Spine decompositions and limit theorems for a class of critical superprocesses

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Wuhan University,
March, 2019

Background

- $(Z_n)_{n\geq 0}$: a critical **Galton-Watson (GW) branching process** with offspring variance $\sigma^2 \in (0, \infty)$.
- Kolmogorov 1938 proved

$$nP(Z_n > 0) \xrightarrow[n\to\infty]{} \frac{2}{\sigma^2}.$$

- Yaglom 1947 proved

$$\left\{\frac{Z_n}{n}; P(\cdot|Z_n > 0)\right\} \xrightarrow[n\to\infty]{d} \frac{\sigma^2}{2}\mathbf{e},$$

where $\mathbf{e}$ is an exponential random variable with mean 1.

Background

For more general Kolmogorov type and Yaglom type results see:

	Analytical proof	Probabilistic proof
GW	Kolmogorov 1938 Yaglom 1947 Kesten, Ney, and Spitzer 1966	Lyons, Pemantle, and Peres 1995 Geiger 1999 Geiger 2000 Ren, Song, and Sun 2018
Multitype GW	Joffe and Spitzer 1967	Vatutin and Dyakonova 2001 (for only Kolmogorov type result)
Continuous time GW	Athreya and Ney 1972	-
Continuous time Multitype GW	Athreya and Ney 1974	-
Branching Markov processes	Asmussen and Hering 1983	Powell 2016
Continuous sate branching processes	Li 2000 Lambert 2007	Ren, Song, and Sun 2019
Superprocesses	Evans and Perkins 1990 Ren, Song, and Zhang 2015	Ren, Song, and Sun 2019

Settings

- E: locally compact separable metric space.
- $\mathcal{M}_f$: the collection of all the finite Borel measures on E equipped with the weak topology.
- **Spatial motion** $\{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in E}\}$: an E-valued Hunt process with transition semigroup $(P_t)_{t\geq 0}$ and lifetime ζ.
- **Branching mechanism** $\Psi : E \times [0,\infty) \to [0,\infty)$ s.t.

$$\Psi(x,z) := -\beta(x)z + \alpha(x)z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(x,dy),$$

where $\beta \in b\mathscr{B}_E$; $\alpha \in bp\mathscr{B}_E$; π is a kernel from E to $(0,\infty)$ s.t.

$$\sup_{x\in E} \int_{(0,\infty)} (y \wedge y^2)\pi(x,dy) < \infty.$$

Superprocesses

- $\mu(f) := \int f d\mu$ for each measure μ and function f, whenever the integral make sense.

Definition (Superprocess, Dynkin 1991)

Say an $\mathcal{M}_f$-valued Markov process $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f(E)}\}$ is a (ξ,ψ)-superprocess if

$$\mathbf{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)},$$

where $(t,x)\mapsto V_t f(x)$ on $[0,\infty)\times E$ is the unique locally bounded positive solution to the equation

$$V_t f(x) + \int_0^t P_{t-s}\psi\big(\cdot, V_s f(\cdot)\big)(x)ds = P_t f(x).$$

Superprocesses

- Superprocess are high-density limits of
  - branching particle systems (Watanabe 1968, Dawson 1975, Dynkin 1991),
  - long-range contact process (Müller and Tribe 1995, Durrett and Perkins 1999),
  - voter model (Cox, Durrett, and Perkins 2000), and
  - long range percolation (Lalley and Zheng 2010).

Example (Super Brownian motion, Watanabe 1968, Dawson 1975)

Consider a Branching Brownian motion with

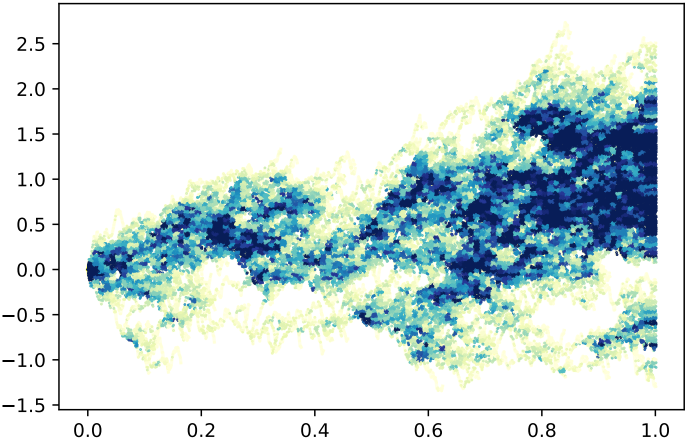
- k initial particles;
- killing rate $2k$;
- critical binary branching.

