Stable Central Limit Theorems for Super Ornstein-Uhlenbeck Processes

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Joint work with **Yan-Xia Ren**, **Renming Song** and **Jianjie Zhao**

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This talk is based on the following two papers:

[1] Y.-X. Ren, R. Song Z. Sun and J. Zhao: Stable central limit theorems for super Ornstein-Uhlenbeck processes. *Elect. J. Probab.*, **24** (2019), No. 141, 1–42

[2] Y.-X. Ren, R. Song Z. Sun and J. Zhao: Y.-X. Ren, R. Song, Z, Sun and J. Zhao Stable central limit theorems for super Ornstein-Uhlenbeck processes, II. https://arxiv.org/pdf/2005.11731.pdf

Outline

Background/CLT with finite second moment

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Some spatial central limit theorems for supercritical branching OU processes with binary branching were proved in Adamczak-Milos (EJP, 2015), and some spatial central limit theorems for supercritical super-processes were proved in Milos (JTP, 2018). These two papers made connections between spatial central limit theorems and branching rate regimes. The results of these two papers have been refined and generalized in a series of papers by Ren-Song-Zhang.

Background/CLT with infinite second moment

There are also central limit theorem type results for supercritical branching processes and branching Markov processes with branching mechanisms of infinite second moment. For earlier papers, see Asmussen, (AOP, 1976) and Heyde (JAP, 1971).

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Our goal is to establish stable central limit theorems for super-OU processes with general stable branching mechanisms.

Outline

Model/Parameters

Suppose spatial motion $\xi = \{(\xi_t)_{t\geq 0}, (\Pi_x)_{x\in\mathbb{R}^d}\}$ is an OU process on $\mathbb{R}^d$ with generator

$$Lf(x) = \frac{1}{2}\sigma^2\Delta f(x) - bx\cdot\nabla f(x)$$

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Suppose that ψ is a branching mechanism of the form

$$\psi(z) = -\alpha z + \rho z^2 + \int_{(0,\infty)}(e^{-zy} - 1 + zy)\pi(dy)$$

where $\alpha > 0$ and $\rho \geq 0$ and π is a measure on $(0,\infty)$ with $\int_{(0,\infty)}(y\wedge y^2)\pi(dy) < \infty$. We call α the branching rate.

Model/Assumptions

Assumption 1

The branching mechanism satisfies Grey's condition, i.e. there is some constant $z' > 0$ such that $\psi(z) > 0$ for all $z > z'$ and that $\int_{z'}^{\infty} \psi(z)^{-1} dz < \infty$.

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Assumption 2

There exist constants $\eta > 0$ and $\beta \in (0, 1)$ such that

$$\int_{(1,\infty)} y^{1+\beta+\delta} \left| \pi(dy) - \frac{\eta \, dy}{\Gamma(-1-\beta) y^{2+\beta}} \right| < \infty,$$

for some $\delta \in (0, 1 - \beta)$.

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for some $\delta \in (0, 1-\beta)$.

Roughly speaking, Assumption 2 says that ψ is "not too far away" from $\widetilde{\psi}(z) = -\alpha z + \eta z^{1+\beta}$ near 0.

Model/Superprocess

Denote by $\mathcal{M}(\mathbb{R}^d)$ ($\mathcal{M}_c(\mathbb{R}^d)$, resp.) the space of all finite Borel measures (of compact support, resp.) on $\mathbb{R}^d$. We suppose that $X = \{(X_t)_{t\geq 0}, (\mathbb{P}_\mu)_{\mu\in\mathcal{M}(\mathbb{R}^d)}\}$ is a superprocess with spatial motion ξ and branching mechanism ψ, i.e., a super-OU process.

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For each non-negative bounded Borel function f on $\mathbb{R}^d$, we have

$$\mathbb{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad t \geq 0, \mu \in \mathcal{M}(\mathbb{R}^d),$$

where $(t, x) \mapsto V_t f(x)$ is the unique locally bounded non-negative solution to the equation

$$V_t f(x) + \Pi_x\Big[\int_0^t \psi(V_{t-s} f(\xi_s))\mathrm{d}s\Big] = \Pi_x[f(\xi_t)], \quad x \in \mathbb{R}^d, t \geq 0.$$

Outline

Main Result/Preliminary

The OU process ξ has an invariant distribution

$$\varphi(x)dx := \Big(\frac{b}{\pi\sigma^2}\Big)^{d/2}\exp\Big(-\frac{b}{\sigma^2}|x|^2\Big)dx, \quad x \in \mathbb{R}^d.$$

Let $L^2(\varphi) := \Big\{h \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}) : \int_{\mathbb{R}^d} |h(x)|^2\varphi(x)dx < \infty\Big\}$. Then, $L^2(\varphi)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\varphi}$.

