Quasi-stationary distributions for subcritical superprocesses

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Superprocess

- Let E be a Polish space.
- Let $\xi = \{(\xi_t)_{t\in[0,\zeta)}; (\Pi_x)_{x\in E}\}$ be a E-valued Borel right process with (sub)Markovian transition semigroup $(P_t)_{t\geq 0}$.
- Let ψ be a function on $E \times \mathbb{R}_+$ such that

$$\psi(x,z) = -\beta(x)z + \sigma(x)^2 z^2 + \int_{(0,\infty)} (e^{-zu} - 1 + zu)\pi(x, \mathrm{d}u)$$

  where $\beta, \sigma \in \mathcal{B}_b(E)$ and $(u \wedge u^2)\pi(x, \mathrm{d}u)$ is a bounded kernel from E to $(0,\infty)$.
- Let $\mathcal{M}_f(E)$ be the space of all finite Borel measures on E equipped with the topology of weak convergence.
- Say a function f on $\mathbb{R}_+ \times E$ is locally bounded if

$$\sup_{s\in[0,t],x\in E} |f(s,x)| < \infty, \quad t \in \mathbb{R}_+.$$

Superprocess

- For a measure μ and a function f, denote by $\mu(f)$ the integral of f w.r.t. μ when it is well-defined.
- $\forall f \in \mathcal{B}_b^+(E)$, $\exists$ a unique locally bounded non-negative Borel function $(t,x) \mapsto V_t f(x)$ on $\mathbb{R}_+ \times E$ such that

$$V_t f + \int_0^t P_s \psi(\cdot, V_{t-s} f(\cdot)) \mathrm{d}s = P_t f \quad \text{ on } E, t \geq 0.$$

- $\exists$ an $\mathcal{M}_f(E)$-valued Borel right process $X = \{(X_t)_{t\geq 0}; (\mathbb{P}_\mu)_{\mu \in \mathcal{M}_f(E)}\}$ such that

$$\mathbb{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad f \in \mathcal{B}_b^+(E), t \geq 0, \mu \in \mathcal{M}_f(E).$$

- We call this process the (ξ, ψ)-superprocess (Watanabe (1968), Ikeda, Nagasawa and Watanabe (1968, 1969), Dawson (1975, 1977)).

Yaglom limit, QLDs and QSDs

- Denote $\mathbf{0}$ the null measure on E. Write $\mathcal{M}_f^o(E) = \mathcal{M}_f(E) \setminus \{\mathbf{0}\}$.
- For any probability measure $\mathbf{P}$ on $\mathcal{M}_f(E)$, define

$$(\mathbf{P}\mathbb{P})[\cdot] := \int_{\mathcal{M}_f(E)} \mathbb{P}_\mu[\cdot]\mathbf{P}(\mathrm{d}\mu).$$

Yaglom limit and QSDs

- Suppose that $\mathbf{Q}$ is a probability measure on $\mathcal{M}_f(E)$ concentrated on $\mathcal{M}_f^o(E)$.
- Say $\mathbf{Q}$ is the Yaglom limit of the superprocess X if

$$\mathbb{P}_\mu(X_t \in \cdot | X_t \neq \mathbf{0}) \xrightarrow[t\to\infty]{\mathrm{d}} \mathbf{Q}(\cdot), \quad \mu \in \mathcal{M}_f^o(E).$$

- Say $\mathbf{Q}$ is a quasi-limit distribution (QLD) of X, if $\exists$ a probability measure $\mathbf{P}$ on $\mathcal{M}_f^o(E)$ such that

$$(\mathbf{P}\mathbb{P})(X_t \in B | X_t \neq \mathbf{0}) \xrightarrow[t\to\infty]{} \mathbf{Q}(B), \quad B \in \mathcal{B}(\mathcal{M}_f^o(E)).$$

- Say $\mathbf{Q}$ is a quasi-stationary distribution (QSD) of X, if

$$(\mathbf{Q}\mathbb{P})(X_t \in B | X_t \neq \mathbf{0}) = \mathbf{Q}(B), \quad t \geq 0, B \in \mathcal{B}(\mathcal{M}_f^o(E)).$$

Motivation

Motivation

We want to investigate those sets: $\{\text{Yaglom limit of X}\}, \{\text{QLDs of X}\}$, and $\{\text{QSDs of X}\}$.