$X_t^{(k)}(A)$: number of particles the Borel set A at time t.

$$\left(\frac{1}{k} X_t^{(k)}(\cdot)\right)_{t\geq 0} \xrightarrow[k\to\infty]{w} (\xi, \Psi)\text{-superprocess}$$

with ξ be a Brownian motion and $\Psi(z) = z^2$.

Branching Brownian Motion with large k



Assumptions

- $X_t(f) := \int_E f(x)X_t(dx), \quad f \in b\mathscr{B}_E, t \geq 0.$
- The mean behavior of Superprocess can be described by the Feynman-Kac transform of (P_t):

$$\mathbf{P}_{\delta_x}[X_t(f)] = P_t^\beta f(x) := \Pi_x[e^{\int_0^t \beta(\xi_r)dr} f(\xi_t)\mathbf{1}_{t<\zeta}],$$

for $x \in E, t \geq 0, f \in b\mathscr{B}_E$.

Assumption 1.

There exist a σ-finite Borel measure m with full support on E and a family of strictly positive, bounded continuous functions $\{p_t(\cdot,\cdot) : t > 0\}$ on $E \times E$ such that,

- $P_t f(x) = \int_E p_t(x,y)f(y)m(dy), \quad t > 0, x \in E, f \in b\mathscr{B}_E,$
- $\int_E p_t(y,x)m(dy) \leq 1, \quad t > 0, x \in E,$
- $\int_E \int_E p_t(x,y)^2 m(dx)m(dy) < \infty, \quad t > 0.$
- $x \mapsto \int_E p_t(x,y)^2 m(dy)$ and $x \mapsto \int_E p_t(y,x)^2 m(dy)$ are both continuous on E.

Assumptions

- $(P_t^\beta)_{t\geq 0}$ and its dual semigroup $(P_t^{\beta *})_{t\geq 0}$ are both strongly continuous semigroups of compact operators in $L^2(E,m)$.
- Transition density p_t^β: $P_t^\beta f(x) = \int_E p_t^\beta(x,y) f(y) m(dy)$
- L and L^*: the generators of $(P_t^\beta)_{t\geq 0}$ and $(P_t^{\beta *})_{t\geq 0}$, respectively.
- $\sigma(L)$ and $\sigma(L^*)$: the spectra of L and L^*, respectively.
- $\lambda := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(L^*))$, a common eigenvalue of multiplicity 1.
- ϕ and ϕ^*: the eigenfunction of L and L^* associated with the eigenvalue λ.
- Normalize ϕ and ϕ^* by $\langle \phi, \phi \rangle_m = \langle \phi, \phi^* \rangle_m = 1$.

Assumption 2. (Critical and Intrinsic Ultracontractive)

- $\lambda = 0$.
- $\forall t > 0, \exists c_t > 0, \forall x, y \in E, \quad p_t^\beta(x,y) \leq c_t \phi(x) \phi^*(y)$.

Assumptions

- $A(x) := 2\alpha(x) + \int_{(0,\infty)} y^2 \pi(x, dy), \quad x \in E.$

Assumption 3 (Second moment)

$A\phi$ is bounded.

- $\nu(dx) := \phi^*(x) m(dx).$

Assumption 4 (Non-persistence)

- $\forall x \in E, t > 0, \quad \mathbf{P}_{\delta_x}(X_t(1) = 0) > 0,$
- $\exists t > 0, \quad \mathbf{P}_{\nu}(X_t(1) = 0) > 0.$

Results

- Let initial measure $\mu \in \mathcal{M}_f$ satisfies $\mu(\phi) < \infty$.

