博士研究生学位论文

题目：临界分支过程与超过程的脊柱分解与极限定理

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摘要

本文主要研究了临界分支过程和一类临界超过程的极限行为，以及它们与多脊柱分解理论的关系。特别地，本文系统地研究了临界分支过程和一类临界超过程的脊柱分解和双脊柱分解定理，以及它们的 Kolmogrov 型、Yaglom 型和 Slack 型极限定理。

本文首先给出了临界 Galton-Watson 树的脊柱分解定理和双脊柱分解定理。这些分解定理描述了 Galton-Watson 树的一阶 size-biased 变换和二阶 size-biased 变换。使用这些分解定理，本文给出了临界 Galton-Watson 过程 Yaglom 定理的一个新的直观概率证明。

接着本文建立了一类临界超过程的脊柱分解和双脊柱分解定理，并讨论了他们和这类超过程极限定理的关系。这两种分解分别刻画了一类具有有限二阶矩的超过程的一阶 size-biased 变换和二阶 size-biased 变换。同时这两类分解可以视为泊松随机测度的一个新的分解定理的特例。使用这些分解，我们给出了一类具有有限二阶矩的临界超过程灭绝概率渐近行为的 Kolmogrov 型和 Yaglom 型极限定理的概率证明。

然后，本文利用这些脊柱分解定理证明了超过程的特征函数满足某个非线性复值积分方程。该方程可以用来估计一类具有稳定分支的上临界超过程的尾概率收敛速度。

最后，本文研究了一类具有空间非齐次的稳定分支的临界超过程 $\{X;\mathbf{P}_\mu\}$ 的 Slack 型极限定理。假设全空间稳定系数的下确界为 $\gamma_0>1$。本文证明了，在一定条件下，t 时刻的不灭绝概率 $\mathbf{P}_\mu(\|X_t\|\neq 0)$ 是以指数为 $(\gamma_0-1)^{-1}$ 正则变化的形式收敛到 0 的；在不灭绝的条件概率下，对于一大类非负测试函数 f, 经过适当的伸缩变换，$X_t(f)$ 会弱收敛到一个 Laplace 变换为

$$E[e^{-u\mathbf{z}^{(\gamma_0-1)}}]=1-(1+u^{-(\gamma_0-1)})^{-1/(\gamma_0-1)}$$

的严格正的随机变量 $\mathbf{z}^{(\gamma_0-1)}$。

关键词：分支过程，Galton-Watson 树，超过程，脊柱分解，双脊柱分解，弱收敛

Spine decompositions and limit theorems for critical branching processes and critical superprocesses

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ABSTRACT

This thesis focuses on the limiting theory of critical Galton-Watson branching processes and a class of critical superprocesses. Properties and relationships between their asymptotic behaviors and the multi-spine theory are considered. In particular, we systematically study the spine decompositions and the two-spine decompositions of critical Galton-Watson trees and a class of critical superprocesses, and their Kolmogorov type, Yaglom type and Slack type limit results.

We begin by proposing a two-spine decomposition of the critical Galton-Watson tree and using that decomposition to give a new probabilistic proof of Yaglom exponential limit law.

Next, we establish a spine decomposition theorem and a 2-spine decomposition theorem for some critical superprocesses. These two kinds of decompositions are unified as a decomposition theorem for size-biased Poisson random measures. We use these decompositions to give probabilistic proofs of the asymptotic behavior of the survival probability and Yaglom exponential limit law for some critical superprocesses with second moments.

Then, using these spine decompositions, we prove that the characteristic functions of superprocesses are mild solutions to a complex-valued integral equation. This equation will help us to estimate the tail probability of a class of supercritical superprocesses with stable branching.

Finally, we consider a critical superprocess $\{X;\mathbf{P}_\mu\}$ with general spatial motion and spatially dependent stable branching mechanism with lowest stability index $\gamma_0 > 1$. We show that, under some conditions, $\mathbf{P}_\mu(\|X_t\| \neq 0)$ converges to 0 as $t \to \infty$ and is regularly varying with index $(\gamma_0 - 1)^{-1}$. Then we prove the Slack type result that for a large class of non-negative testing functions f, the distribution of $\{X_t(f);\mathbf{P}_\mu(\cdot|\|X_t\| \neq 0)\}$, after appropriate rescaling, converges weakly to a positive random variable $\mathbf{z}^{(\gamma_0-1)}$ with Laplace transform $E[e^{-u\mathbf{z}^{(\gamma_0-1)}}] = 1 - (1 + u^{-(\gamma_0-1)})^{-1/(\gamma_0-1)}$.

KEYWORDS: Branching process, Galton-Watson tree, superprocess, spine decomposition, 2-spine decomposition, weak convergence

Contents

Chapter 1 Introduction

1.1 Backgrounds

Superprocess is a very important measure-valued Markov process. It was introduced by Watanabe [80], Ikeda, Nagasawa and Watanabe [40, 41, 42], and Dawson [17, 18]. It belongs to a large class of stochastic processes called Markovian branching processes. This class includes other models such as Galton-Watson processes, multitype Galton-Watson processes, continuous time Galton-Watson processes, multitype continuous time Galton-Watson processes, branching random walks, branching Markov processes and continuous state branching processes. Nowadays the theory of Markovian branching processes is one of the most important subjects in modern probability theory. On the applied side, they are inspired by and used to model various genetic and biological systems. On the theoretical side, they are closely related to nonlinear PDE's, stochastic PDE's, stochastic analysis and many other branches of modern mathematics.

The asymptotic behavior of the extinction probability and the size of the population is a fundamental problem in the theory of Markovian branching processes. Roughly, there are three different cases to consider: in the supercritical case, the expectation of the population grows exponentially; in the subcritical case, the expectation of the population decays exponentially; in the critical case, the exponential grow rate (or decay rate) of the expectation of the population is 0.

The limiting behavior of Galton-Watson processes is well known, see [5] for example. In the critical case, Slack [75] considered Galton-Watson processes with offspring distribution belonging to the domain of attraction of an α-stable law with $\alpha \in (1,2]$. He showed that the total population, after an appropriate rescaling and conditioning, converges weakly to a random variable $\mathbf{z}^{(\alpha)}$ with Laplace transform $E[e^{-u\mathbf{z}^{(\alpha)}}] = 1-(1+u^{-\alpha})^{-1/\alpha}$. In the case $\alpha = 2$, this result is first obtained by Yaglom [81], and therefore, is known as Yaglom's theorem.

It turns out that the Slack type result is universal, in the sense that, for almost all the Markovian branching processes mentioned above, similar Slack type weak limit results are true. For those results under various different names see table 1.1.

Evans and Perkins [31] established a Yaglom type result for a critical superprocess with quadratic branching mechanism. Recently, Ren, Song and Zhang [68] generalized this to a class of critical superprocesses with more general branching mechanisms and more general

Table 1.1 Kolmogorov, Yaglom and Slack type results

	$\alpha = 2$: Analytical method	$\alpha = 2$: Probabilistic method	$\alpha \in (1, 2)$
Galton-Watson processes	[48] A. Kolmogorov (1938) [81] A. Yaglom (1947) [46] H. Kesten, P. Ney and F. Spitzer (1966)	[58] R. Lyons, R. Pemantle and Y. Peres (1995) [33] J. Geiger (1999) [32] J. Geiger (2000) [63] Y.-X. Ren, R. Song and Z. Sun (2018a)	[75] R. Slack (1968)
Multitype Galton-Watson processes	[44] A. Joffe and F. Spitzer (1967)	[79] V. Vatutin and E. Dyakonova (2001)	[35] M. Goldstein and F. Hoppe (1978)
Continuous time Galton-Watson processes	[5] K. Athreya and P. Ney (1972)	-	[78] V. Vatutin (1977)
Continuous time multitype Galton-Watson processes	[6] K. Athreya and P. Ney (1974)	-	[78] V. Vatutin (1977)
Branching Markov processes	[4] S. Asmussen and H. Hering (1983)	[62] E. Powell (2015)	[4] S. Asmussen and H. Hering (1983)
Continuous-state branching processes	[55] Z. Li (2000) [54] A. Lambert (2007)	[65] Y.-X. Ren, R. Song and Z. Sun (2019)	[50] A. Kyprianou and J. Pardo (2008) [70] Y.-X. Ren, T. Yang and G.-H. Zhao (2014)
Superprocesses	[31] Evans and Perkins (1990) [68] Y.-X. Ren, R. Song and R. Zhang (2015)	[65] Y.-X. Ren, R. Song and Z. Sun (2019)	[64] Y.-X. Ren, R. Song and Z. Sun (2018b+)

spatial motions. For critical superprocesses without second-moment conditions, it is natural to ask whether Slack type result is valid. Also, for critical superprocesses with second-moment condition, since the methods used in [31] and [68] are all analytic, it is natural to ask whether an intuitive probabilistic proof exists.

The main topic of this thesis is to give positive answers to both of these questions. We consider the asymptotic behaviors of branching processes and superprocesses in the critical case using a method called multi-spine decomposition. The idea of using the spine method to study the limiting behavior of branching processes is due to Lyons, Pemantle and Peres [58]. For spine method in general branching processes and its applications under a variety of names, see [2, 3, 9, 12, 26, 27, 28, 34, 39, 54, 57, 69] for example. The multi-spine is first investigated by Harris and Roberts [37] in the context of branching Markov processes. Our main contribution is that we find a generic relationship between the multi-spine theory and the limiting behaviors for both branching processes and superprocesses, in the critical case.

Roughly speaking, the spine is the trajectory of an immortal particle, and the k-spine-skeleton is the combination of k spines. The multi-spine decomposition says that the size-biased measure transformations of a Markovian branching process can be decomposed as branching immigrations along with some multi-spine-skeleton. These decomposition theorems are important at least for two reasons. The first is that they capture the interplays between the original branching processes and their measure-transformed counterparts. This provides new probabilistic points of view for characterizing properties of the original processes. The second is that they are flexible and generic, in the sense that almost all the models mentioned earlier can be decomposed under different measure transformations.

Now, in order to be more precise about all these results and methods, we first introduce the models considered in this thesis.

1.2 Models

This thesis focuses on two models: Galton-Watson branching processes and superprocesses.

1.2.1 Galton-Watson branching processes

Let $(\xi_i^n)_{i,n\geq 1}$ be i.i.d. $\mathbb{Z}_+$-valued random variables. Define a sequence $(Z_n)_{n\geq 0}$ by $Z_0 = 1$ and

$$Z_{n+1} = \mathbf{1}_{Z_n>0} \sum_{k=1}^{Z_n} \xi_k^n. \tag{1.2.1}$$

$(Z_n)_{n\geq 1}$ is called a Galton-Watson process. The idea behind the definition is that Z_n is the number of individuals in the n-th generation, and each member of the n-th generation gives birth independently to an identically distributed number of children. $\mu(k) = P(\xi_i^n = k)$ is called the offspring distribution. Let $m = E[\xi_i^n] \in (0, \infty)$. It can be verified easily that $M_n := (Z_n/m^n)_{n\geq 0}$ is a non-negative martingale with respect to the natural filtration of (Z_n). So, this martingale has an a.s. limit which is denoted as M_∞.

If $m < 1$, then it is easy to see that

$$P(Z_n > 0) \leq E(Z_n; Z_n > 0) = E(Z_n) = \mu^n \xrightarrow[n\to\infty]{} 0.$$

Therefore, in this case we have almost surely that $Z_n = 0$ for n large enough. This also says that $M_\infty = 0$. If $m = 1$, then (Z_n) itself is a non-negative martingale. Since $(Z_n)_{n\geq 0}$ are integer valued, so we have $Z_n = M_\infty$ for large n. If our process (Z_n) is non-trivial, or equivalently speaking, if $P(\xi_i^n = 1) < 1$, then from (1.2.1) we have

$$P(Z_n = k, \quad \forall n \geq N) = 0$$

for all $N \geq 0$ and $k \geq 1$. So, in this case we also have $Z_n = 0$ for all n sufficiently large and that $M_\infty = 0$.

Denote by $\theta_m = P(Z_m = 0)$. Then it can be verified directly from (1.2.1) that (θ_m) satisfies the following regression equation

$$\theta_n = \varphi(\theta_{n-1}),$$

where $\varphi(s) := \sum_{k\geq 0} \mu(k)s^k$. If $m > 1$, then from the above equation, it can be verified that $\rho := P(Z_n = 0, \exists n \geq 0) = \lim_{m\to\infty} \theta_m$ is the unique fixed point of φ in $[0, 1)$.

We will call $(Z_n)_{n\geq 0}$ a μ-Galton-Watson processes, and say it is subcritical, critical and supercritical according to $m < 1$, $m = 1$ and $m > 1$. In this thesis, we mainly focus on the asymptotic behavior of critical branching processes. For the limiting behavior of the subcritical and supercritical cases, we refer our reader to [5].