Here are some basic facts (Méléard and Villemonais (2012)):

- $\#\{\text{Yaglom limit of X}\} \leq 1$.
- $\{\text{Yaglom limit of X}\} \subset \{\text{QLDs of X}\} = \{\text{QSDs of X}\}$.
- For any $\mathbf{Q} \in \{\text{QSDs of X}\}$, there exists an $r \in (0, \infty)$ such that $(\mathbf{Q}P)(X_t \neq \mathbf{0}) = e^{-rt}$. We say r is the decay rate of $\mathbf{Q}$.

Criticality of superprocesses

- The mean semigroup $(P_t^\beta)_{t\geq 0}$ of X is given by

$$P_t^\beta f(x) := \Pi_x\left[e^{\int_0^t \beta(\xi_r)\mathrm{d}r} f(\xi_t)\mathbf{1}_{t<\zeta}\right] = \mathbb{P}_{\delta_x}[X_t(f)],$$

where $f \in \mathcal{B}_b(E), t \geq 0$ and $x \in E$.

- **Assumption 0:** $\exists$ a constant $\lambda < 0$, a strictly positive $\phi \in \mathcal{B}_b(E)$, and a probability measure ν with full support on E such that for each $t \geq 0$,

$$P_t^\beta \phi = e^{\lambda t}\phi, \quad \nu P_t^\beta = e^{\lambda t}\nu, \quad \nu(\phi) = 1.$$

- The assumption $\lambda < 0$ says that the mean of $(X_t(\phi))_{t\geq 0}$ decay exponentially with rate $-\lambda > 0$. In this case the superprocess X is called subcritical.

Intrinsic ultracontractive and non-persistent

- Denote by $L_1^+(\nu)$ the collection of all function $f \in \mathcal{B}_b^+(E)$ which are integrable w.r.t. ν.
- **Assumption 1:** For all $t > 0, x \in E$ and $f \in L_1^+(\nu)$, it holds that

  $$P_t^\beta f(x) = e^{\lambda t}\phi(x)\nu(f)(1 + C_{t,x,f})$$

  for some real $C_{t,x,f}$ with

  $$\sup_{x\in E, f\in L_1^+(\nu)} |C_{t,x,f}| < \infty$$

  and

  $$\lim_{t\to\infty} \sup_{x\in E, f\in L_1^+(\nu)} |C_{t,x,f}| = 0.$$
- **Assumption 2:** There exists $T \geq 0$ such that $\mathbb{P}_\nu(X_t = \mathbf{0}) > 0$ for all $t > T$.

Main Results

Theorem (Liu, Ren, Song and S. (2020))

If Assumptions 0-2 hold, then the Yaglom limit $\mathbf{Q}$ *of* X *exists.*

Theorem (Liu, Ren, Song and S. (2020))

Suppose that Assumptions 0-2 hold. Then

- $\forall r > -\lambda$, *there is no QSD of* X *with decay rate* r*;*
- $\forall r \in (0, -\lambda]$, $\exists$ *a unique QSD* $\mathbf{Q}_r$ *of* X *with decay rate* r*;*
- $\mathbf{Q}$, *the Yaglom limit* $= \mathbf{Q}_{-\lambda}$, *the QSD with the highest decay rate.*

Literature

- Yaglom limit for Galton-Watson processes were studied by Yaglom (1947), Heathcote, Seneta and Vere-Jones (1967), and Joff (1967).
- QSDs for Galton-Watson processes were studied by Hoppe and Seneta (1976).
- Yaglom limit for Multitype Galton-Watson processes are studied by Hoppe (1975), Hoppe and Seneta (1978), Joffe (1967).
- For Yaglom limit and QSDs for branching Markov processes, see Asmussen and Hering's book: *branching processes* (1983) and the references therein.
- Yaglom limit for continuous-state branching process (a degenerated superprocess) were studied by Li (2000) and Lambert (2007).
- QSDs for continuous-state branching processes were studied by Lambert (2007).

Thanks!