Effect of small noise on the speed of reaction-diffusion equation with non-Lipschitz drift

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- Recall: we say the centered Gaussian process $\{B_t : t \geq 0\}$ is a Brownian motion if $\mathbb{E}[B_t B_s] = \int 1_{[0,t]}(r) \cdot 1_{[0,s]}(r)\, dr$.
- We say the centered Gaussian process $\{W_t(\phi) : t \geq 0, \phi \in L^2(\mathbb{R})\}$ is a Space-time white noise if

$$\mathbb{E}[W_t(\phi) W_s(\psi)] = \iint 1_{[0,t]}(r)\, \phi(x) \cdot 1_{[0,s]}(r)\, \psi(x)\, dr\, dx.$$

Walsh (1986)

Similar to Itô calculus, given a space-time white noise W and a predictable random field $\{g_s(x) : t \geq 0, x \in \mathbb{R}\}$ such that

$$\iint_0^t g_s(x)^2\, ds\, dx < +\infty, \quad t \geq 0, \quad \text{a.s.}$$

One can define a continuous local martingale

$$\iint_0^t g_s(x)\, W(ds\, dx), \quad t \geq 0.$$

- **Stochastic reaction-diffusion equation**

$$(*) \qquad \partial_t u = \frac{1}{2}\partial_x^2 u + f(u) + \sigma(u)\dot{W}, \qquad t \geq 0,\ x \in \mathbb{R}$$

where W is a space-time white noise.

Walsh (1986)

We say a random field $\{u_s(x) : s \geq 0, x \in \mathbb{R}\}$ and a white noise W solve $(*)$ if $\forall\ \varphi \in C_c^\infty(\mathbb{R})$ and $t \geq 0$, almost surely

$$\int u_t(x)\varphi(x)\,dx - \int u_0(x)\varphi(x)\,dx$$

$$= \iint_0^t u_s(x)\frac{\partial_x^2}{2}\varphi(x)\,ds\,dx + \iint_0^t f(u_s(x))\varphi(x)\,ds\,dx + \iint_0^t \sigma(u_s(x))\varphi(x)\,W(ds\,dx).$$

Iwata (1987), Reimers (1989)

When f and σ are Lipschitz continuous, $(*)$ has a pathwise unique strong solution.

When f and σ are continuous function with linear growth, $(*)$ has weak solution.

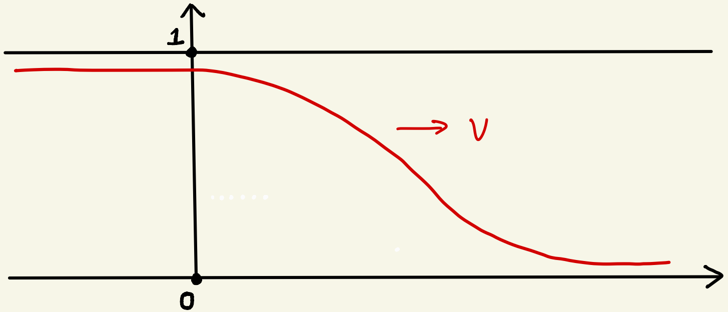
FKPP equation

$$(0) \qquad \partial_t u = \frac{1}{2}\partial_x^2 u + u(1-u), \quad t \geq 0,\ x \in \mathbb{R}; \quad 0 \leq u_t(x) \leq 1.$$

Fisher (1937), Kolmogorov, Petrovsky & Piscuonv (1937)