For critical branching processes, the following result is well known:

Theorem 1.2.1 ([46]). *Let (Z_n) be a critical Galton-Watson branching process with* $\mathrm{Var}(Z_1) = \sigma^2 \in (0, \infty)$. *Then*

1. $nP(Z_n > 0) \xrightarrow[n\to\infty]{} 2/\sigma^2$;
2. $\{n^{-1}Z_n; P(\cdot|Z_n > 0)\} \xrightarrow[n\to\infty]{d} Y$,

where Y is an exponential random variable with mean $\sigma^2/2$.

Under a third moment assumption, assertions (1) and (2) of Theorem 1.2.1 are due to Kolmogorov [48] and Yaglom [81] respectively. Therefore, Theorem 1.2.1(1) is usually called Kolmogorov's theorem, and Theorem 1.2.1(2) is usually called Yaglom's theorem. For probabilistic proofs of the above results, we refer our readers to [33], [32], [58] and [63].

Slack [75] considered critical Galton-Watson branching processes without the finite variance condition, and he obtain the following:

Theorem 1.2.2. *Suppose that $\{(Z_n)_{n\geq 0}; P\}$ is a critical Galton-Watson process. Assume that the generating function $f(s)$ of its offspring distribution is of the form*

$$f(s) = s + (1 - s)^{1+\alpha} l(1 - s), \quad s \geq 0,$$

where $\alpha \in (0, 1]$ and l is a function slowly varying at 0. *Then*

$$P(Z_n > 0) = n^{-1/\alpha} L(n),$$

where L is a function slowly varying at ∞, and

$$P\big(P(Z_n > 0)Z_n \leq y | Z_n > 0\big) \xrightarrow[n\to\infty]{} H_\alpha(y),$$

where H_α is a probability distribution function on $\mathbb{R}_+$ with Laplace transform given by

$$\int_{[0,\infty]} e^{-\theta y} dH_\alpha(y) = 1 - (1 + \theta^{-\alpha})^{-1/\alpha}, \quad \theta \in \mathbb{R}_+.$$

We will call the above Slack's theorem. Note that, while $\alpha = 2$, Slack's theorem actually reduces to Kolmogorov's and Yaglom's theorem. As have been mentioned in the first subsection, ever since these pioneering papers of Kolmogorov, Yaglom and Slack, lots of analogous results have been obtained for more general critical branching processes. This includes continuous time branching processes, discrete time multitype branching processes, continuous time multitype branching processes, branching Markov processes, continuous-state branching processes and superprocess. See table 1.1 for the literature in this direction.

A large part of this thesis is devoted to give a new probabilistic proof of Kolmogorov type and Yaglom type results for a class of critical superprocesses with finite second moment condition, and to give a proof of Slack type result for a class of critical superprocesses without the finite second moment condition. We now introduce the superprocesses.

1.2.2 Superprocesses

We first give the definition of superprocesses, and then give some explanation. Let E be a locally compact separable metric space. Denote by $\mathcal{M}(E)$ the space of all finite measures on E. For any measurable function f and a measure μ on some measurable space, we write $\mu(f)$ for the integration $\int f d\mu$ whenever it exists.

A process $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}(E)}\}$ is said to be a (ξ,ψ)-superprocess if

- the spatial motion $\xi = \{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in E}\}$ is an E-valued Hunt process with its lifetime denoted by ζ;
- the branching mechanism $\psi : E \times [0,\infty) \to \mathbb{R}$ is given by
$$\psi(x,z) = -\beta(x)z + \alpha(x)z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(x,dy),$$
where $\beta \in \mathcal{B}_b(E)$, $\alpha \in \mathcal{B}_b(E,\mathbb{R}_+)$ and $\pi(x,dy)$ is a kernel from E to $(0,\infty)$ such that $\sup_{x\in E}\int_{(0,\infty)}(y\wedge y^2)\pi(x,dy) < \infty$.
- $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}(E)}\}$ is an $\mathcal{M}(E)$-valued Hunt process with transition probability determined by
$$\mathbf{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad t \geq 0, \mu \in \mathcal{M}(E), f \in \mathcal{B}_b^+(E),$$
where for each $f \in \mathcal{B}_b(E)$, the function $(t,x) \mapsto V_t f(x)$ on $[0,\infty)\times E$ is the unique locally bounded positive solution to the equation
$$V_t f(x) + \Pi_x\Big[\int_0^{t\wedge\zeta} \psi(\xi_s, V_{t-s}f(\xi_s))ds\Big] = \Pi_x[f(\xi_t)\mathbf{1}_{t<\zeta}], \quad t\geq 0, x \in E. \qquad (1.2.2)$$

We refer our readers to [56] for the existence of such processes. To avoid triviality, we always assume that $\psi(x,z)$ is not identically equal to $-\beta(x)z$. This definition is quite technical, so we give some examples below.

Example 1.2.3. Suppose that $E = \{x_0\}$ is a space which has only one point. Let $\xi_t \equiv x_0$ be the trivial process. Let the branching mechanism be

$$\psi(x_0,z) := \psi(z) := -bz + az^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\mu(dy), \quad z \geq 0, \tag{1.2.3}$$

where $b \in \mathbb{R}, a \geq 0$ and μ is a measure on $(0,\infty)$ with $\int_{(0,\infty)} y \wedge y^2 \pi(dy) < \infty$. Note that the (ξ,ψ) superprocess is an $\mathcal{M}(E)$-valued process. Therefore, there is a non-negative process $(Y_t)_{t\geq 0}$ such that

$$X_t = Y_t \delta_{x_0}, \quad t \geq 0.$$

This process (Y_t) is called a continuous-state branching process with branching mechanism ψ. It is easy to verify that (Y_t) is also a Markov process, and its transition probability $(P_y)_{y\geq 0}$ satisfies the following branching property:

$$P_y[e^{-\lambda Y_t}] = P_{y_1}[e^{-\lambda Y_t}]P_{y_2}[e^{-\lambda Y_t}], \quad t \geq 0, \lambda \geq 0,$$

where $y = y_1 + y_2$ and $y_1, y_2 \geq 0$. If the branching mechanism takes the form of

$$\psi(z) = z^2, \quad z \geq 0, \tag{1.2.4}$$

then $(Y_t)_{t\geq 0}$ is also known as Feller's diffusion, and it is the solution to the SDE

$$dY_t = \sqrt{Y_t}dB_t, \quad t \geq 0,$$

where (B_t) is a standard Brownian motion on $\mathbb{R}$. See [56] for more details about this example.

Example 1.2.4. Suppose that $E = \mathbb{R}^d$. Let the spatial motion (ξ_t) be a standard Brownian motion in $\mathbb{R}^d$. Let the branching mechanism takes the form of

$$(x,z) \mapsto z^2, \quad x \in \mathbb{R}^d, z \geq 0.$$

In this case, the (ξ,ψ)-superprocess $\{(X_t); (\mathbf{P}_\mu)_{\mu\in\mathbb{R}^d}\}$ is called a super Brownian motion with branching mechanism $\psi(x) = z^2$. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and f be a continuous nonnegative bounded Borel function on $\mathbb{R}^d$. Then we have

$$\mathbf{P}_\mu[e^{-X_t(f)}] = e^{-\mu(v_t)}, \quad t \geq 0,$$

where the function $v : (t,x) \mapsto v_t(x)$ on $\mathbb{R}_+ \times \mathbb{R}^d$ is the unique solution to the PDE

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - v^2, \quad v_0 = f.$$

Note that the total mass of this measure-valued process $Y_t := X_t(1)$ is actually the Feller's diffusion mentioned above.

Besides its connection with non-linear PDEs, superprocesses can also been obtained as the scaling limits of several discrete stochastic particle systems. This includes branching particle systems [80, 17, 22], long-range contact process [60, 21], voter model and Lotka-Volterra model [13, 14] and long range percolation [53]. We will not give the full picture in this direction here. Instead, we present the following example which says that the scaling limit of a binary branching Brownian motion is the super Brownian motion with branching mechanism $\psi(z) = z^2$. This result will not be directly used in this thesis. We present it here, because it gives an interpretation of superprocesses, and shows how superprocesses and branching processes are connected.

Here, by a binary branching Brownian motion $(X_t)_{t\geq 0}$, we mean the following model:

- at the beginning, there are several particles living in $\mathbb{R}^d$;
- independent of other particles, each particle in the system performs standard Brownian motion and is killed at a constant rate $r > 0$;
- independent of other particles, each particle in the system, at the end of its life, dies with no offspring or splits into two new particles, with equal probability;
- each particle in the system has a same weight $m > 0$; for $t \geq 0$ and any measurable subset A of $\mathbb{R}^d$, $X_t(A)$ is the total weight of all the particles positioned in A at time t.

The follwoing result is due to [17, 80].

Theorem 1.2.5. *Fix a $\mu \in \mathcal{M}(\mathbb{R}^d)$. For every $n \in \mathbb{N}$, consider a binary branching Brownian motion (X^n_t) in $\mathbb{R}^d$, with branching rate $2n$ and particle weights n^{-1}, with its initial configuration X^n_0 satisfying that nX^n_0 is a Poisson random measure on $\mathbb{R}^d$ with intensity $n\mu$. Then $(X^n_t)_{t\geq 0} \xrightarrow[n\to\infty]{d} (X_t)_{t\geq 0}$ in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^d))$, based on the topology of weak convergence in $\mathcal{M}(\mathbb{R}^d)$, where $(X_t)_{t\geq 0}$ is a super Brownian motion with branching mechanism $\psi(z) = z^2$ and initial configuration μ.*

As mentioned earlier, analogues results of Theorem 1.2.1 and Theorem 1.2.2 were obtained for a lots of Markovian branching processes. The main interests of this thesis is to prove those results for a large class of general superprocesses. Our approach for Kolmogorov type and Yaglom type results for the superprocesses are different from the aforementioned works [31] and [68], and is more intuitive and probabilistic. The statements of those results for the general superprocesses is quite technical. For the sake of simplicity, in this subsection, we only present our results for the continuous-state branching processes. More precise statements of the theory will be presented in Chapter 3 and Chapter 5. We will also give some intuitions of our methods in the next section.

Suppose that $\{(Y_t); P_x\}$ is a continuous-state branching process with branching mechanism ψ given by (1.2.3). Then its Laplace transform satisfies that

$$P_x[e^{-\lambda Y_t}] = e^{-xv_t(\lambda)}, \quad x \in \mathbb{R}^+, t \geq 0, \lambda \in \mathbb{R}_+,$$

where for each $\lambda \geq 0$, $t \mapsto v_t(\lambda)$ is the unique positive solution to the equation

$$v_t(\lambda) - \int_0^t \psi(v_s(\lambda))ds = \lambda, \quad t \geq 0.$$

Taking derivative with respect to λ on the both side, and letting $\lambda = 0$, we get

$$\frac{\partial v_t}{\partial \lambda}(0) - \int_0^t b\frac{\partial v_s}{\partial \lambda}(0)ds = 1,$$

which says that

$$P_x[Y_t] = x\frac{\partial v_t}{\partial \lambda}(0) = xe^{bt}, \quad t \geq 0.$$

If $b > 0$, then the expectation of (Y_t) will grows exponentially; if $b = 0$, then the expectation of (Y_t) will be a constant; if $b < 0$, then the expectation of (Y_t) will be decrease exponentially. So we say the CSBP (Y_t) is subcritical, critical, supercritical according to $b > 0$, $b = 0$ and $b < 0$.

The following Kolmogorov and Yaglom type results for the critical CSBP are due to [55].

Theorem 1.2.6. *Let $\{(Y_t)_{t\geq 0}; (P_x)_{x\geq 0}\}$ be a continuous state branching process with branching mechanism ψ given in* (1.2.4)*. Suppose that $\beta = 0$ and*

$$\sigma := \psi''(0+) < \infty.$$

Then we have

$$tP_x(Y_t > 0) \xrightarrow[t\to\infty]{} 2x/\sigma, \quad x > 0,$$

and

$$P_x(Y_t/t \geq u | Y_t > 0) \xrightarrow[t\to\infty]{} e^{-2u/\sigma}, \quad u \geq 0.$$

The Slack type result for CSBP is due to [50] and [70]:

Theorem 1.2.7. *Let $\{(Y_t)_{t\geq 0}; (P_x)_{x\geq 0}\}$ be a continuous state branching process with branching mechanism ψ given in* (1.2.4)*. Suppose that $\psi(\lambda) = \lambda^{1+\alpha}L(1/\lambda)$ where $\alpha \in (0,1]$ and L is slowly varying at infinity. Then $F(t) := P_1(Y_t > 0)$ converges to 0 as $t \to \infty$, and is regularly varying with index $-1/\alpha$. Furthermore, for each $x > 0$ and $y \geq 0$, it holds that*

$$P_x(F(t)Y_t \leq y | Y_t > 0) \xrightarrow[t\to\infty]{} P(\mathbf{z}^{(\alpha)} \leq y)$$

where $\mathbf{z}^{(\alpha)}$ *is a random variable with Laplace transform given by*

$$E[e^{-\theta\mathbf{z}^{(\alpha)}}] = 1 - (1+\theta^{-\alpha})^{-1/\alpha}, \quad \theta \geq 0.$$

1.3 Method

1.3.1 Spine method for Galton-Watson processes

Let $(Z_n)_{n\geq 0}$ be a Galton-Watson branching process with offspring distribution μ. The spine methods for branching processes are initiated in Lyons, Pemantle and Peres [58], where they gave a probabilistic proof of Theorem 1.2.1 using the so-called size-biased μ-Galton-Watson tree. In this thesis, by *size-biased transform* we mean the following: Let X be a random variable and $g(X)$ be a Borel function of X with $P(g(X) \geq 0) = 1$ and $E[g(X)] \in (0,\infty)$. We say a random variable W is a $g(X)$-size-biased transform (or simply $g(X)$-transform) of X if

$$E[f(W)] = \frac{E[g(X)f(X)]}{E[g(X)]}$$

for each positive Borel function f. An X-transform of X is sometimes called a size-biased transform of X.