Theorem (Kolmogorov type result, Ren, Song, and Zhang 2015, Ren, Song, and Sun 2019)

$$t\mathbf{P}_\mu(X_t(1) > 0) \xrightarrow[t\to\infty]{} \frac{\mu(\phi)}{\frac{1}{2}\langle A\phi, \phi\phi^*\rangle_m}.$$

- Let testing function f satisfies $\phi^{-1}f \in b\mathscr{B}_E$.

Theorem (Yaglom type result, Ren, Song, and Zhang 2015, Ren, Song, and Sun 2019)

$$\left\{t^{-1}X_t(f); \mathbf{P}_\mu(\cdot|X_t(1) > 0)\right\} \xrightarrow[t\to\infty]{d} \frac{1}{2}\langle \phi^*, f\rangle_m \langle A\phi, \phi\phi^*\rangle_m \mathbf{e},$$

where $\mathbf{e}$ *is an exponential random variable with mean 1.*

Size-biased transform (definition)

- $(\Omega, \mathscr{F}, Q)$: a measure space.
- $G \in \mathscr{F}^+$: $Q(G) \in (0, \infty)$.

Definition

A probability measure Q^G is called the G-size-biased transform (or simply, G-transform) of Q if

$$dQ^G = \frac{G}{Q(G)} dQ.$$

- Let Q be a probability measure. $\{(X_t)_{t\in\Gamma}; Q\}$ be a stochastic process.
- We say a process $(Y_t)_{t\in\Gamma}$ is the G-transform of process (X_t) if

$$(Y_t)_{t\in\Gamma} \stackrel{d}{=} \{(X_t)_{t\in\Gamma}; Q^G\}.$$

Size-biased transform (exponential r.v.)

- Y: Strictly positive random variable with finite mean.
- $\dot{Y}$: Y-transform of Y.
- U: a uniform r.v. on $[0, 1]$.
- $\dot{Y}$ and U are independent.

Lemma (Pakes and Khattree 1992)

Y is exponentially distributed iff $Y \stackrel{d}{=} U \cdot \dot{Y}$.

- Further assume Y has finite variance.
- $\dot{Y}'$: copy of $\dot{Y}$.
- $\ddot{Y}$: a Y^2-transform of Y.
- $\dot{Y}, \dot{Y}', \ddot{Y}$ and U are independent.

Lemma (Ren, Song, and Sun 2018)

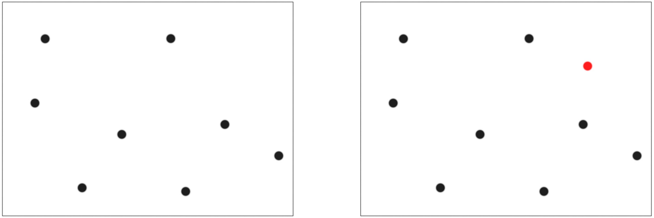
Y is exponentially distributed iff $\ddot{Y} \stackrel{d}{=} \dot{Y} + U \cdot \dot{Y}'$.

Size-biased transform (Poisson random measure)

- $\mathcal{N}$: a Poisson random measure (PRM) on a measurable space $(S, \mathscr{S})$ with intensity measure N.
- $F \in \mathscr{S}^+$: $0 < N(F) < \infty$.

Theorem (Ren, Song, and Sun 2019)

$$\{\mathcal{N}; P^{\mathcal{N}(F)}\} \overset{d}{=} \{\mathcal{N} + \delta_s; P \otimes N^F(ds)\}.$$