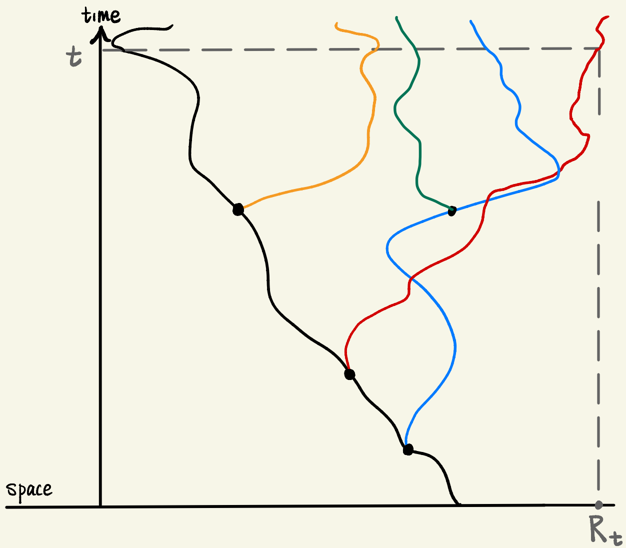
1. For $\forall\, v \geq \sqrt{2}$, $\exists$ a wave profile F_v s.t. $u_t(x) = F_v(x - vt)$ solves (0).
2. If $u_0(x) = \mathbb{1}_{(-\infty,0)}(x)$, then $\exists$ a centering term $m(t)$ s.t.

$$u_t(x + m(t)) \xrightarrow[t\to+\infty]{} F_{\sqrt{2}}(x) \quad \& \quad m(t)/t \xrightarrow[t\to\infty]{} \sqrt{2}.$$



Bramson (1978)

$$m(t) = \sqrt{2}\,t - \frac{3}{2\sqrt{2}}\log t + O(1).$$



Branching Brownian motion

- ∀ particle moves as Brownian motion.
- ∀ particle branches into 2 particles with rate 1.

Independently

Mckean (1975)

If R_t is the position of the right-most particle, then $u_t(x) := \mathbb{P}(R_t > x)$ solves the FKPP equation.

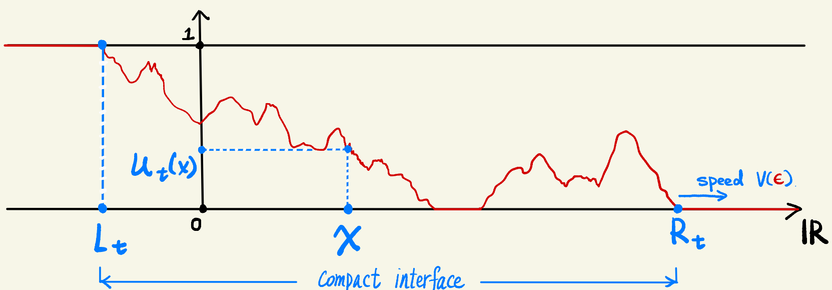
Stochastic FKPP equation

$$(1) \quad \partial_t u = \tfrac{1}{2}\partial_x^2 u + u(1-u) + \underbrace{\epsilon}_{\text{noise strength}} \underbrace{\sqrt{u(1-u)}\,\dot{W}}_{\text{Wright-Fisher white noise}} \qquad (t,x)\in \mathbb{R}_+ \times \mathbb{R}\ ; \quad \epsilon > 0.$$

Shiga (1988), Mueller & Sowers (1995)

The weak existence and weak uniqueness of SPDE (1) hold in

$C_I := \{$ Continuous $[0,1]$-valued functions on $\mathbb{R}$ with compact interface $\}$.



Four equivalent solution concepts:

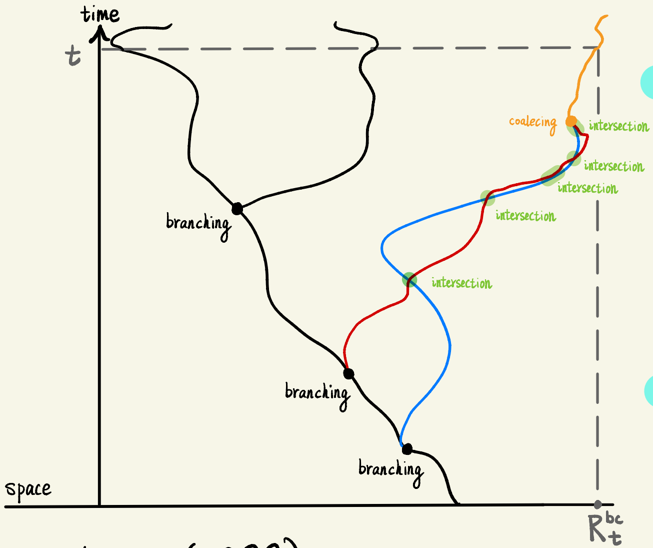
- Mild form
- Schwartz distribution
- Martingale problem

Walsh (1984), Iwata (1987)

- Dual process of branching-coalescing Brownian motion. — Shiga (1988)

Mueller & Sowers (1995), Conlon & Doering (2005)

$\forall \epsilon > 0$, $\exists$ a deterministic $V(\epsilon) \in \mathbb{R}$ s.t. $\dfrac{R_t}{t} \xrightarrow[t\to\infty]{} V(\epsilon)$ a.s.



Branching-coalescing Brownian motion

Independently
- $\forall$ particle moves as Brownian motion.
- $\forall$ Particle branches into 2 particles with rate 1.
- $\forall$ pair of particles coalesce into 1 particle with rate ϵ^2 according to their intersection local time.

Notation:

$I_t = \{$ Particles alive at time $t\}$.

$X^i_t =$ Position of Particle $i \in I_t$ at time t.

Shiga (1988)

If the branching-coalescing Brownian motion is independent of the weak solution $u_t(x)$,

then $\mathbb{E} \prod_{i \in I_0} [1 - u_t(X^i_0)] = \mathbb{E} \prod_{i \in I_t} [1 - u_0(X^i_t)]$.

$\xrightarrow{\text{LHS}}$ finite moments of u_t $\xrightarrow{\text{determine}}$ distribution of u_t $\longrightarrow$ weak uniqueness.

$\xrightarrow{\text{taking } u_0 = 1_{(-\infty,0)}}$ $\mathbb{P}(R^{bc}_t > x) = \mathbb{E}\, u_t(x)$ $\longrightarrow$ propagation of branching-coalescing Brownian motion.

N-branching random walk

Bérard & Gouéré (2010).

Branching-coalescing Brownian motion

Mueller, Mytnik & Quastel (2011).

N-branching Brownian motion

Maillard (2012). ....

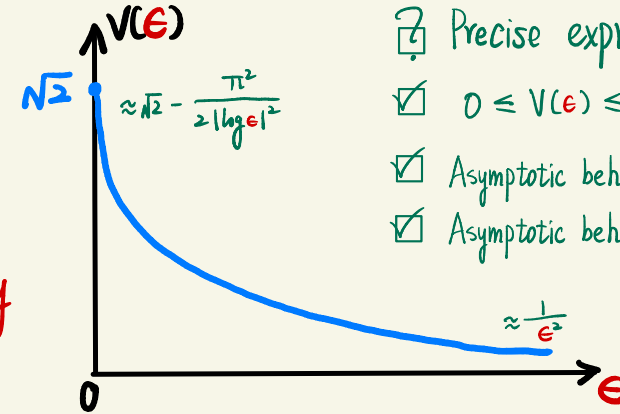
Branching Brownian motion in a strip

Berestycki, Berestycki & Schweinsberg (2013).