We now recall the size-biased μ-Galton-Watson tree introduced in [58]. Let L be a random variable with distribution μ. Denote by $\dot{L}$ an L*-transform* of L. The celebrated *size-biased* μ*-Galton-Watson tree* is then constructed as follows:

- there is an initial particle which is marked;
- any marked particle gives independent birth to a random number of children according to $\dot{L}$. Pick one of those children randomly as the new marked particle while leaving the other children as unmarked particles;
- any unmarked particle gives birth independently to a random number of unmarked children according to L;
- the evolution goes on.

Notice that the marked particles form a descending family line which will be referred as the *spine*. Let $\dot{Z}_n$ be the population of the nth generation in the size-biased tree. It is proved in [58] that the process $(\dot{Z}_n)_{n\geq 0}$ is a martingale transform of the process $(Z_n)_{n\geq 0}$ via the martingale $(Z_n)_{n\geq 0}$. That is, for any generation number n and any bounded Borel function g on $\mathbb{N}_0^n$,

$$E[g(\dot{Z}_1,\dots,\dot{Z}_n)] = \frac{E[Z_n g(Z_1,\dots,Z_n)]}{E[Z_n]}. \tag{1.3.1}$$

It is natural to consider probabilistic proofs of analogous results of Theorem 1.2.1 for more general critical branching processes. Vatutin and Dyakonova [79] gave a probabilistic proof of Theorem 1.2.1(1) for multitype critical branching processes. As far as we know, there is no probabilistic proof of Yaglom's theorem for multitype critical branching processes. It seems that it is difficult to adapt the probabilistic proofs in [32] and [58] for monotype branching processes to more general models, such as multitype branching processes, branching Hunt processes and superprocesses.

In my joint paper with Ren and Song [63], we propose a $k(k-1)$-type size-biased μ-Galton-Watson tree equipped with a two-spine skeleton, which serves as a change-of-measure of the original μ-Galton-Watson tree; and with the help of this two-spine technique, in the next chapter, we give a new probabilistic proof of Theorem 1.2.1(2), i.e. Yaglom's theorem. The main motivation for developing this new proof for the classical Yaglom's theorem is that this new method is generic, in the sense that it can be generalized to more complicated critical branching systems. In fact, in Chapter 3, based on our follow-up paper [65], we show that, in a similar spirit, a two-spine structure can be constructed for a class of critical superprocesses, and a probabilistic proof of a Yaglom type theorem can be obtained for those processes.

Another aspect of our new proof is that we take advantage of a fact that the exponential distribution can be characterized by a particular x^2-type size-biased distributional equation. An intuitive explanation of our method, and a comparison with the methods of [32] and [58], will be made shortly. We think this new point of view of convergence to the exponential law provides an alternative insight on the classical Yaglom's theorem.

We now give a formal construction of our $k(k-1)$-type size-biased μ-Galton-Watson tree. Suppose that μ has mean 1 and finite variance, i.e.,

$$\sum_{k=0}^{\infty} k\mu(k) = 1 \tag{1.3.2}$$

and

$$0 < \sigma^2 := \sum_{k=0}^{\infty}(k-1)^2\mu(k) = \sum_{k=0}^{\infty} k(k-1)\mu(k) < \infty. \tag{1.3.3}$$

Denote by $\dot{L}$ an L-transform of L, and by $\ddot{L}$ an $L(L-1)$-transform of L. Fix a generation number n and pick a random generation number K_n uniformly among $\{0,\dots,n-1\}$. The *$k(k-1)$-type size-biased μ-Galton-Watson tree with height n* is then defined as a particle system such that:

- there is an initial particle which is marked;
- before or after generation K_n, any marked particle gives birth independently to a random number of children according to $\dot{L}$; pick one of those children randomly as the new

marked particle while leaving the other children as unmarked particles;

- the marked particle at generation K_n, however, gives birth, independent of other particles, to a random number of children according to $\ddot{L}$; pick two different particles randomly among those children as the new marked particles while leaving the other children as unmarked particles;
- any unmarked particle gives birth independently to a random number of unmarked children according to L;
- the system stops at generation n.

If we track all the marked particles, it is clear that they form a *two-spine skeleton* with K_n being the last generation where those two spines are together. It would be helpful to consider this skeleton as two disjoint spines, where *the longer spine* is a family line from generation 0 to n and *the shorter spine* is a family line from generation $K_n + 1$ to n.

For any $0 \le m \le n$, denote by $\ddot{Z}_m^{(n)}$ the population of the mth generation in the $k(k-1)$-type size-biased μ-Galton-Watson tree with height n. The main reason for proposing such a model is that the process $(\ddot{Z}_m^{(n)})_{0\le m\le n}$ can be viewed as a $Z_n(Z_n-1)$-transform of the process $(Z_m)_{0\le m\le n}$. This is made precise in the result below which will be proved in Section 2.1.1.

Theorem 1.3.1. *Let $(Z_m)_{m\ge 0}$ be a μ-Galton-Watson process and $(\ddot{Z}_m^{(n)})_{0\le m\le n}$ be the population of a $k(k-1)$-type size-biased μ-Galton-Watson tree with height n. Suppose that μ has mean 1 and finite variance. Then, for any bounded Borel function g on $\mathbb{N}_0^n$,*

$$E[g(\ddot{Z}_1^{(n)},\dots,\ddot{Z}_n^{(n)})] = \frac{E[Z_n(Z_n-1)g(Z_1,\dots,Z_n)]}{E[Z_n(Z_n-1)]}.$$

The idea of considering a branching particle system with more than one spine is not new. A particle system with k spines was constructed in [37] and used in the many-to-few formula for branching Markov processes and branching random walks. Inspired by [37], we use a two-spine model to characterize the $k(k-1)$-type size-biased branching process.

Suppose that X is a non-negative random variable with $E[X]\in(0,\infty)$. Then its distribution conditioned on $\{X>0\}$ can be characterized by its conditional expectation $E[X|X>0]$ and its size-biased transform $\dot{X}$. In fact, for each $\lambda \ge 0$,

$$\begin{aligned} E[1-e^{-\lambda X}|X>0] &= \frac{E[1-e^{-\lambda X}]}{P(X>0)} \qquad (1.3.4)\\ &= \frac{1}{P(X>0)}\int_0^\lambda E[Xe^{-sX}]ds = E[X|X>0]\int_0^\lambda E[e^{-s\dot{X}}]ds. \end{aligned}$$

As a consequence, Theorem 1.2.1 is equivalent to

$$E\Big[\frac{Z_n}{n}|Z_n > 0\Big] \underset{n\to\infty}{\longrightarrow} \frac{\sigma^2}{2} \tag{1.3.5}$$

and

$$E[e^{-s\frac{\dot{Z}_n}{n}}] \underset{n\to\infty}{\longrightarrow} E[e^{-s\dot{Y}}]. \tag{1.3.6}$$

where $\dot{Y}$ is a Y-transform of the exponential random variable Y. Indeed, since $E[Z_n] = 1$, (1.3.5) is equivalent to Theorem 1.2.1(1); and assuming (1.3.5), according to (1.3.4), we can see that (1.3.6) is equivalent to Theorem 1.2.1(2). In Section 2.2, for completeness, we will simplify the argument of [33] and [79], and give a proof of Theorem 1.2.1(1).

Our method of proving (1.3.6) takes advantage of a fact that the exponential distribution is characterized by an x^2-type size-biased distributional equation. This is made precise in the next lemma, which will be proved in Section 2.2:

Lemma 1.3.2. *Let Y be a strictly positive random variable with finite second moment. Then Y is exponentially distributed if and only if*

$$\ddot{Y} \overset{d}{=} \dot{Y} + U \cdot \dot{Y}', \tag{1.3.7}$$

where $\dot{Y}$ and $\dot{Y}'$ are both Y-transforms of Y, $\ddot{Y}$ is a Y^2-transform of Y, U is a uniform random variable on $[0, 1]$*, and $\dot{Y}$, $\dot{Y}'$, $\ddot{Y}$ and U are independent.*

With this lemma and Theorem 1.3.1, we can give an intuitive explanation of the exponential convergence in Yaglom's Theorem. From the construction of the $k(k-1)$-type size-biased μ-Galton-Watson tree $(\ddot{Z}_m^{(n)})_{0\le m\le n}$, we see that the population $\ddot{Z}_n^{(n)}$ in the nth generation can be separated into two parts: descendants from the longer spine and descendants from the shorter spine. Due to their construction, the first part, the descendants from the longer spine at generation n, is distributed approximately like $\dot{Z}_n$, while the second part, the descendants from the shorter spine at generation n, is distributed approximately like $\dot{Z}_{\lfloor U\cdot n\rfloor}$. Those two parts are approximately independent of each other. So, after a renormalization, we have roughly that

$$\frac{\ddot{Z}_n^{(n)}}{n} \overset{d}{\approx} \frac{\dot{Z}_n}{n} + U \cdot \frac{\dot{Z}'_{\lfloor Un\rfloor}}{Un}, \tag{1.3.8}$$

where the process $(\dot{Z}'_m)$ is an independent copy of $(\dot{Z}_m)$. Suppose that $\dot{Z}_n/n$ converges weakly to a random variable $\dot{Y}$, and $\ddot{Z}_n/n$ converges weakly to a random variable $\ddot{Y}$. Then, according to [58, Lemma 4.3], $\ddot{Y}$ is a size-biased transform of $\dot{Y}$. Therefore, letting $n \to \infty$ in (1.3.8), $\dot{Y}$ should satisfy (1.3.7), which, by Lemma 1.3.2, suggests that (1.3.6) is true.

It is interesting to compare this method of proving exponential convergence with the methods used in [32] and [58]. In [58], Lyons, Pemantle and Peres characterize the exponential distribution by a different but well-known x-type size-biased distributional equation: A nonnegative random variable Y with positive finite mean is exponentially distributed if and only if it satisfies that

$$Y \stackrel{d}{=} U \cdot \dot{Y} \tag{1.3.9}$$

where $\dot{Y}$ is a Y-transform of Y, and U is a uniform random variable on $[0,1]$, which is independent of $\dot{Y}$. With the help of the size-biased tree, they then show that $\lceil U \cdot \dot{Z}_n \rceil$ is distributed approximately like Z_n conditioned on $\{Z_n > 0\}$. So, after a renormalization, they have roughly that

$$\left\{\frac{Z_n}{n}; P(\cdot|Z_n > 0)\right\} \stackrel{d}{\approx} U \cdot \frac{\dot{Z}_n}{n}. \tag{1.3.10}$$

Suppose that $\{Z_n/n; P(\cdot|Z_n > 0)\}$ converges weakly to a random variable Y, and $\dot{Z}_n/n$ converges weakly to a random variable $\dot{Y}$. Then, according to [58, Lemma 4.3], $\dot{Y}$ is the size-biased transform of Y. Therefore, letting $n \to \infty$ in (1.3.10), Y should satisfy (1.3.9), which suggests that Y is exponentially distributed.