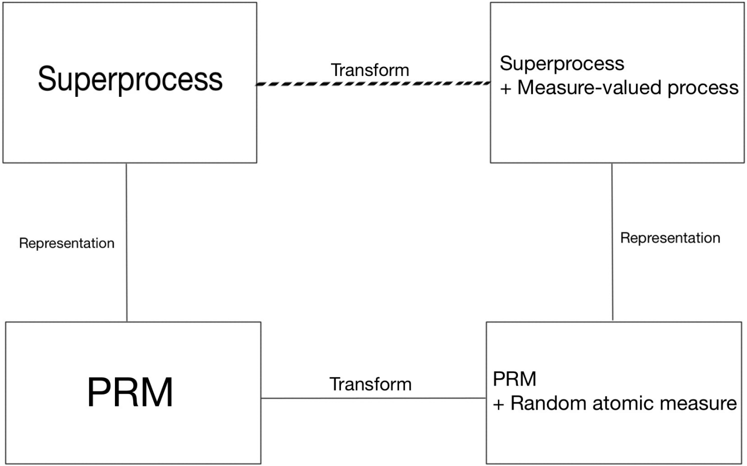
Superprocesses as PRMs

- $\mathcal{W}$: Skorokhod space of $\mathcal{M}_f$-valued càdlàg paths.
- $(\mathbb{N}_x)_{x\in E}$: Kuznetsov measure (N-measure, excursion measure) of superprocess (X_t).
- $\mu \in \mathcal{M}_f$.
- $\mathcal{N}_\mu$: a Poisson random measure on $\mathcal{W}$ with intensity measure $\int_E \mathbb{N}_x[\cdot]\mu(dx)$.

Theorem (Superprocesses as PRMs, see Li 2011 Theorem 8.24 for example.)

$$\{(X_t)_{t>0}; \mathbf{P}_\mu\} \overset{d}{=} \left(\int_{\mathcal{W}} w_t(\cdot)\mathcal{N}_\mu(dw)\right)_{t>0}.$$

Idea


Superprocess
Transform
Superprocess
+ Measure-valued process
Representation
Representation
PRM
Transform
PRM
+ Random atomic measure

Size-biased transforms of Superprocesses

- F: a non-negative measurable function on $\mathcal{W}$ s.t. $\mathbb{N}_\mu[F] \in (0, \infty)$.

Theorem (Ren, Song, and Sun 2019)

$$\{(X_t)_{t\geq 0}; \mathbf{P}_\mu^{\mathcal{N}(F)}\} \overset{d}{=} \{(X_t + w_t)_{t\geq 0}; \mathbf{P}_\mu \otimes \mathbb{N}_\mu^F(dw)\}.$$

- While considering the $\mathcal{N}(F)$-transform of superprocesses, we only have to characterize $\mathbb{N}^F$ the corresponding transform of the $\mathbb{N}$-measures.
- $F(w) = w_t(\phi)$: the classical spine decomposition theorem developed by Engländer and Kyprianou 2004, Liu, Ren, and Song 2011, and Eckhoff, Kyprianou, and Winkel 2015.
- $F(w) = w_t(f)$ for a general testing function f: a general spine decomposition theorem developed Ren, Song, and Sun 2019.
- $F(w) = w_t(\phi)^2$: a 2-spine decomposition theorem developed by Ren, Song, and Sun 2019.
- ......

Spine decomposition: An example

Example (Engländer and Kyprianou 2004)

Suppose the branching mechanism $\psi(x,z)=z^2$ and the underlying process is conservative. Then $\phi(x)\equiv 1$.

- **The spine** (ξ_t): a process with law Π_x;
- **The immigration** $\mathbf{n}^{\xi}_T(ds,dw)$: conditioned on (ξ_t), a Poisson random measure on $(0,T]\times\mathcal{W}$ with intensity measure $2ds\times\mathbb{N}_{\xi_s}(dw)$;

Then

$$\{(w_t)_{0<t\le T};\mathbb{N}_x^{w_t(1)}(dw)\}\overset{d}{=}\left(\int_{(0,t]\times\mathcal{W}}w_{t-s}\mathbf{n}^{\xi}_T(ds,dw)\right)_{0<t\le T}.$$

2-Spine decomposition: An example

Example (Ren, Song, and Sun 2019)

- **The main spine** $(\xi_t)_{t\geq 0}$: a process with law Π_x;
- **The splitting time** κ: conditioned on above, an uniform r.v. in $[0,T]$.
- **The auxiliary spine** $(\xi'_{\kappa+t})_{t\geq 0}$: conditioned on above, a process with law Π_{ξ_κ};
- **The main immigration** $\mathbf{n}^{\xi}_T$: conditioned on above, a Poisson random measure on $(0,T]\times\mathcal{W}$ with intensity measure $2ds\times\mathbb{N}_{\xi_s}(dw)$.
- **The auxiliary immigration** $\mathbf{n}^{\xi'}_T$: conditioned on above, a Poisson random measure on $(\kappa,T]\times\mathcal{W}$ with intensity measure $2ds\times\mathbb{N}_{\xi'_s}(dw)$.