L-branching Brownian motion

Pain (2016).

universality



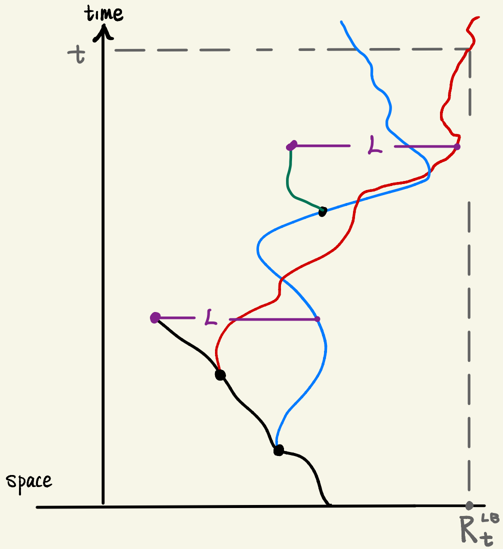
- ? Precise expression.
- ☑ $0 \leq V(\epsilon) \leq \sqrt{2}$.
- ☑ Asymptotic behavior for small ϵ.
- ☑ Asymptotic behavior for large ϵ.

Brunet & Derrida conjecture (1997), Mueller, Mytnik & Quastel (2011)

$$V(\epsilon) = \sqrt{2} - \frac{\pi^2}{2|\log \epsilon|^2} + O\left(\frac{\log|\log \epsilon|}{|\log \epsilon|^3}\right), \qquad \epsilon \to 0.$$

Conlon & Doering (2005), Mueller, Mytnik & Ryzhik (2021)

$$\lim_{\epsilon \to +\infty} \epsilon^2 V(\epsilon) = 1.$$



L-Branching Brownian motion.

Independently,

- $\forall$ Particle moves as Brownian motion
- $\forall$ Particle branches into 2 particles with rate 1.
- $\forall$ Particle is killed if its position $\leq R_t - L$.

Pain (2016)

$$\lim_{t\to\infty} \frac{R_t^{LB}}{t} =: V_L = \sqrt{2} - \frac{\pi^2}{2\sqrt{2}\,L^2} + o\left(\frac{1}{L^2}\right), \quad L \to +\infty.$$

Benguria & Depassier (1996): In application, people are interested in reaction-diffusion equation
$\partial_t u = \frac{1}{2}\partial_x^2 u + f(u)$ with more general reaction term.
And in many cases when f is lipschitz, the system has finite propagation speed.

Aguirre & Escobedo (1986): If $f(u) = u^p(1-u)$ with $0<p<1$, then the system doesn't have finite propagation.
non-Lipschitz

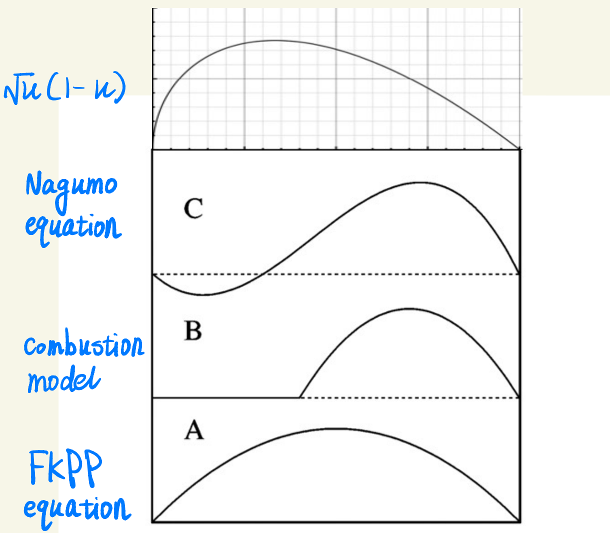


FIG. 1. The three basic types of reaction terms that arise in different applications.

Image of $f(u)$ © Benguria & Depassier (1996)

Athreya & Tribe (2000):

A large class of stochastic reaction-diffusion equations
$$\partial_t u = \frac{1}{2}\partial_x^2 u + f(u) + \sigma(u)\dot{W}$$
with Lipschitz f and σ^2, have duality relation to some branching-coalescing type particle system.