In [32], Geiger characterizes the exponential distribution by another distributional equation: If $Y^{(1)}$ and $Y^{(2)}$ are independent copies of a random variable Y with positive finite variance, and U is an independent uniform random variable on $[0,1]$, then Y is exponentially distributed if and only if

$$Y \stackrel{d}{=} U(Y^{(1)} + Y^{(2)}). \tag{1.3.11}$$

Geiger then shows that for (Z_n), conditioned on non-extinction at generation n, the distribution of the generation of the most recent common ancestor (MRCA) of the particles at generation n is asymptotically uniform among $\{0, 1, \ldots, n\}$ (a result due to [83], see also [33]), and there are asymptotically two children of the MRCA, each with at least 1 descendant in generation n. After a renormalization, roughly speaking, Geiger has that

$$\left\{\frac{Z_n}{n}; P(\cdot|Z_n > 0)\right\} \stackrel{d}{\approx} U \cdot \frac{Z^{(1)}_{\lfloor Un \rfloor}}{Un} + U \cdot \frac{Z^{(2)}_{\lfloor Un \rfloor}}{Un}, \tag{1.3.12}$$

where for each m, $Z^{(1)}_m$ and $Z^{(2)}_m$ are independent copies of $\{Z_m; P(\cdot|Z_m > 0)\}$. Therefore, if $\{Z_n/n; P(\cdot|Z_n > 0)\}$ converges weakly to a random variable Y, then Y should satisfy (1.3.11), which suggests that Y is exponentially distributed.

From this comparison, we see that all the methods mentioned above share one similarity: They all establish the exponential convergence via some particular distributional equation.

However, since the equations (1.3.7), (1.3.9) and (1.3.11) are different, the actual way of proving the convergence varies. In [58], an elegant tightness argument is made along with (1.3.10). However, it seems that this tightness argument is not suitable for (1.3.12), due to a property that the conditional convergence for some subsequence Z_{n_k}/n_k implies the convergence of $U\cdot\dot{Z}_{n_k}/n_k$, but does not imply the convergence of $Z^{(i)}_{\lfloor Un_k\rfloor}/Un_k, i=1,2$. Instead, a contraction type argument in the L^2-Wasserstein metric is used in [32].

For similar reasons, in Chapter 2, to actually prove the exponential convergence using (1.3.8) and (1.3.7), some efforts also must be made. We observe that the distributional equation (1.3.8) admits a so-called size-biased add-on structure, which is related to Lèvy's theory of infinitely divisible distributions: Suppose that X is a nonnegative random variable with $a := E[X]\in(0,\infty)$; then X is infinitely divisible if and only if there exists a nonnegative random variable A independent of X such that $\dot{X}\overset{d}{=}X+A$ where $\dot{X}$ is the X-transform of X. In fact, the Laplace exponent of X can be expressed as

$$-\ln E[e^{-\lambda X}] = a\alpha(\{0\})\lambda + a\int_{(0,\infty)}\frac{1-e^{-\lambda y}}{y}\alpha(dy),$$

where α is the distribution of A. Moreover, if A is strictly positive, then

$$-\ln E[e^{-\lambda X}] = a\int_0^\lambda E[e^{-sA}]ds.$$

From this point of view, after considering the Laplace transforms of (1.3.8) and (1.3.7), we can establish the convergence of $E[e^{-\lambda\dot{Z}_n/n}]$ to $E[e^{-\lambda\dot{Y}}]$, which will eventually lead us to Yaglom's theorem. This is made precise in Section 2.2.

1.3.2 Spine methods for CSBP

The spine decomposition of size-biased superprocesses is constructed in [25, 28, 57] under different settings. Roughly speaking, the spine is the trajectory of an immortal moving particle and the spine decomposition theorem says that, after a size-biased transform, the transformed superprocess can be decomposed in law as the sum of a copy of the original superprocess and an immigration processes along this spine. We will develop this result to more general settings and give a general spine decomposition theorem for the superprocesses in Chapter 3. We will also develop a 2-spine decomposition for a class of critical superprocesses in Chapter 3. The precise statements of those decomposition theorem are quite technical. In order to have a simple overview of the spine theory for the superprocesses, we only consider CSBP in this section.

Let $\{(Y_t);P_x\}$ be a CSBP with branching mechanism $\psi(z)=z^2$. It is helpful to consider

$Y = (Y_t)$ as a random element taking values in the following Skorokhod spaces:

$$\mathbb{D} := \{w = (w_t) : w \text{ is an } \mathbb{R}_+\text{-valued càdlàg paths on } [0, \infty) \text{ with } 0 \text{ as a trap.}\}$$

The branching property of $Y = (Y_t)$ now says that Y can be considered as an infinitely divisible $\mathbb{D}$-valued random element. According to [24], there is a σ-finite measure $\mathbb{N}$ on $\mathbb{D}$ which can be considered as the "Lévy measure" of this infinitely divisible random element Y. Such measure can be characterized by the following properties:

- $\mathbb{N}(\forall t > 0, Y_t = 0) = 0$;
- $\mathbb{N}(Y_0 \neq 0) = 0$;
- for any $\mu \in \mathcal{M}(E)$, if $\mathcal{N}$ is Poisson random measure defined on some probability space with intensity $y\mathbb{N}$ with $y > 0$, then the CSBP $\{(Y_t); P_y\}$ can be realized by $\tilde{Y}_0 := y$ and $\tilde{Y}_t := \mathcal{N}[w_t]$ for each $t > 0$.

We refer to $\mathbb{N}$ as the Kuznetsov measure for the CSBP. And with some abuse of notation, we will always assume that our CSBP $\{(Y_t); P_x\}$ is given by $Y_t = \mathcal{N}[w_t], t \geq 0$ for some Poisson random measure $\{\mathcal{N}; P_x\}$ with intensity $x\mathbb{N}$.

Similar to the size-biased decomposition of infinitely divisible non-negative random variables mentioned earlier, the CSBP has the following spine decomposition: For each measure μ and a non-negative measurable function f with $\mu(f) \in (0, \infty)$, we define the f-transform of μ as the following probability measure

$$d\mu^f := \frac{f}{\mu[f]} d\mu.$$

For each fixed $x \in \mathbb{R}$ and $t > 0$, denote by $P_x^{Y_t}$ the Y_t-transform of P_x. We say $\{Y, Z, n; Q_x^{(t)}\}$ is a spine representation of $P_x^{Y_t}$ if

- $\{Y = (Y_s)_{0 \leq s \leq t}; Q_x^{(t)}\}$ is a copy of the original CSBP $\{(Y_s)_{0 \leq s \leq t}; P_x\}$;
- independent of $\{(Y_s)_{0 \leq s \leq t}; Q_x^{(t)}\}$, $n(ds, dw)$ is a Poisson random measure on $[0, t] \times \mathbb{D}$ with intensity
$$2ds \times \mathbb{N}(dw);$$
- $(Z_s)_{0 \leq s \leq t}$ is a non-negative process defined by
$$Z_s = \int_0^s w_{s-r} n(dr, dw), \quad 0 \leq s \leq t. \tag{1.3.13}$$

Theorem 1.3.3 ([25, 28, 57]). *Suppose that $\{Y, Z, n; Q_x^{(t)}\}$ is a spine representation of $P_x^{Y_t}$, then we have $\{(Y_s)_{0 \leq s \leq t}; P_x^{Y_t}\} \overset{law}{=} \{(Y_s + Z_s)_{0 \leq s \leq t}; Q_x^{(t)}\}$.*

Let us explain some intuition about the above spine representations: The Poisson random measure $n(ds, dw)$ there can actually be interpreted as an immigration process. Note that it can be represented as the summation of (possibly infinite many) atomic measures on $[0, t] \times \mathbb{D}$,

$$n(ds, dw) = \sum_{s^i \in \mathcal{D}} \delta_{(s^{(i)}, w^{(i)})},$$

where, roughly speaking, at time $s^i \in \mathcal{D}$, there is a bunch of population immigrated into the system whose evolution afterwards are determined by $w^{(i)}$. Here the set $\mathcal{D}$ is the set of the times of all the immigration events. $\mathcal{D}$ is obviously countable since $n(ds, dw)$ is a Poisson random measure. Therefore, Z_s given by (1.3.13) is well defined, since it is a summation of at most countably many positive values. Z_s can actually be interpreted as the total contribution of all immigrations at time s.

Note that, the CSBP $\{(Y_t); P_x\}$ itself is a martingale. So the Y_t-transform of probability P_x can be considered as Doob's martingale transformation. In Chapter 3, we develop this theory further to include other type of size-biased transformation which may not be Doob's martingale transformation: Let F be a functional of the path $(w_s)_{0\leq s\leq t}$ where $w \in \mathbb{D}$. Suppose that this functional satisfies that $\mathbb{N}[F(w)] \in (0, \infty)$. Then from the mean formula for the Poisson random measure, we have

$$P_x[\mathcal{N}(F)] = x\mathbb{N}[F(w)] \in (0, \infty), \quad x > 0.$$

Therefore, both $P_x^{\mathcal{N}(F)}$–the $\mathcal{N}(F)$-transform of probability P_x and $\mathbb{N}^F$–the F-transform of measure $\mathbb{N}$ are all well defined probability measure on $\mathbb{D}$. We say $\{(Y_s)_{0\leq s\leq t}, (Z_s)_{0\leq s\leq t}; Q_x^{(t)}\}$ is a size-biased representation of $P_x^{\mathcal{N}(F)}$ if

- $\{(Y_s)_{0\leq s\leq t}; Q_x^{(t)}\}$ is a copy of the original CSBP $\{(Y_s)_{0\leq s\leq t}; P_x\}$;
- independent of $\{(Y_s)_{0\leq s\leq t}; Q_x^{(t)}\}$, $\{(Z_s)_{0\leq s\leq t}; Q_x^{(t)}\}$ is a process which has the same law as $\{(w_s)_{0\leq s\leq t}; \mathbb{N}^F\}$.

If we take $F(w) = w_t$, then it will be proved in Chapter 3 that the process $\{(Z_s)_{0\leq s\leq t}; Q_x^{(t)}\}$ given in (1.3.13) has the exactly the same law as $\{(w_s)_{0\leq s\leq t}; \mathbb{N}^{w_t}\}$. In other words, if we know $\{Y, Z, n; Q_x^{(t)}\}$ is a spine representation of $P_x^{Y_t}$, then $\{Y, Z; Q_x^{(t)}\}$ is a size-biased representation of $P_x^{Y_t}$. The following theorem explained the naming:

Theorem 1.3.4. *Suppose that* $\{(Y_s)_{0\leq s\leq t}, (Z_s)_{0\leq s\leq t}; Q_x^{(t)}\}$ *is a size-biased representation of* $P_x^{\mathcal{N}(F)}$. *Then we have* $\{(Y_s)_{0\leq s\leq t}; P_x^{\mathcal{N}(F)}\} \overset{law}{=} \{(Y_s + Z_s)_{0\leq s\leq t}; Q_x^{(t)}\}$.

In Chapter 3, we will prove that, if F takes the form of $F(w) = w_t^2$, and $\{Y, Z; Q_x^{(t)}\}$ is the corresponding size-biased representation of $P_x^{\mathcal{N}[w_t^2]}$, then $\{Z; Q_x^{(t)}\}$ can be decomposed further as a immigration processes along a 2-spine skeleton. More precisely, we say

$$\{(Y_s)_{0\le s\le t},(Z_s)_{0\le s\le t},\kappa,n_1,n_2;Q_x^{(t)}\}$$

is a 2-spine representation of $P_x^{\mathcal{N}[w_t^2]}$ if

- $\{Y:=(Y_s)_{0\le s\le t};Q_x^{(t)}\}$ is a copy of the original CSBP $\{(Y_s)_{0\le s\le t};P_x\}$;
- independent of $\{(Y_s)_{0\le s\le t};Q_x^{(t)}\}$, $n_1(ds,dw)$ is a Poisson random measure on $[0,t]\times\mathbb{D}$ with intensity
$$2ds\times\mathbb{N}(dw);$$
- independent of Y and n_1, κ is a random time selected uniformly in the time interval $[0,t]$; and conditioned on the κ, $n_2(ds,dw)$ is a Poisson random measure on $[\kappa,t]\times\mathbb{D}$ with intensity
$$2\mathbf{1}_{s\in[\kappa,t]}ds\times\mathbb{N}(dw);$$
- $(Z_s)_{0\le s\le t}$ is a non-negative process defined by
$$Z_s=\int_0^s w_{s-t}n_1(dr,dw)+\mathbf{1}_{s\ge\kappa}\int_\kappa^s w_{s-r}n_2(dr,dw),\quad 0\le s\le t.$$

The Z_s above can be interpreted as the total contribution of two different type of immigrations at time s. The first type of immigrations are directed by Poisson random measure n_1 and the second type are directed by Poisson random measure n_2. It would be helpful to imagine that there is a "spine particle" with life time $[0,t]$ who "generates" new mass of branching populations into the system according to a certain Poissonian way, and at a random time κ, there suddenly appears another "spine particle" with life time $[\kappa,t]$ who also "generates" new mass of branching populations into the system according to a certain Poissonian way. In Chapter 3 we will prove the following 2-spine decomposition for the CSBP:

Theorem 1.3.5. *Suppose that* $\{(Y_s)_{0\le s\le t},(Z_s)_{0\le s\le t},\kappa,n_1,n_2;Q_x^{(t)}\}$ *is a 2-spine representation of* $P_x^{\mathcal{N}[w_t^2]}$*, then* $\{(Y_s)_{0\le s\le t};P_x^{\mathcal{N}[w_t^2]}\}\overset{law}{=}\{(Y_s+Z_s)_{0\le s\le t};Q_x^{(t)}\}$.