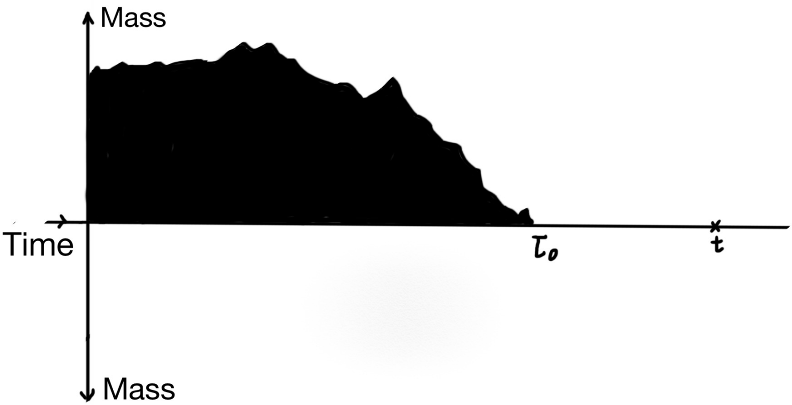
Then

$$\{(w_t)_{0<t\leq T};\mathbb{N}_x^{w_t(1)^2}(dw)\}\overset{d}{=}\left(\int_{(0,t]\times\mathcal{W}} w_{t-s}\,(\mathbf{n}^{\xi}_T+\mathbf{n}^{\xi'}_T)(ds,dw)\right)_{0<t\leq T}.$$

Sketch of the proof of Yaglom type result

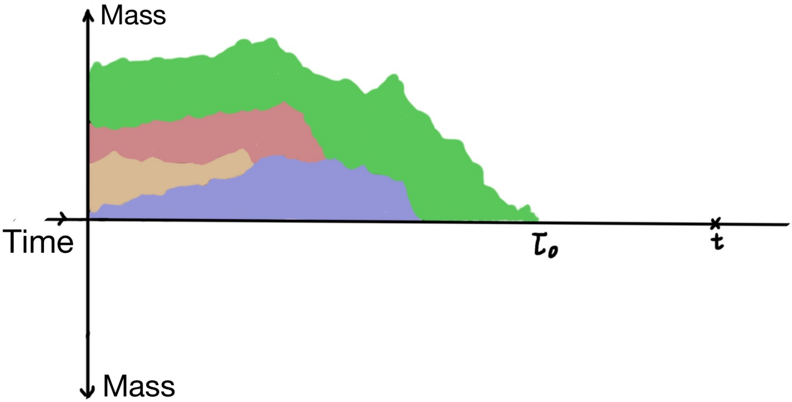
Let $F(w) = w_t(1)$. Consider the total mass of the superprocess

$$Z_t := X_t(1) = \mathcal{N}[F].$$



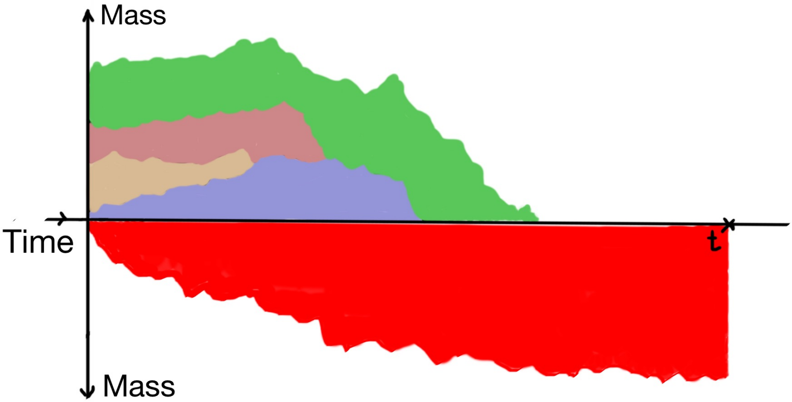
Sketch of the proof of Yaglom type result

Process (Z_t) can be decomposed as a PRM:



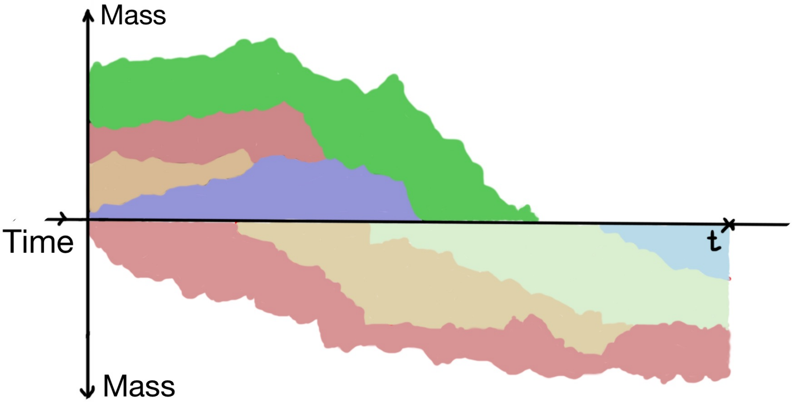
Sketch of the proof of Yaglom type result

Z_t-transform of the total mass: $\dot{Z}_t \overset{d}{=} \{Z_t + w_t(1); \mathbf{P} \otimes \mathbb{N}^{w_t(1)}(dw)\}$



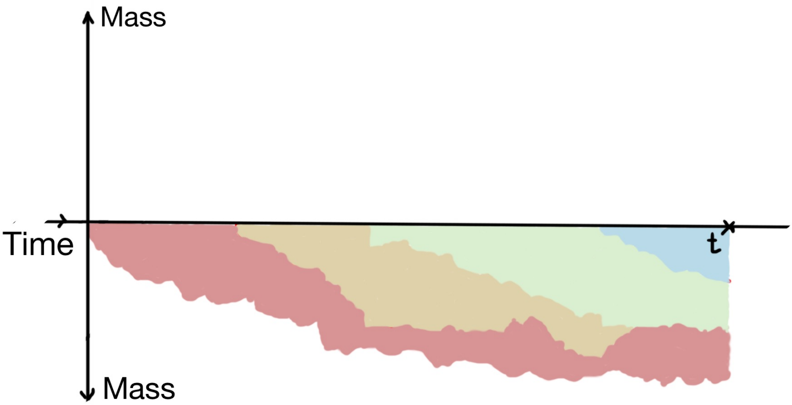
Sketch of the proof of Yaglom type result

Spine decomposition theorem:



Sketch of the proof of Yaglom type result

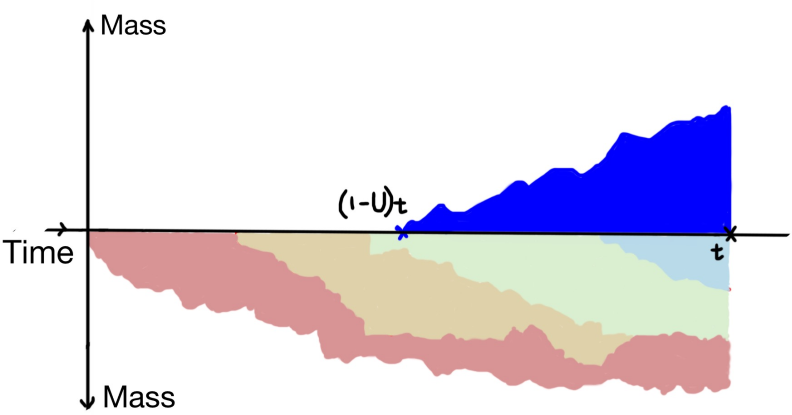
$$\dot{Z}_t \overset{d}{\approx} \{w_t(1); \mathbb{N}^{w_t(1)}(dw)\}.$$