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The OU operator L has discrete spectrum $\sigma(L) = \{-bk : k \in \mathbb{Z}_+\}$. The eigenfunctions of L consists a family of polynomials $\{\phi_p : p \in \mathbb{Z}_+^d\}$ which forms a complete orthonormal basis of $L^2(\varphi)$. For each $p \in \mathbb{Z}_+^d$, ϕ_p is an eigenfunction of L corresponding to the eigenvalue $b|p|$, where $|p| := \sum_{k=1}^d p_k$.

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Lemma 1

For any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $(H_t^p)_{t \geq 0}$ is a $\mathbb{P}_\mu$-martingale. Furthermore, if $\alpha\tilde{\beta} > |p|b$, then for any $\gamma \in (0, \beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $(H_t^p)_{t \geq 0}$ is a $\mathbb{P}_\mu$-martingale bounded in $L^{1+\gamma}(\mathbb{P}_\mu)$; thus $H_\infty^p := \lim_{t \to \infty} H_t^p$ exists $\mathbb{P}_\mu$-almost surely and in $L^{1+\gamma}(\mathbb{P}_\mu)$.

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Fixing $\beta \in (0,1)$, $p \in \mathbb{Z}_+^d$ and $b > 0$, if the branching rate α is large enough so that $\alpha\tilde{\beta} > |p|b$ then we say we are in the large branching rate regime; if $\alpha\tilde{\beta} = |p|b$ then we are in the critical branching rate regime; if $\alpha\tilde{\beta} < |p|b$ then we are in the small branching rate regime.

Main Result/Preliminary

Denote by $\mathcal{P} \subset L^2(\varphi)$ the class of functions of polynomial growth on $\mathbb{R}^d$:

$$\left\{ f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}) : \exists\ C > 0, n \in \mathbb{Z}_+ \text{ s.t. } \forall x \in \mathbb{R}^d,\ |f(x)| \le C(1+|x|)^n \right\}.$$

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Then we have the decomposition $\mathcal{P} = \mathcal{C}_{\mathrm{s}} \otimes \mathcal{C}_{\mathrm{c}} \otimes \mathcal{C}_{\mathrm{l}}$ where

$$\mathcal{C}_{\mathrm{s}} := \mathcal{P} \cap \overline{\mathrm{Span}}\{\phi_p : \alpha\tilde{\beta} < |p|b\}$$
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Note that $\mathcal{C}_{\mathrm{s}}$ is an infinite dimensional space, $\mathcal{C}_{\mathrm{l}}$ and $\mathcal{C}_{\mathrm{c}}$ are finite dimensional spaces, and $\mathcal{C}_{\mathrm{c}}$ might be empty.

Main Result/Preliminary

Define a semigroup

$$T_t f := \sum_{p\in\mathbb{Z}_+^d} e^{-\left||p|b-\alpha\tilde{\beta}\right|t}\langle f,\phi_p\rangle_\varphi \phi_p, \quad t\geq 0, f\in\mathcal{P},$$

and a family of functionals

$$m_t[f] := \eta\int_0^t \mathrm{d}u \int_{\mathbb{R}^d}\big(-iT_u f(x)\big)^{1+\beta}\varphi(x)\mathrm{d}x, \quad 0\leq t<\infty, f\in\mathcal{P}.$$

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Proposition 2

For each $f \in \mathcal{P}$, there exists a $(1+\beta)$-stable random variable ζ^f with characteristic function $\theta \mapsto e^{m[\theta f]}, \theta \in \mathbb{R}$, where

$$m[f] := \begin{cases} \lim_{t\to\infty} m_t[f], & f \in \mathcal{C}_\mathrm{s} \oplus \mathcal{C}_\mathrm{l}, \\ \lim_{t\to\infty} \frac{1}{t} m_t[f], & f \in \mathcal{P} \setminus (\mathcal{C}_\mathrm{s} \oplus \mathcal{C}_\mathrm{l}). \end{cases}$$

Main Result/Preliminary

For $f \in \mathcal{P}$, note that

$$\mathrm{X}_t(f) = \sum_{p\in\mathbb{Z}_+^d} \langle f, \phi_p\rangle_\varphi e^{(\alpha-|p|b)t} H_t^p, \quad t \geq 0.$$

Define the centering

$$\mathrm{x}_t(f) := \sum_{p\in\mathbb{Z}_+^d:\alpha\tilde{\beta}>|p|b} \langle f, \phi_p\rangle_\varphi e^{(\alpha-|p|b)t} H_\infty^p, \quad t \geq 0.$$

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Let $D := \{\exists t \geq 0, \text{ such that } \|X_t\| = 0\}$ be the extinction event.