The reason that those decompositions for size-biased CSBP is useful for proving Yaglom type results is similar to that for the Galton-Watson branching processes. In fact, we can see that Theorem 1.3.3 characterized the size-biased transformation of the CSBP, while Theorem 1.3.5 characterized the double size-biased transformation of the CSBP. To distinguish those two characterization, we will use $\{Y,Z,n;Q_x^{(t)}\}$ to denote a spine representation of $P_x^{Y_t}$, and use $\{\tilde{Y},\tilde{Z},\kappa,n_1,n_2;\tilde{Q}_x^{(t)}\}$ to denote a 2-spine representation of $P_x^{\mathcal{N}[w_t^2]}$. The construction of the

2-spine representation actually says that

$$\tilde{Z}_t \overset{law}{=} Z_t + Z'_{t-Ut},$$

where Z' is a copy of Z, U is an uniform distributed random variable in $[0,1]$, and Z, Z' and U are independent. Note that Z_t has the same law as $\{w_t; \mathbb{N}^{w_t}\}$, and $\tilde{Z}_t$ has the same law as $\{w_t; \mathbb{N}^{w_t^2}\}$. This actually implies that $\tilde{Z}_t$ is the size-biased transform of Z_t. Suppose that Z_t/t converges weakly to a random variable X, and $\tilde{Z}_t/t$ converges weakly to a random variable $\tilde{X}$. Then, according to [58, Lemma 4.3], $\tilde{X}$ is an X-transform of X, and we should have

$$\tilde{X} = X + U \cdot X',$$

where X' is a copy of X, U is a uniform random variable on $[0,1]$, and X, X', U are independent. With this observation and Lemma 1.3.2, we can see why Yaglom type result for CSBP should be true. The precise proofs of both Kolmogorov type and Yaglom type results for a large class of critical superprocesses using a 2-spine method will be presented in Chapter 3.

A proof of Slack type result for a large class of critical superprocess without the second moment condition will be presented in Chapter 5. We mention here that the 2-spine decomposition for the critical superprocesses requires the second moment condition, so we can not use it anymore in Chapter 5. The general one-spine decomposition theorem developed in Chapter 3 still plays a central role though.

1.4 Organization of the thesis

The rest of the thesis is organized as follows: Chapter 2 is based on my work [63] in collaboration with Yan-Xia Ren and Renming Song. We give a relatively short and self-contained application of the multi-spine techniques providing a new proof of Yaglom's theorem for the critical Galton-Watson processes. We show that the double-size-biased transformation of a critical Galton-Watson tree corresponds to a branching tree with 2 distinguishable spines. Note that, we already explained intuitively why Yaglom's theorem should be true using this 2-spine method earlier. In Chapter 2, we translate this intuition into mathematics. This is useful both for giving a new point of view on Yaglom type theorem and a new application to multi-spine theory. Our method is generic in the sense that it can be applied to much more complicated branching systems such that superprocesses.

Chapter 3 is based on my work [65] in collaboration with Yan-Xia Ren and Renming Song. In that chapter, we give a probabilistic proof of Yaglom type results for a class of critical superprocesses using a newly developed general size-biasing technique for the superprocesses. First, we establish a general framework for size-biased decomposition theorems for the superprocesses using their Poissonian representations. Second, under this framework, we establish a spine decomposition theorem and a 2-spine decomposition theorem for critical superprocesses. Third, we give a proof of the Kolmogorov type and Yaglom type result using those spine decompositions. Compared to the analytical methods used by Perkins [31] and Ren, Song and Zhang [68], our probabilistic proof is more intuitive and gives results under weaker conditions. Also, our general framework connects the spine theorem to the Poissonian representation of the superprocesses. This connection is fundamental and seems has not been fully exploited before in the literature.

In Chapter 4, we consider the characteristic function of superprocesses. We prove that the characteristic exponent of $\langle X_t, f\rangle$ is the mild solution to a non-linear complex-valued PDE where $(X_t)_{t\geq 0}$ is a general non-persistent superprocess and f is a testing function. This is more general than the classical theory about the Laplace exponents of a superprocess satisfying a non-linear real-valued PDE, because we allow our testing function f to take both positive and negative values. The general spine decomposition theorem in Chapter 4 is used to prove this result. In the follow-up work [71] in collaboration with Yan-Xia Ren, Renming Song and Jianjie Zhao, we use this result to prove several stable central limit theorems for supercritical super Ornstein-Uhlenbeck processes.

Chapter 5 is based on the work [64] in collaboration with Yan-Xia Ren and Renming Song. In that chapter, we establish Slack type results for a class of critical superprocesses with spatially dependent stable branching. Using the general spine theory for the superprocess developed in Chapter 3, we could establish rate of decay of the survival probability. We can also show that the Laplace transform of the one-dimensional distributions of the superprocess, after proper rescaling, can be characterized by a non-linear delay equation. We then show that the Laplace transform of Slack's random variable $\mathbf{z}^{\alpha}$ can also be characterized by a similar non-linear equation. As far as we know, this characterization of Slack's random variable is new. The desired Slack type results can then be showed by comparison of those equations. That the stable index is spatially inhomogeneous and that the second moment is infinite make the arguments challenging. This work adds more results to the theory of critical superprocess and provides a new point of view for Slack type universal results.

Chapter 2 A 2-spine Decomposition of the Critical Galton-Watson Tree

2.1 Trees and their decompositions

2.1.1 Spaces and measures

Suppose that μ is an offspring distribution with mean 1 and finite variance. In this subsection, we give a proof of Theorem 1.3.1. Consider *particles* as elements in the space

$$\mathcal{U} := \{\emptyset\} \cup \bigcup_{k=1}^{\infty} \mathbb{N}^k,$$

where $\mathbb{N} := \{1, 2, \dots\}$. Therefore elements in $\mathcal{U}$ are of the form 213, which we read as the individual being the 3rd child of the 1st child of the 2nd child of the initial ancestor $\emptyset$. For two particles $u = u_1 \dots u_n, v = v_1 \dots v_m \in \mathcal{U}$, uv denotes the concatenated particle $uv := u_1 \dots u_n v_1 \dots v_m$. We use the convention $u\emptyset = \emptyset u = u$ and $u_1 \dots u_n = \emptyset$ if $n = 0$. For any particle $u := u_1 \dots u_{n-1} u_n$, we define its *generation* as $|u| := n$ and its *parent particle* as $\overleftarrow{u} := u_1 \dots u_{n-1}$. For any particle $u \in \mathcal{U}$ and any subset $\mathbf{a} \subset \mathcal{U}$, we define the *number of children of u in* $\mathbf{a}$ as $l_u(\mathbf{a}) := \#\{\alpha \in \mathbf{a} : \overleftarrow{\alpha} = u\}$. We also define the *height* of $\mathbf{a}$ as $|\mathbf{a}| := \sup_{\alpha \in \mathbf{a}} |\alpha|$ and its *population in the nth generation* as $X_n(\mathbf{a}) := \#\{u \in \mathbf{a} : |u| = n\}$. A *tree* $\mathbf{t}$ is defined as a subset of $\mathcal{U}$ such that there exists an $\mathbb{N}_0$-valued sequence $(l_u)_{u \in \mathcal{U}}$, indexed by $\mathcal{U}$, satisfying

$$\mathbf{t} = \{u_1 \dots u_m \in \mathcal{U} : m \geq 0, u_j \leq l_{u_1 \dots u_{j-1}}, \forall j = 1, \dots, m\}.$$

A *spine* $\mathbf{v}$ on a tree $\mathbf{t}$ is defined as a sequence of particles $\{v^{(k)} : k = 0, 1, \dots, |\mathbf{t}|\} \subset \mathbf{t}$ such that $v^{(0)} = \emptyset$ and $\overleftarrow{v^{(k)}} = v^{(k-1)}$ for any $k = 1, \dots, |\mathbf{t}|$. In the case that $|\mathbf{t}| = \infty$, we simply write $k = 0, 1, \dots$ as $k = 0, 1, \dots, |\mathbf{t}|$.

Fix a generation number $n \in \mathbb{N}$. Define the following spaces.

- *The space of trees with height no more than n*,

$$\mathbb{T}_{\leq n} := \{\mathbf{t} : \mathbf{t} \text{ is a tree with } |\mathbf{t}| \leq n\}.$$

- *The space of n-height trees with one distinguishable spine*,

$$\dot{\mathbb{T}}_n := \{(\mathbf{t}, \mathbf{v}) : \mathbf{t} \text{ is a tree with } |\mathbf{t}| = n, \mathbf{v} \text{ is a spine on } \mathbf{t}\}.$$

- *The space of n-height trees with two different distinguishable spines*,

$$\ddot{\mathbb{T}}_n := \{(\mathbf{t}, \mathbf{v}, \mathbf{v}') : (\mathbf{t}, \mathbf{v}) \in \dot{\mathbb{T}}_n, (\mathbf{t}, \mathbf{v}') \in \dot{\mathbb{T}}_n, \mathbf{v} \neq \mathbf{v}'\}.$$

Let $(L_u)_{u\in\mathcal{U}}$ be a collection of independent random variables with law μ, indexed by $\mathcal{U}$. Denote by T the random tree defined by

$$T := \{u_1 \dots u_m \in \mathcal{U} : 0 \leq m \leq n, u_j \leq L_{u_1\dots u_{j-1}}, \forall j = 1, \dots, m\}.$$

We refer to T as a *μ-Galton-Watson tree with height no more than n* since its population $(X_m(T))_{0\leq m\leq n}$ is a μ-Galton-Watson process stopped at generation n. Define the *μ-Galton-Watson measure* $\mathbf{G}_n$ on $\mathbb{T}_{\leq n}$ as the law of the random tree T. That is, for any $\mathbf{t} \in \mathbb{T}_{\leq n}$,

$$\mathbf{G}_n(\mathbf{t}) := P(T = \mathbf{t}) = P(L_u = l_u(\mathbf{t}) \text{ for any } u \in \mathbf{t} \text{ with } |u| < n) = \prod_{u\in\mathbf{t}:|u|<n} \mu(l_u(\mathbf{t})).$$

Recall that $\dot{L}$ is an L-transform of L. Define $\dot{C}$ as a random number which, conditioned on $\dot{L}$, is uniformly distributed on $\{1, \dots, \dot{L}\}$. Independent of $(L_u)_{u\in\mathcal{U}}$, let $(\dot{L}_u, \dot{C}_u)_{u\in\mathcal{U}}$ be a collection of independent copies of $(\dot{L}, \dot{C})$, indexed by $\mathcal{U}$. We then use $(L_u)_{u\in\mathcal{U}}$ and $(\dot{L}_u, \dot{C}_u)_{u\in\mathcal{U}}$ as the building blocks to construct the size-biased μ-Galton-Watson tree $\dot{T}$ and its distinguishable spine $\dot{V}$ following the steps described in Section 1.2. We use L_u as the number of children of particle u if u is unmarked and use $\dot{L}_u$ if u is marked. In the latter case, we always set the $\dot{C}_u$-th child of u, i.e. particle $u\dot{C}_u$, as the new marked particle. For convenience, we stop the system at generation n. To be precise, the random spine $\dot{V}$ is defined by

$$\dot{V} := \{v_1 \dots v_m \in \mathcal{U} : 0 \leq m \leq n, v_j = \dot{C}_{v_1\dots v_{j-1}}, \forall j = 1, \dots, m\},$$

and the random tree $\dot{T}$ is defined by

$$\dot{T} := \{u_1 \dots u_m \in \mathcal{U} : 0 \leq m \leq n, u_j \leq \tilde{L}_{u_1\dots u_{j-1}}, \forall j = 1, \dots, m\},$$

where, for any $u \in \mathcal{U}$, $\tilde{L}_u := L_u \mathbf{1}_{u\notin\dot{V}} + \dot{L}_u \mathbf{1}_{u\in\dot{V}}$.

We now consider the distribution of the $\dot{\mathbb{T}}_n$-valued random element $(\dot{T}, \dot{V})$. For any $(\mathbf{t}, \mathbf{v}) \in \dot{\mathbb{T}}_n$, the event $\{(\dot{T}, \dot{V}) = (\mathbf{t}, \mathbf{v})\}$ occurs if and only if:

- $L_u = l_u(\mathbf{t})$ for each $u \in \mathbf{t} \setminus \mathbf{v}$ with $|u| < n$ and
- $(\dot{L}_{v_1\dots v_m}, \dot{C}_{v_1\dots v_m}) = (l_{v_1\dots v_m}(\mathbf{t}), v_{m+1})$ for each $v_1 \dots v_{m+1} \in \mathbf{v}$ with $0 \leq m \leq n-1$.