Sketch of the proof of Yaglom type result

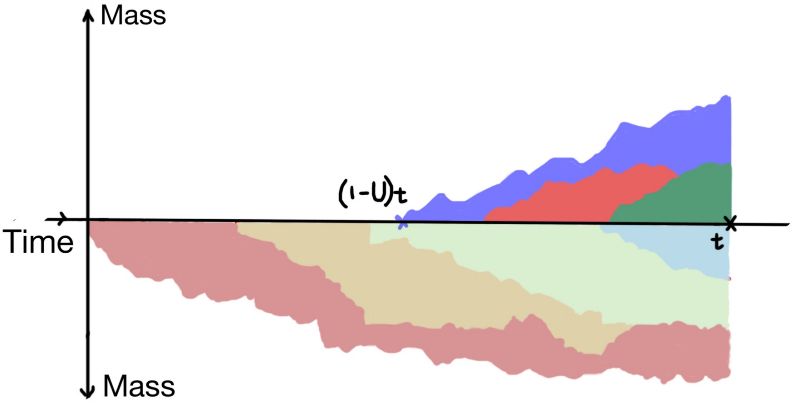
Z_t^2-transform of Z_t:

$$\ddot{Z}_t \overset{d}{\approx} \{w_t; \mathbb{N}^{w_t^2}(dw)\}.$$

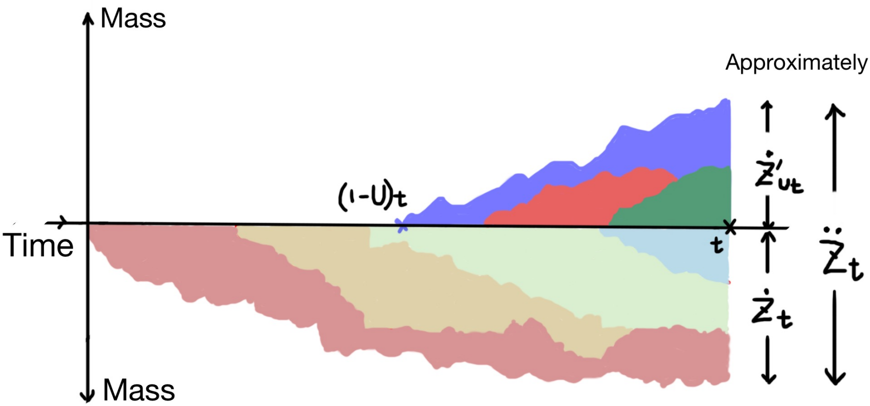


Sketch of the proof of Yaglom type result

The 2-spine decomposition theorem:



Key observation



Sketch of the proof of Yaglom type result

- Approximately:

$$\ddot{Z}_t \overset{d}{\approx} \dot{Z}_t + \dot{Z}'_{Ut}$$

- Renormalization:

$$\frac{\ddot{Z}_t}{t} \overset{d}{\approx} \frac{\dot{Z}_t}{t} + U \cdot \frac{\dot{Z}'_{Ut}}{Ut}$$

- Let $t \to \infty$:

$$\ddot{Y} \overset{d}{=} \dot{Y} + U\dot{Y}'.$$

- Y should be exponential random variable.

Results

- Let initial measure $\mu \in \mathcal{M}_f$ satisfies $\mu(\phi) < \infty$.

Theorem (Kolmogorov type result, Ren, Song, and Zhang 2015, Ren, Song, and Sun 2019)

$$t\mathbf{P}_\mu(X_t(1) > 0) \xrightarrow[t\to\infty]{} \frac{\mu(\phi)}{\frac{1}{2}\langle A\phi, \phi\phi^*\rangle_m}.$$

- Let testing function f satisfies $\phi^{-1}f \in b\mathscr{B}_E$.

Theorem (Yaglom type result, Ren, Song, and Zhang 2015, Ren, Song, and Sun 2019)

$$\left\{\frac{X_t(f)}{t}; \mathbf{P}_\mu(\cdot|X_t(1) > 0)\right\} \xrightarrow[t\to\infty]{d} \frac{1}{2}\langle \phi^*, f\rangle_m \langle A\phi, \phi\phi^*\rangle_m \mathbf{e},$$

where $\mathbf{e}$ *is an exponential random variable with mean 1.*

Remarks

- The idea of multi-spine decomposition is not new. It is first introduced by Harris and Roberts 2017 in the context of branching Markov processes. Then appeared in Abraham and Debs 2018; Ren, Song, and Sun 2018 for (discrete time) GW tree, and in Harris, Johnston, and Roberts 2017; Johnston 2017 for (continuous time) GW tree.
- Our Kolmogorov type and Yaglom type results for critical superprocesses are established under slightly weaker conditions than Ren, Song, and Zhang 2015.

Thank You!

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