Conjecture

Shiga's duality holds between $\partial_t u = \frac{1}{2}\partial_x^2 u + u^p(1-u) + \epsilon\sqrt{u(1-u)}\,\dot{W}$ and branching-coalescing Brownian motion whose offspring law has generating function $u-(1-u)^p u$. $(0<p<1)$

Examples: types A, B, C & $f(u) = u^p(1-u)$ with $\frac{1}{2} \le p \le 1$.

(2) $\partial_t u = \frac{1}{2}\partial_x^2 u + f(u) + \epsilon\sqrt{u(1-u)}\,\dot{W}$ where $f \in C([0,1])$ and $\sup\limits_{u\in(0,1)} \frac{|f(u)|}{\sqrt{u(1-u)}} < +\infty$

Mueller, Mytnik & Ryzhik (2021)

- $\forall \epsilon > 0$, the weak existence and weak uniqueness of SPDE (2) hold in $C_I := \{$ Continuous $[0,1]$-valued functions on $\mathbb{R}$ with compact interface $\}$.
- $\forall \epsilon > 0$, $\exists$ a deterministic $V_f(\epsilon) \in \mathbb{R}$ s.t. $\frac{R_t}{t} \xrightarrow[t\to\infty]{} V_f(\epsilon)$ a.s.

Idea: $\partial_t u = \frac{1}{2}\partial_x^2 u + \epsilon\sqrt{u(1-u)}\,\dot{W}$.

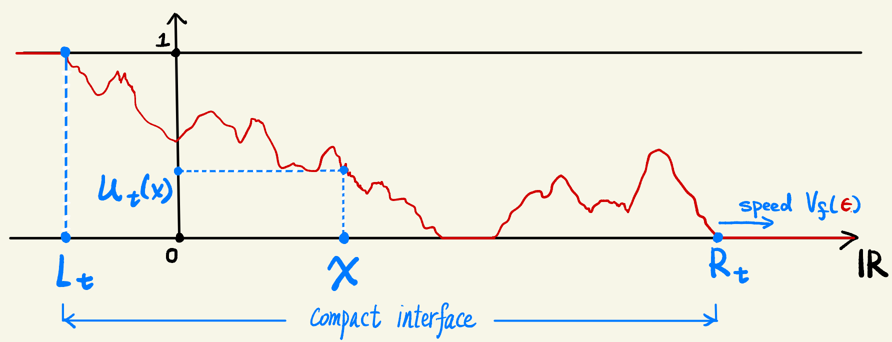
(2) Girsanov transformation

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$$

$$M_t = \int_{L_t}^{R_t}\int_0^t \frac{f(u)}{\sqrt{u(1-u)}}\,dW$$

bounded

Finite propagation $\xleftarrow{\text{due to}}$ Noise $\epsilon > 0$



Mueller, Mytnik & Ryzhik (2021)

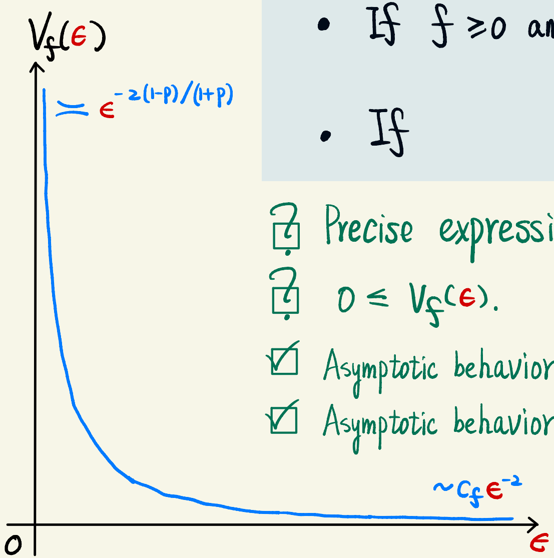
$\forall\ f \in C([0,1])$ satisfying a "slightly stronger" condition than $\sup_{u\in(0,1)} \frac{|f(u)|}{\sqrt{u(1-u)}} < +\infty$,

$\exists$ a deterministic $C_f \in \mathbb{R}$ s.t. $\lim_{\epsilon\to+\infty} \epsilon^2 V_f(\epsilon) = C_f$.