Therefore, the distribution of $(\dot{T}, \dot{V})$ can be determined by

$$P((\dot{T}, \dot{V}) = (\mathbf{t}, \mathbf{v})) = \prod_{u\in\mathbf{t}\setminus\mathbf{v}:|u|<n} \mu(l_u(\mathbf{t})) \cdot \prod_{u\in\mathbf{v}:|u|<n} l_u(\mathbf{t})\mu(l_u(\mathbf{t}))\frac{1}{l_u(\mathbf{t})} = \mathbf{G}_n(\mathbf{t}). \tag{2.1.1}$$

The *size-biased μ-Galton-Watson measure* $\dot{\mathbf{G}}_n$ on $\mathbb{T}_{\le n}$ is then defined as the law of the $\mathbb{T}_{\le n}$-valued random element $\dot{T}$. That is, for any $\mathbf{t} \in \mathbb{T}_{\le n}$,

$$\begin{aligned}\dot{\mathbf{G}}_n(\mathbf{t}) := P(\dot{T} = \mathbf{t}) &= \sum_{\mathbf{v}:(\mathbf{t},\mathbf{v})\in\dot{\mathbb{T}}_n} P((\dot{T},\dot{V}) = (\mathbf{t},\mathbf{v})) \\ &= \#\{\mathbf{v} : (\mathbf{t},\mathbf{v}) \in \dot{\mathbb{T}}_n\} \cdot \mathbf{G}_n(\mathbf{t}) = X_n(\mathbf{t}) \cdot \mathbf{G}_n(\mathbf{t}).\end{aligned} \tag{2.1.2}$$

Equations (2.1.1), (2.1.2) and their consequence (1.3.1) were first obtained in [58]. We use these equations to help us to understand how the $k(k-1)$-type size-biased μ-Galton-Watson tree can be represented.

Recall that K_n is a random generation number uniformly distributed on $\{0,\dots,n-1\}$, and $\ddot{L}$ is an $L(L-1)$-transform of L. Define $(\ddot{C},\ddot{C}')$ as a random vector which, conditioned on $\ddot{L}$, is uniformly distributed on $\{(i,j) \in \mathbb{N}^2 : 1 \le i \ne j \le \ddot{L}\}$. Suppose that $(L_u)_{u\in\mathcal{U}}, (\dot{L}_u,\dot{C}_u)_{u\in\mathcal{U}}$, $(\ddot{L},\ddot{C},\ddot{C}')$ and K_n are independent of each other. We now use these elements to build the $k(k-1)$-type size-biased μ-Galton-Watson tree $\ddot{T}$ and its two different distinguishable spines $\ddot{V}$ and $\ddot{V}'$ following the steps described in Section 1.2. Write $C_u := \dot{C}_u\mathbf{1}_{|u|\ne K_n} + \ddot{C}\mathbf{1}_{|u|=K_n}$ and $C'_u := \dot{C}_u\mathbf{1}_{|u|\ne K_n} + \ddot{C}'\mathbf{1}_{|u|=K_n}$. We define the random spines $\ddot{V}$ and $\ddot{V}'$ as

$$\begin{aligned}\ddot{V} &:= \{v_1\dots v_m \in \mathcal{U} : 0 \le m \le n, v_j = C_{v_1\dots v_{j-1}}, \forall j = 1,\dots,m\},\\ \ddot{V}' &:= \{v_1\dots v_m \in \mathcal{U} : 0 \le m \le n, v_j = C'_{v_1\dots v_{j-1}}, \forall j = 1,\dots,m\},\end{aligned}$$

and the random tree $\ddot{T}$ as

$$\ddot{T} := \{u_1\dots u_m \in \mathcal{U} : 0 \le m \le n, u_j \le L''_{u_1\dots u_{j-1}}, \forall j = 1,\dots,m\},$$

where, for any $u \in \mathcal{U}$, $L''_u := L_u\mathbf{1}_{u\notin\ddot{V}\cup\ddot{V}'} + \dot{L}_u\mathbf{1}_{u\in\ddot{V}\cup\ddot{V}',|u|\ne K_n} + \ddot{L}\mathbf{1}_{u\in\ddot{V}\cup\ddot{V}',|u|=K_n}$.

We now consider the distribution of $(\ddot{T},\ddot{V},\ddot{V}')$. For any $(\mathbf{t},\mathbf{v},\mathbf{v}') \in \ddot{\mathbb{T}}_n$, the event $\{(\ddot{T},\ddot{V},\ddot{V}') = (\mathbf{t},\mathbf{v},\mathbf{v}')\}$ occurs if and only if:

- $K_n = k_n := |\mathbf{v} \cap \mathbf{v}'|$,
- $L_u = l_u(\mathbf{t})$ for each $u \in \mathbf{t} \setminus (\mathbf{v} \cup \mathbf{v}')$ with $|u| < n$,
- $(\dot{L}_{v_1\dots v_m}, \dot{C}_{v_1\dots v_m}) = (l_{v_1\dots v_m}(\mathbf{t}), v_{m+1})$ for each $v_1\dots v_m v_{m+1} \in \mathbf{v}\cup\mathbf{v}'$ with $k_n \ne m < n$ and
- $(\ddot{L},\ddot{C},\ddot{C}') = (l_{v_1\dots v_{k_n}}(\mathbf{t}), v_{k_n+1}, v'_{k_n+1})$ for $v_1\dots v_{k_n}v_{k_n+1} \in \mathbf{v}$ and $v_1\dots v_{k_n}v'_{k_n+1} \in \mathbf{v}'$.

Using this analysis, we get that

$$\begin{aligned}P\big((\ddot{T},\ddot{V},\ddot{V}') = (\mathbf{t},\mathbf{v},\mathbf{v}')\big) = \frac{1}{n} \cdot &\prod_{u\in\mathbf{t}\setminus(\mathbf{v}\cup\mathbf{v}'):|u|<n} \mu(l_u(\mathbf{t})) \cdot \prod_{u\in\mathbf{v}\cup\mathbf{v}':k_n\ne|u|<n} l_u(\mathbf{t})\mu(l_u(\mathbf{t}))\frac{1}{l_u(\mathbf{t})}\\ &\cdot \prod_{u\in\mathbf{v}\cup\mathbf{v}':|u|=k_n} \frac{l_u(\mathbf{t})(l_u(\mathbf{t})-1)\mu(l_u(\mathbf{t}))}{\sigma^2}\frac{1}{l_u(\mathbf{t})(l_u(\mathbf{t})-1)}\end{aligned}$$

$$= \frac{1}{n\sigma^2}\mathbf{G}_n(\mathbf{t}).$$

The $k(k-1)$*-type size-biased* μ*-Galton-Watson measure* $\ddot{\mathbf{G}}_n$ on $\mathbb{T}_{\le n}$ is then defined as the law of the random element $\ddot{T}$. That is, for any $\mathbf{t} \in \mathbb{T}_{\le n}$,

$$\begin{aligned}\ddot{\mathbf{G}}_n(\mathbf{t}) := P(\ddot{T} = \mathbf{t}) &= \sum_{(\mathbf{v},\mathbf{v}'):(\mathbf{t},\mathbf{v},\mathbf{v}')\in\ddot{\mathbb{T}}_n} P\big((\ddot{T},\ddot{V},\ddot{V}') = (\mathbf{t},\mathbf{v},\mathbf{v}')\big) \\ &= \#\{(\mathbf{v},\mathbf{v}') : (\mathbf{t},\mathbf{v},\mathbf{v}') \in \ddot{\mathbb{T}}_n\} \cdot \frac{\mathbf{G}_n(\mathbf{t})}{n\sigma^2} = \frac{X_n(\mathbf{t})(X_n(\mathbf{t})-1)}{n\sigma^2} \cdot \mathbf{G}_n(\mathbf{t}).\end{aligned} \tag{2.1.3}$$

We note in passing that, because of the way they are constructed, the measures $(\ddot{\mathbf{G}}_n)_{n\ge1}$ are not consistent, that is, the measure $\ddot{\mathbf{G}}_n$ is not the restriction of $\ddot{\mathbf{G}}_{n+1}$. This implies that the change of measure in Theorem 1.3.1 is not a martingale change of measure.

Proof of Theorem 1.3.1. Note that

$$\{(X_m(\mathbf{t}))_{0\le m\le n}; \mathbf{G}_n\} \overset{d}{=} (Z_m)_{0\le m\le n} \quad \text{and} \quad \{(X_m(\mathbf{t}))_{0\le m\le n}; \ddot{\mathbf{G}}_n\} \overset{d}{=} (\ddot{Z}_m)_{0\le m\le n}.$$

According to (2.1.3), for any bounded Borel function g on $\mathbb{N}_0^n$, we can verify that

$$\begin{aligned}E[g(\ddot{Z}_1^{(n)},\dots,\ddot{Z}_n^{(n)})] &= \ddot{\mathbf{G}}_n[g(X_1(\mathbf{t}),\dots,X_n(\mathbf{t}))] \\ &= \mathbf{G}_n\Big[\frac{X_n(\mathbf{t})(X_n(\mathbf{t})-1)}{n\sigma^2} g(X_1(\mathbf{t}),\dots,X_n(\mathbf{t}))\Big] \\ &= \frac{1}{n\sigma^2} E[Z_n(Z_n-1)g(Z_1,\dots,Z_n)].\end{aligned} \tag{2.1.4}$$

Taking $g \equiv 1$ in equation (2.1.4), we get that

$$E[Z_n(Z_n-1)] = E[\dot{Z}_n - 1] = n\sigma^2. \tag{2.1.5}$$

□

2.1.2 Double size-biased transform for Galton-Watson tree

Using the notation introduced in the previous section, we are now ready to give a precise meaning to (1.3.8):

Proposition 2.1.1. *Let* $(\dot{Z}_m)_{0\le m\le n}$ *be the population of a size-biased* μ*-Galton Watson tree and* $(\ddot{Z}_m^{(n)})_{0\le m\le n}$ *be the population of a* $k(k-1)$*-type size-biased* μ*-Galton-Watson tree with height* n*. Suppose that* μ *satisfies* (1.3.2) *and* (1.3.3)*. Then*

$$E[e^{-\lambda \ddot{Z}_n^{(n)}}] = E[e^{-\lambda \dot{Z}_n}]E[g(\lambda,\lfloor Un\rfloor)e^{-\lambda \dot{Z}_{\lfloor Un\rfloor}}],$$

where U is a uniform random variable on $[0,1]$ independent of $\{\dot{Z}_m : 0 \le m \le n\}$; and $g(\lambda,m)$ is a function on $[0,\infty) \times \mathbb{N}_0$ such that $g(\lambda,m) \to 1$, uniformly in λ as $m \to \infty$.

Proof. For any particle $u = u_1 \dots u_n$, we define $[\emptyset,u] := \{u_1 \dots u_j : j = 0,\dots,n\}$ as the *descending family line from $\emptyset$ to u*. The particles in $\dot{T}$ can be separated according to their nearest spine ancestor. For each $k = 0,\dots,n$, we write $\dot{A}_k := \{u \in \dot{T} : |[\emptyset,u] \cap \dot{V}| = k\}$. Then

$$X_n(\dot{T}) = \sum_{k=0}^{n} X_n(\dot{A}_k). \tag{2.1.6}$$

Notice that the right side of the above equation is a sum of independent random variables; and from their construction, we see that $X_n(\dot{A}_k) \overset{d}{=} Z^{(\dot{L}-1)}_{n-k-1}$. Here, $Z^{(\dot{L}-1)}_{(-1)} := 1$ and $(Z^{(\dot{L}-1)}_m)_{m\in\mathbb{N}_0}$ denotes a μ-Galton-Watson process with $Z^{(\dot{L}-1)}_0$ distributed according to $\dot{L}-1$. Taking Laplace transforms on both sides of (2.1.6) we get

$$E[e^{-\lambda \dot{Z}_n}] = \prod_{k=0}^{n} E[e^{-\lambda Z^{(\dot{L}-1)}_{n-k-1}}]. \tag{2.1.7}$$

Similarly, we consider the $k(k-1)$-type size-biased μ-Galton-Watson tree $(\ddot{T},\ddot{V},\ddot{V}')$. Write

$$\ddot{A}^l_k := \{u \in \ddot{T} : |[\emptyset,u] \cap \ddot{V}| = k, [\emptyset,u] \cap (\ddot{V}' \setminus \ddot{V}) = \emptyset\}$$

and

$$\ddot{A}^s_k := \{u \in \ddot{T} : |[\emptyset,u] \cap \ddot{V}'| = k, [\emptyset,u] \cap (\ddot{V}' \setminus \ddot{V}) \neq \emptyset\}.$$

Then,

$$X_n(\ddot{T}) = \sum_{k=0}^{n} X_n(\ddot{A}^l_k) + \sum_{k=K_n+1}^{n} X_n(\ddot{A}^s_k). \tag{2.1.8}$$

Notice that, conditioning on $K_n = m$ with $m \in \{0,\dots,n-1\}$, the right side of the above equation is a sum of independent random variables; and from their construction, we see that $X_n(\ddot{A}^l_k) \overset{d}{=} Z^{(\dot{L}-1)}_{n-k-1}$ for each $k \neq m$; $X_n(\ddot{A}^l_m) \overset{d}{=} Z^{(\ddot{L}-2)}_{n-m-1}$; and $X_n(\ddot{A}^s_k) \overset{d}{=} Z^{(\dot{L}-1)}_{n-k-1}$ for each $k \ge m+1$. Here, $Z^{(\ddot{L}-2)}_{(-1)} := 1$ and $(Z^{(\ddot{L}-2)}_k)_{k\in\mathbb{N}_0}$ is a μ-Galton-Watson process with initial population distributed according to $\ddot{L}-2$.