Main Result

Theorem 3 (Ren, Song, S. and Zhao, arXiv:2005.11731)

If $\mu \in \mathcal{M}_c(\mathbb{R}^d) \setminus \{0\}$, $f_s \in \mathcal{C}_s$, $f_c \in \mathcal{C}_c$ and $f_l \in \mathcal{C}_l$, then under $\mathbb{P}_\mu(\cdot|D^c)$,

$$S(t) := \left(e^{-\alpha t}\|X_t\|, \frac{X_t(f_s)}{\|X_t\|^{1-\tilde{\beta}}}, \frac{X_t(f_c)}{\|tX_t\|^{1-\tilde{\beta}}}, \frac{X_t(f_l) - \mathrm{x}_t(f_l)}{\|X_t\|^{1-\tilde{\beta}}} \right)$$

$$\xrightarrow[t\to\infty]{d} (\widetilde{H}_\infty, \zeta^{f_s}, \zeta^{f_c}, \zeta^{-f_l}),$$

where $\widetilde{H}_\infty$ has the distribution of $\{H^0_\infty; \widetilde{\mathbb{P}}_\mu\}$; ζ^{f_s}, ζ^{f_c} and ζ^{-f_l} are the $(1+\beta)$-stable random variables described in Proposition 2; $\widetilde{H}_\infty$, ζ^{f_s}, ζ^{f_c} and ζ^{-f_l} are independent.

Main Result

Corollary 4

Let $\mu \in \mathcal{M}_c(\mathbb{R}^d) \setminus \{0\}$ and $f \in \mathcal{P}$ with $f = f_{\mathrm{s}} + f_{\mathrm{c}} + f_{\mathrm{l}}$ where $f_{\mathrm{s}} \in \mathcal{C}_{\mathrm{s}}$, $f_{\mathrm{c}} \in \mathcal{C}_{\mathrm{c}}$ and $f_{\mathrm{l}} \in \mathcal{C}_{\mathrm{l}}$. Then under $\mathbb{P}_\mu(\cdot|D^c)$, it holds that

1. if $f_{\mathrm{c}} \equiv 0$, then

$$\frac{X_t(f) - \mathrm{x}_t(f)}{\|X_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{s}}} + \zeta^{-f_{\mathrm{l}}},$$

   where $\zeta^{f_{\mathrm{s}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are the $(1+\beta)$-stable random variables described in Proposition 2, $\zeta^{f_{\mathrm{s}}}$ and $\zeta^{-f_{\mathrm{l}}}$ are independent;

2. if $f_{\mathrm{c}} \not\equiv 0$, then

$$\frac{X_t(f) - \mathrm{x}_t(f)}{\|tX_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta^{f_{\mathrm{c}}}.$$

   where $\zeta^{f_{\mathrm{c}}}$ is the $(1+\beta)$-stable random variable described in Proposition 2.

Outline

Intuition

The central limit theorem for $X_t(\phi_p)$ takes different forms depending on whether it's in the large branching regime ($\alpha\tilde{\beta} > |p|b$), or in the critical branching regime ($\alpha\tilde{\beta} = |p|b$), or in the small branching regime ($\alpha\tilde{\beta} < |p|b$). We now give some intuitive explanation of this phase transition.

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The phase transition is due to an interplay of two competing effects in the system: coarsening and smoothing. The coarsening effect corresponds to the increase of the spatial inequality and is a consequence of the branching. The smoothing effect corresponds to the decrease of the spatial inequality and is a consequence of the mixing property of the OU processes.

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The branching rate α captures the mean intensity of the branching in the system. Therefore, the lager the branching rate α, the stronger the coarsening effect.

The tail index β describes the heaviness of the tail of the offspring distribution. When β is smaller i.e. the tail is heavier, then it is more likely that one particle can suddenly have a large amount of offspring. In other words, the larger the tail index β, the smaller the fluctuation of offspring number, and then the stronger the coarsening effect.

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The order $|p|$ is related to the capability of ϕ_p capturing the mixing property of the OU particles. In particular, in the case that $|p| = 0$, no mixing property can be captured. (Since $\phi_0 \equiv 1$, we are only considering the total mass $X_t(\phi_0) = \|X_t\|$). In general, the higher the order $|p|$, the more mixing property can be captured by ϕ_p, and therefore the stronger the smoothing effect.

Thank you!