Mytnik, Barnes & S. (2021+)

$\forall\ f \in C([0,1])$ s.t. $\sup_{u\in(0,1)} \frac{|f(u)|}{\sqrt{u(1-u)}} < +\infty$:

- If $f \geq 0$ and $\exists\ \frac{1}{2} \leq p \leq 1$ s.t. $\liminf_{u\downarrow 0} u^{-p} f(u) > 0$, then $\liminf_{\epsilon\downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_f(\epsilon) > 0$;
- If $\exists\ \frac{1}{2} \leq p \leq 1$ s.t. $\limsup_{u\downarrow 0} u^{-p} f(u) < +\infty$, then $\limsup_{\epsilon\downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_f(\epsilon) < +\infty$.



- ? Precise expression.
- ? $0 \leq V_f(\epsilon)$.
- ☑ Asymptotic behavior for small ϵ.
- ☑ Asymptotic behavior for large ϵ.

For example, if $f(u) = u^p(1-u)$ with $\frac{1}{2} \leq p \leq 1$, then

$\exists\ c, C > 0$ & $\epsilon_0 > 0$ s.t.

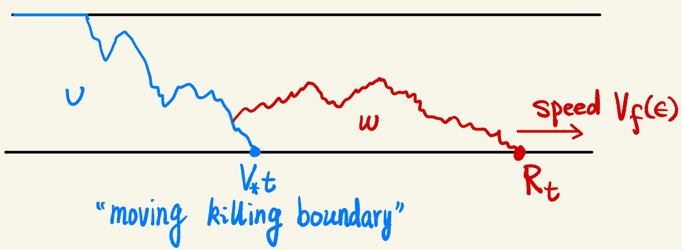
$$c\,\epsilon^{-2\frac{1-p}{1+p}} \leq V_f(\epsilon) \leq C\,\epsilon^{-2\frac{1-p}{1+p}}, \quad 0 < \epsilon \leq \epsilon_0$$

Same exponent, can not be improved.

Step 1: Choose a killing boundary.

Decompose $u = v + w$ where
$$\begin{cases} \partial_t v = \frac{1}{2}\partial_x^2 v + f(v) + \epsilon\sqrt{v(1-v)}\,\dot{W}', & x < V_* t \\ v = 0, & x \geq V_* t \end{cases}$$

left to be chosen



If we choose:

- $V_* \gg V_f(\epsilon) \xrightarrow{\text{then}} w$ will be small,
- $V_* \ll V_f(\epsilon) \xrightarrow{\text{then}} w$ will be large.

Idea: If $\exists\, V_*$ s.t. w is neither too small nor too large,
Then $V_* \approx V_f(\epsilon)$.

Criticality

An insight: The balanced value for V_* can be predicted by finding the solution (F, V_*) so that

$$\begin{cases} F'(0-) = -\epsilon^2, & \\ F(-\infty) = 1, & \\ \rho(t,x) := F(x - V_* t), & \forall (t,x), \\ \partial_t \rho = \frac{1}{2}\partial_x^2 \rho + f(\rho), & x < V_* t, \\ \rho = 0, & x \geq V_* t. \end{cases}$$

phase plane (PDE) argument

$$V_* \sim \epsilon^{-2\frac{1-p}{1+p}}.$$

Only a prediction.
Still need to analyze w when $V_* \sim \epsilon^{-2\frac{1-p}{1+p}}$.

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Step 2: Analyze w.

$A_t = \int_0^t \dot{A}_s \, ds$ is the total mass of v killed at the boundary up to time t.