Taking Laplace transform on both sides of (2.1.8) and using (2.1.7), we get

$$\begin{aligned} E[e^{-\lambda \ddot{Z}^{(n)}_n}] &= \frac{1}{n}\sum_{m=0}^{n-1}\Big(\prod_{k=0,k\neq m}^{n} E[e^{-\lambda Z^{(\dot{L}-1)}_{n-k-1}}]\Big) \cdot E[e^{-\lambda Z^{(\ddot{L}-2)}_{n-m-1}}] \cdot \Big(\prod_{k=m+1}^{n} E[e^{-\lambda Z^{(\dot{L}-1)}_{n-k-1}}]\Big) \\ &= E[e^{-\lambda \dot{Z}_n}]\frac{1}{n}\sum_{m=0}^{n-1} \frac{E[e^{-\lambda Z^{(\ddot{L}-2)}_{n-m-1}}]}{E[e^{-\lambda Z^{(\dot{L}-1)}_{n-m-1}}]} \cdot E[e^{-\lambda \dot{Z}_{n-m-1}}] \end{aligned}$$

$$= E[e^{-\lambda \dot{Z}_n}]\frac{1}{n}\sum_{m=0}^{n-1}\frac{E[e^{-\lambda Z_m^{(\dot{L}-2)}}]}{E[e^{-\lambda Z_m^{(\dot{L}-1)}}]}\cdot E[e^{-\lambda \dot{Z}_m}] = E[e^{-\lambda \dot{Z}_n}]E[g(\lambda,\lfloor Un\rfloor)e^{-\lambda \dot{Z}_{\lfloor Un\rfloor}}],$$

where

$$P(Z_m^{(\dot{L}-2)}=0)\le g(\lambda,m):=\frac{E[e^{-\lambda Z_m^{(\dot{L}-2)}}]}{E[e^{-\lambda Z_m^{(\dot{L}-1)}}]}\le P(Z_m^{(\dot{L}-1)}=0)^{-1}.$$

Notice that, from the criticality, $P(Z_m^{(\dot{L}-2)}=0)$ and $P(Z_m^{(\dot{L}-1)}=0)^{-1}$ converge to 1. □

2.2 Proof of Theorem 1.2.1

Proof of Theorem 1.2.1(1). Denote by B_n^j the event that the Galton-Watson process $(Z_n)_{n\ge 0}$ survives up to generation n, and the left-most particle in the n-th generation is a descendant of the jth particle of the first generation. Write $q_n = P[Z_n = 0] = f^{(n)}(0)$ and $p_n = 1 - q_n$ where f is the probability generating function of the offspring distribution μ. Then

$$E[Z_n|Z_n>0]=\sum_{k=1}^{\infty}E[Z_n;Z_1=k|Z_n>0]=p_n^{-1}\sum_{k=1}^{\infty}E[Z_n;Z_1=k;Z_n>0] \tag{2.2.1}$$

$$=p_n^{-1}\sum_{k=1}^{\infty}\sum_{j=1}^{k}E[Z_n;Z_1=k;B_n^j]=p_n^{-1}\sum_{k=1}^{\infty}\sum_{j=1}^{k}P[Z_1=k;B_n^j]E[Z_n|Z_1=k,B_n^j]$$

$$=p_n^{-1}\sum_{k=1}^{\infty}\sum_{j=1}^{k}P[Z_1=k;B_n^j]\Big(E[Z_{n-1}|Z_{n-1}>0]+k-j\Big)$$

$$=E[Z_{n-1}|Z_{n-1}>0]+\frac{p_{n-1}}{p_n}\sum_{k=1}^{\infty}\sum_{j=1}^{k}\mu(k)q_{n-1}^{j-1}(k-j).$$

The criticality implies that $q_n \uparrow 1$ as $n\to\infty$, and that

$$\frac{p_n}{p_{n-1}}=\frac{1-f^{(n)}(0)}{1-f^{(n-1)}(0)}=\frac{1-f(q_{n-1})}{1-q_{n-1}}\xrightarrow[n\to\infty]{}f'(1)=1.$$

By the monotone convergence theorem,

$$\frac{p_{n-1}}{p_n}\sum_{k=1}^{\infty}\sum_{j=1}^{k}\mu(k)q_{n-1}^{j-1}(k-j)\xrightarrow[n\to\infty]{}\sum_{k=1}^{\infty}\sum_{j=1}^{k}\mu(k)(k-j)=\sum_{k=1}^{\infty}\mu(k)k(k-1)/2=\frac{\sigma^2}{2}.$$

Now combining (2.2.1) with the above, we get

$$\frac{1}{nP(Z_n>0)}=\frac{1}{n}E[Z_n|Z_n>0]$$

$$=\frac{1}{n}E[Z_0|Z_0>0]+\frac{1}{n}\sum_{m=1}^{n}\frac{p_{m-1}}{p_m}\sum_{k=1}^{\infty}\sum_{j=1}^{k}\mu(k)q_{m-1}^{j-1}(k-j)$$

$$\xrightarrow[n\to\infty]{}\frac{\sigma^2}{2}.$$

□

In order to compare distributions using their size-biased add-on structures, we need the following lemma:

Lemma 2.2.1. *Let X_0 and X_1 be two non-negative random variables with the same mean $a = E[X_0] = E[X_1] \in (0,\infty)$. Let F_0 be defined by $E[e^{-\lambda \dot{X}_0}] = E[e^{-\lambda X_0}]F_0(\lambda)$, where $\dot{X}_0$ is an X_0-transform of X_0, and F_1 be defined by $E[e^{-\lambda \dot{X}_1}] = E[e^{-\lambda X_1}]F_1(\lambda)$, where $\dot{X}_1$ is an X_1-transform of X_1. Then,*

$$\left|E[e^{-\lambda X_0}] - E[e^{-\lambda X_1}]\right| \le a\int_0^\lambda |F_0(s) - F_1(s)|ds, \quad \lambda \ge 0.$$

Proof. Since $\dot{X}_0$ is an X_0-transform of X_0, we have

$$\partial_\lambda(-\ln E[e^{-\lambda X_0}]) = \frac{E[X_0 e^{-\lambda X_0}]}{E[e^{-\lambda X_0}]} = \frac{aE[e^{-\lambda \dot{X}_0}]}{E[e^{-\lambda X_0}]} = aF_0(\lambda).$$

Similarly, $\partial_\lambda(-\ln E[e^{-\lambda X_1}]) = aF_1(\lambda)$. Therefore, since $x - \ln x$ is decreasing on $[0,1]$,

$$\begin{aligned}\left|E[e^{-\lambda X_0}] - E[e^{-\lambda X_1}]\right| &\le \left|\ln E[e^{-\lambda X_0}] - \ln E[e^{-\lambda X_1}]\right| = a\Big|\int_0^\lambda F_0(s)ds - \int_0^\lambda F_1(s)ds\Big| \\ &\le a\int_0^\lambda |F_0(s) - F_1(s)|ds\end{aligned}$$

as desired. □

We are now ready to prove Lemma 1.3.2. It is elementary to verify that if Y is exponentially distributed, then it satisfies (1.3.7). So we only need to show that if Y is a strictly positive random variable with finite second moment, then (1.3.7) implies that it is exponentially distributed. The following lemma will be used to prove this.

Lemma 2.2.2. *Suppose that $c > 0$ is a constant, and F is a non-negative bounded function on $[0,\infty)$ satisfying that, for any $\lambda \ge 0$,*

$$F(\lambda) \le \frac{1}{c}\int_0^1 du \int_0^\lambda F(us)ds. \tag{2.2.2}$$

Then $F \equiv 0$.

Proof. By dividing both sides of (2.2.2) by $\|F\|_\infty$, without loss of any generality, we can assume F is bounded by 1. We prove this lemma by contradiction. Assume that

$$\rho := \inf\{x \ge 0 : F(x) \ne 0\} < \infty, \tag{2.2.3}$$

with the convention $\inf \emptyset = \infty$. Then, for each $\lambda \geq 0$,

$$F(\rho+\lambda) = \frac{1}{c}\int_0^1 du \int_0^{\rho+\lambda} F(us)ds = \frac{1}{c}\int_0^1 du \int_\rho^{\rho+\lambda} F(us)ds \leq \frac{\lambda}{c}.$$

Using this new upper bound, we have

$$F(\rho+\lambda) = \frac{1}{c}\int_0^1 du \int_\rho^{\rho+\lambda} F(us)ds \leq \frac{1}{c}\int_0^1 du \int_\rho^{\rho+\lambda} \frac{\lambda}{c} ds \leq \frac{\lambda^2}{c^2}.$$

Repeating this process, we have $F(\rho+\lambda) \leq \frac{\lambda^m}{c^m}$ for each $m \in \mathbb{N}$, which implies that $F = 0$ on $[\rho, \rho+c)$. This, however, contradicts (2.2.3). □

Proof of Lemma 1.3.2. Suppose that Y is a strictly positive random variable with finite second moment, and (1.3.7) is true. Define $a := E[\dot{Y}] \in (0,\infty)$. Consider an exponential random variable $\mathbf{e}$ with mean $a/2$. It is elementary to verify that $\mathbf{e}$ satisfies (1.3.7), in the sense that $\ddot{\mathbf{e}} \stackrel{d}{=} \dot{\mathbf{e}} + U\dot{\mathbf{e}}'$, where $\dot{\mathbf{e}}$ and $\dot{\mathbf{e}}'$ are both $\mathbf{e}$-transforms of $\mathbf{e}$, $\ddot{\mathbf{e}}$ is an $\mathbf{e}^2$-transform of $\mathbf{e}$, U is a uniform random variable on $[0,1]$, and $\dot{\mathbf{e}}$, $\dot{\mathbf{e}}'$, $\ddot{\mathbf{e}}$ and U are independent. Notice that $E[\dot{\mathbf{e}}] = a$, therefore we can compare the distribution of $\dot{Y}$ with that of $\dot{\mathbf{e}}$ using Lemma 2.2.1. This gives that

$$\left|E[e^{-\lambda\dot{Y}}] - E[e^{-\lambda\dot{\mathbf{e}}}]\right| \leq a\int_0^\lambda \int_0^1 \left|E[e^{-su\dot{Y}}] - E[e^{-su\dot{\mathbf{e}}}]\right| duds, \quad \lambda \geq 0,$$

which, according to Lemma 2.2.2, says that $\dot{Y} \stackrel{d}{=} \dot{\mathbf{e}}$. Since Y and $\mathbf{e}$ are strictly positive, according to (1.3.4), we have

$$E[1-e^{-\lambda Y}]/E[Y] = E[1-e^{-\lambda\mathbf{e}}]/E[\mathbf{e}], \quad \lambda \geq 0.$$

Letting $\lambda \to \infty$, we get $E[Y] = E[\mathbf{e}]$. Therefore, $Y \stackrel{d}{=} \mathbf{e}$ as desired. □

Proof of Theorem 1.2.1(2). Consider an exponential random variable Y with mean $\sigma^2/2$. Let $\dot{Y}$ be a Y-transform of Y. As in Section 1.3.1, we only need to prove that $\dot{Z}_n/n$ converge weakly to $\dot{Y}$. From Proposition 2.1.1, we know that

$$E[e^{-\lambda \ddot{Z}_n^{(n)}}] = E[e^{-\lambda\dot{Z}_n}]E[g(\lambda, \lfloor Un\rfloor)e^{-\lambda\dot{Z}_{\lfloor Un\rfloor}}],$$

where U is a uniform random variable on $[0,1]$ independent of $\{\dot{Z}_m : 0 \leq m \leq n\}$; and $g(\lambda,m)$ is a function on $[0,\infty)\times\mathbb{N}_0$ such that $g(\lambda,m) \to 1$, uniformly in λ as $m \to \infty$. After a renormalization, we have that

$$E[e^{-\lambda\frac{\ddot{Z}_n^{(n)}-1}{n}}] = E[e^{-\lambda\frac{\dot{Z}_n-1}{n}}]E\left[g\left(\frac{\lambda}{n}, \lfloor Un\rfloor\right)e^{-\lambda U\frac{\dot{Z}_{\lfloor Un\rfloor}}{Un}}\right], \quad \lambda \geq 0.$$

