t,r}, f \rangle \xrightarrow[t \to +\infty]{} \langle Q_{\infty,r}, f \rangle$, $\forall$ bounded continuous function f on $M_F(E)$.

$\langle Q^{\mu}_{t,r}, f \rangle \xrightarrow[r \to +\infty]{} \langle Q^{\mu}_{t,\infty}, f \rangle$, $\forall$ bounded measurable function f on $M_F(E)$.

Heathcot, Seneta & Vere-Jones' $L\log L$ theorem & Pakes' double limit theorem

- Define

$$\mathcal{E} := \int_E \nu(dx) \int_{(0,+\infty)} u\phi(x) \log^+(u\phi(x))\, \pi(x,du).$$

Liu, Ren, Song, S. (2022+)

$\forall \mu \in \mathcal{M}_F(E)\setminus\{0\}$, the following are equivalent:

$$\lim_{t\to\infty} e^{-\lambda t}\mathbb{P}_\mu(X_t \neq 0) > 0. \quad \Leftrightarrow \quad \mathcal{E} < +\infty.$$

$\Leftrightarrow$ $Q_{\infty,r}$ converges weakly when $r \to +\infty$.

$\Leftrightarrow$ $Q^\mu_{t,\infty}$ converges strongly when $t \to +\infty$.

Further more, if one of the above condition holds, then $\exists$ a probability measure $Q_{\infty,\infty}$ s.t.

$$Q_{\infty,r} \xrightarrow{\text{weakly}} Q_{\infty,\infty}, \quad r \to +\infty, \quad \text{and}$$

$$\forall \mu,\ Q^\mu_{t,\infty} \xrightarrow{\text{strongly}} Q_{\infty,\infty}, \quad t \to +\infty.$$

$$\underset{t\to+\infty}{\text{s-lim}}\ \underset{r\to+\infty}{\text{w-lim}}\ Q^\mu_{t,r} = \underset{r\to+\infty}{\text{w-lim}}\ \underset{t\to+\infty}{\text{s-lim}}\ Q^\mu_{t,r}$$

Comment

- The $L\log L$ condition $\mathcal{E} < +\infty$, appeared in Liu, Ren, Song (2009) in the study of supercritical superdiffusion.
- If $E = \{e\}$, then $X_t = Y_t \delta_e$ where Y_t is a continuous-state branching process.
  In this case, some of our results appeared in Lambert (2007), Grey (1974), & Li (2000).
- If $E = \{e_1, \cdots, e_n\}$ and $\pi(x, du) = 0$, then
  $$X_t = \sum_{i=1}^{n} Y_t^i \delta_{e_i} \quad \text{where} \quad (Y_t^1, \cdots, Y_t^n) \text{ is a multitype Feller diffusion.}$$
  In this case, the double limit theorem appeared in Champagnat & Roelly (2008).

They share the same law!

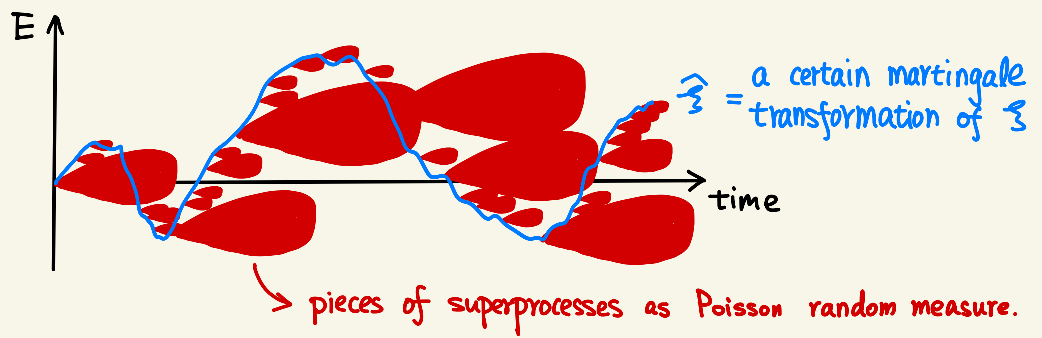
The Q-process $(\tilde{X}_s)_{s\geq 0}$:

$$\mathbb{P}\left((\tilde{X}_s)_{s\leq t}\in\cdot\right)=\text{w-}\lim_{T\to+\infty}\mathbb{P}\left((X_s)_{s\leq t}\in\cdot \mid X_T\neq 0\right).$$

The martingale transformation $(\hat{X}_s)_{s\geq 0}$:

$$\mathbb{P}\left((\hat{X}_s)_{s\leq t}\in\cdot\right)=\mathbb{E}\left[\underbrace{\frac{\langle X_t,\phi\rangle}{\mathbb{E}\langle X_t,\phi\rangle}}_{\text{martingale}}\mathbb{1}\{(X_s)_{s\leq t}\in\cdot\}\right].$$

The superprocess with immgrations along a spine $(X_s+Y_s)_{s\leq t}$:



This talk is based on a joint work with

Rongli Liu, Yan-Xia Ren, and Renming Song.

The manuscript is avaliable at arXiv: 2112.15184.

Thanks !