From $\quad \partial_t u = \partial_x^2 u + f(u) + \epsilon \sqrt{u(1-u)}\, \dot{W}$

subtract $\quad \partial_t v = \partial_x^2 v + f(v) + \epsilon \sqrt{v(1-v)}\, \dot{W}' - \dot{A}_t \delta_{V_t^*}(x)$,

We get $\quad \partial_t w = \partial_x^2 w + f(u) - f(v) + \epsilon \sqrt{u(1-u)}\, \dot{W} - \epsilon \sqrt{v(1-v)}\, \dot{W}' + \dot{A}_t \delta_{V_t^*}(x)$.

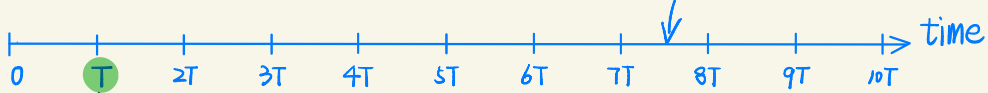
Approximately $\quad \partial_t w \approx \partial_x^2 w + C_f w^p + \epsilon \sqrt{w}\, \dot{W}'' + \dot{A}_t \delta_{V_t^*}(x)$

Girsanov transformation $\quad \partial_t w \approx \partial_x^2 w + \epsilon \sqrt{w}\, \dot{W}'' + \dot{A}_t \delta_{V_t^*}(x)$

critical super-Brownian motion with immigration

! Girsanov transformation doesn't preserve long time behavior. ☑ properties of support.

So we perform the transformation on each of the small intervals $[nT, (n+1)T)$.



left to be chosen
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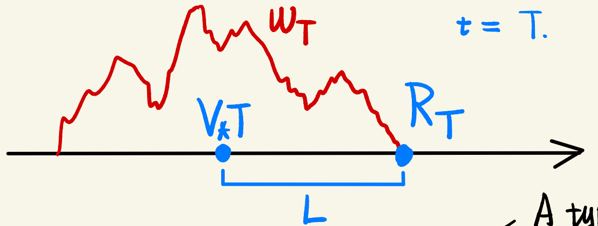
- can't be too large, otherwise: Transformed $w \not\approx$ Original w.
- can't be too small, otherwise: Small time random fluctuation $\xrightarrow{\text{Covers}}$ info. of propagation speed.

Step 3: Choose T.



A typical distance for the support of w to travel in a time interval with length T.

We want to choose T s.t.:

- L can be explained by the thermal diffusivity, i.e. $L \sim \sqrt{T}$.
- w does not give excess speed, i.e. $L/T \lesssim V_* \sim \epsilon^{-2\frac{1-p}{1+p}}$.
- T is as small as possible $\xleftarrow{\text{in order}}$ Transformed $w \approx$ Original w.

$$\Rightarrow \quad T \sim \epsilon^{4\frac{1-p}{1+p}}.$$

Further questions:

- $\lim_{u \downarrow 0} u^{-p} f(u)$ exists $\Rightarrow$ $\lim_{\epsilon \downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_f(\epsilon)$ exists.
- Duality for $\partial_t u = \frac{1}{2}\partial_x^2 u + u^p(1-u) + \epsilon\sqrt{u(1-u)}\dot{W}$ $(0<p<1)$.
- Weak uniqueness for $\partial_t u = \frac{1}{2}\partial_x^2 u + u^p(1-u) + \epsilon\sqrt{u(1-u)}\dot{W}$ $(0<p<\frac{1}{2})$.
- Speed of L_t = Speed of R_t $\left(\overset{\text{duality}}{=}\right.$ Speed of Branching-coalescing Brownian motion$\left.\right)$.
- $(u_t(R_t+x) : x \in \mathbb{R}) \xrightarrow[t \to +\infty]{\text{weakly}}$ a stationary (random) wave profile.
- Speed of N-BBM & L-BBM with heavy-tailed offspring law (universality).

This talk is based on a joint work with

&

Leonid Mytnik

Clayton Barnes.

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Thanks!!