According to Theorem 1.3.1, one can verify that $(\ddot{Z}_n^{(n)}-1)/n$ is a $(\dot{Z}_n-1)/n$ transform of $(\dot{Z}_n-1)/n$. Therefore, the above equation can be viewed as the size-biased add-on structure for the random variable $(\dot{Z}_n - 1)/n$. It is easy to see that the mean of $\dot{Y}$ is σ^2. According to (2.1.5), the mean of $(\dot{Z}_n - 1)/n$ is also σ^2. Then comparing the distribution of $(\dot{Z}_n - 1)/n$ with that of $\dot{Y}$, and using Lemma 2.2.1, we get that

$$\left|E[e^{-\lambda\frac{\dot{Z}_n - 1}{n}}] - E[e^{-\lambda\dot{Y}}]\right| \leq \sigma^2 \int_0^\lambda ds \int_0^1 \left|g(\frac{s}{n}, \lfloor un\rfloor)E[e^{-su\frac{\dot{Z}_{\lfloor un\rfloor}}{un}}] - E[e^{-su\dot{Y}}]\right|du.$$

Taking $n \to \infty$ and using the reverse Fatou's lemma, we arrive at

$$M(\lambda) \leq \sigma^2 \int_0^1 du \int_0^\lambda M(us)ds, \quad \lambda \geq 0,$$

where $M(\lambda) := \limsup_{n\to\infty} |E[e^{-\lambda\frac{\dot{Z}_n}{n}}] - E[e^{-\lambda\dot{Y}}]|$. Thus by Lemma 2.2.2, we have $M \equiv 0$, which says that $\dot{Z}_n/n$ converges weakly to $\dot{Y}$. □

Chapter 3 Spine decompositions of critical superprocesses: Yaglom type result

3.1 Introduction

3.1.1 Motivation

As mentioned in Chapter 1, it is well known that for a critical Galton-Watson process $\{(Z_n)_{n\in\mathbb{N}}; P\}$, we have

$$nP(Z_n > 0) \xrightarrow[n\to\infty]{} \frac{2}{\sigma^2} \tag{3.1.1}$$

and

$$\left\{\frac{Z_n}{n}; P(\cdot|Z_n > 0)\right\} \xrightarrow[n\to\infty]{law} \frac{\sigma^2}{2}\mathbf{e}, \tag{3.1.2}$$

where σ^2 is the variance of the offspring distribution and $\mathbf{e}$ is an exponential random variable with mean 1. The result (3.1.1) was first proved by Kolmogorov in [48] under a third moment condition, and the result (3.1.2) is due to Yaglom [81]. For further references to these results, see [38, 46]. Ever since these pioneering papers of Kolmogorov and Yaglom, lots of analogous results have been obtained for more general critical branching processes. For continuous time critical branching processes, see [5]; for discrete time multitype critical branching processes, see [5, 44]; for continuous time multitype critical branching processes, see [6]; and for critical branching Markov processes, see [4]. We will call results like (3.1.1) Kolmogorov type results and results like (3.1.2) Yaglom type results. Similar results have also been obtained for some superprocesses. Evans and Perkins [31] obtained both Kolmogorov type and Yaglom type results for critical superprocesses when the branching mechanism is $(x, z) \mapsto z^2$ and the spatial motion satisfies some ergodicity conditions. Recently, Ren, Song and Zhang [68] obtained similar limit results for a class of critical superprocesses with general branching mechanisms and general spatial motions.

The proofs of the limit results in the papers mentioned above are all analytic in nature and thus not very transparent. More intuitive probabilistic proofs would be very helpful. This was first accomplished for critical Galton-Watson processes, see [33, 58] for probabilistic proofs of (3.1.1), and [32, 58, 63] for probabilistic proofs of (3.1.2). For more general models, Vatutin and Dyakonova [79] gave a probabilistic proof of a Kolmogorov type result for multitype critical branching processes. Recently, Powell [62] gave probabilistic proofs of both Kolmogorov type and Yaglom type results for a class of critical branching diffusions.

In this chapter, we will use the spine method to give probabilistic proofs of both Kolmogorov type and Yaglom type results for a class of critical superprocesses. We will first establish a size-biased decomposition theorem for superprocesses (Theorem 3.1.2) which will serve as a general framework for the spine method. Then, we will establish a spine decomposition theorem for superprocesses (Theorem 3.1.5) which is more general than those previously considered in [25, 28, 57]. We will also establish a 2-spine decomposition theorem for a class of critical superprocesses (Theorem 3.1.9). Those spine decompositions are all special forms of the aforementioned size-biased decomposition. Finally, we use these tools to give probabilistic proofs of a Kolmogorov type result (Theorem 3.1.10) and a Yaglom type result (Theorem 3.1.11) for critical superprocesses under slightly weaker conditions than [68]. To develop our decomposition for critical superprocesses, we first prove a size-biased decomposition theorem for Poisson random measures (Theorem 3.1.3), which we think is of independent interest. Before we present our main results, we first give a brief review of earlier results on the spine method.

The spine method was first introduced in [58]. Roughly speaking, the spine decomposition theorem says that the size-biased transform of the branching process can be interpreted as an immigration branching process along with an immortal particle. This spine approach is generic in the sense that it can be adapted to a variety of general branching processes and is powerful in studying limit behaviors due to its relation with the size-biased transforms. In this chapter, by the *size-biased transform of a stochastic process* we mean the following: Suppose that we are given, on some probability space $(\Omega, \mathscr{F}, P)$, a process $(X_t)_{t\in\Gamma}$, with Γ being an arbitrary index set, and a non-negative random variable G with $P[G] \in (0,\infty)$. We say a process $\{(\dot{X}_t)_{t\in\Gamma}; \dot{P}\}$ is a *G-transform* of the process $\{(X_t)_{t\in\Gamma}; P\}$ if $\{(\dot{X}_t)_{t\in\Gamma}; \dot{P}\} \overset{f.d.d.}{=} \{(X_t)_{t\in\Gamma}; P^G\}$, where P^G is a probability measure on Ω given by $dP^G := (G/P[G])dP$. (This also gives the definition of a *size-biased transform of a random variable* since a random variable can be considered as a stochastic process whose index is a singleton.)

Using the spine decomposition theorem for the Galton-Watson process $(Z_n)_{n\geq 0}$, Lyons, Pemantle and Peres [58] investigated the Z_n-transform of the process $(Z_k)_{0\leq k\leq n}$, which is denoted by $(\dot{Z}_k)_{0\leq k\leq n}$. Their key observation in the critical case is that $U\cdot \dot{Z}_n$ is distributed approximately like Z_n conditioned on $\{Z_n > 0\}$, where U is an independent uniform random variable on $[0,1]$. If one denotes by X the weak limit of $\frac{Z_n}{n}$ conditioned on $\{Z_n > 0\}$, and by $\dot{X}$ the weak limit of $\frac{\dot{Z}_n}{n}$, then [58] proved that $\dot{X}$ is the X-transform of the positive random variable X and $X \overset{law}{=} U\cdot\dot{X}$, which implies that X is an exponential random variable.

The spine method is also used by Powell [62] to establish results parallel to (3.1.1) and (3.1.2) for a class of critical branching diffusion $\{(Y_t)_{t\geq 0}; (P_x)_{x\in D}\}$ in a bounded smooth domain $D \subset \mathbb{R}^d$. As have been discussed in [62], a direct study of the partial differential equation satisfied by the survival probability $(t, x) \mapsto P_x(\|Y_t\| \neq 0)$ is tricky. Instead, by using a spine decomposition approach, Powell [62] showed that the survival probability decays like $a(t)\phi(x)$, where $\phi(x)$ is the principal eigenfunction of the mean semigroup of (Y_t) and $a(t)$ is a function capturing the uniform speed. In this chapter, our proof of the Kolmogorov type result for critical superprocesses follows a similar argument.

The spine method for superprocesses was developed in [25, 28, 57] and is very useful in studying limit behaviors of supercritical superprocesses. Heuristically, the spine is the trajectory of an immortal moving particle and the spine decomposition theorem says that, after a martingale change of measure, the transformed superprocess can be decomposed in law as an immigration process along this spine. The spine decomposition theorem established in this chapter is more general than those in [25, 28, 57]. We will say more about this in the next subsection.

In chapter 2, we developed a 2-spine decomposition technique (see also [63]) for critical Galton-Watson processes and used it to give a new probabilistic proof of Yaglom's result (3.1.2). One of the facts we used in Chapter 2 is that, if X is a strictly positive random variable with finite second moment, then X is an exponential random variable if and only if

$$\ddot{X} \overset{law}{=} \dot{X} + U \cdot \dot{X}' \tag{3.1.3}$$

where $\dot{X}$ and $\dot{X}'$ are independent X-transforms of X; $\ddot{X}$ is the X^2-transform of X; and U is again an independent uniform random variable on $[0, 1]$. We then proved in Chapter 2 that the $Z_n(Z_n - 1)$-transform of the critical Galton-Watson process $(Z_k)_{0\leq k\leq n}$, which is denoted as $(\ddot{Z}_k^{(n)})_{0\leq k\leq n}$, can be interpreted as an immigration branching process along a 2-spine skeleton. One of those two spines is longer than the other. The spirit of our proof in Chapter 2 is to show that the immigration along the longer spine at generation n is distributed approximately like $\dot{Z}_n$, while the immigration along the shorter spine at generation n is distributed approximately like $\dot{Z}'_{[U\cdot n]}$. Here $\dot{Z}_n$ and $\dot{Z}'_n$ are independent Z_n-transforms of Z_n. Roughly speaking, we have $\ddot{Z}_n^{(n)} \overset{law}{\approx} \dot{Z}_n + \dot{Z}'_{[U\cdot n]}$, and therefore, if X is the weak limit of $\frac{Z_n}{n}$ conditioned on $\{Z_n > 0\}$, then X is a positive random variable satisfying (3.1.3). In this Chapter, we adapt the method of Chapter 2 to develop a 2-spine decomposition for critical superprocesses and then use this 2-spine decomposition to give probabilistic proofs of Kolmogorov type and Yaglom type results for superprocesses. The spirit of this chapter is similar to that of Chapter 2, but the arguments are more complicated.

The idea of multi-spine decomposition is not new. It was first introduced by Harris and Roberts [37] in the context of branching processes. Our 2-spine methods for Galton-Watson trees [63] and for superprocesses in this chapter are both inspired by [37]. An analogous k-spine decomposition theorem also appeared in [36] and [45] in the context of continuous time Galton-Watson processes. The k-th size-biased transform of Galton-Watson trees is also considered in [1]. A closely related infinite spine decomposition is also established in [1] for the supercritical Galton-Watson tree.

There is another decomposition theorem for supercritical Galton-Watson trees with infinite spines which is first introduced in [5, Section 12] and is now known as the skeleton decomposition. The infinite spines in [1] and the skeleton decomposition in [5, Section 12] are two different decomposition theorems. Our 2-spine methods for Galton-Watson trees [63] and for superprocesses in this chapter are more relevant to [1].

We mention here that the analog of the skeleton decomposition in [5, Section 12] for supercritical superprocesses is also available and is very popular. Heuristically, the skeleton is the trajectories of all the prolific individuals, that is, individuals with infinite lines of descent. The skeleton decomposition says that the supercritical superprocess itself can be decomposed in law as an immigration process along this skeleton. For the skeleton methods and its applications under a variety of names, see [7, 8, 19, 25, 29, 30, 51, 52, 59, 66]. If we consider critical superprocesses conditioned to be never extinct, then we will get the transformed superprocesses (after a Doob's h-transformation) considered in [25, 28, 57] for the classical spine decomposition theorem. In this situation, there will be only one prolific individual which is exactly the spine particle. So the natural analog of the skeleton decomposition in the critical case is the classical spine decomposition. The skeleton decomposition will not be used in this thesis.

3.1.2 Main results

Let E be a locally compact separable metric space. We will use $b\mathscr{B}_E$ and $p\mathscr{B}_E$ to denote the collection of all bounded Borel functions and positive Borel functions on E respectively. We write $bp\mathscr{B}_E$ for $b\mathscr{B}_E \cap p\mathscr{B}_E$. For any functions f, g and measure μ on E, we write $\|f\|_\infty := \sup_{x\in E} |f(x)|$, $\mu(f) := \langle \mu, f\rangle := \int_E f d\mu$ and $\langle f, g\rangle_\mu := \int_E fg d\mu$ as long as they have meanings. We use $\mathbf{0}$ to denote the null measure and use $f \equiv 0$ to mean that f is the zero function. If $g(t, x)$ is a function on $[0, \infty) \times E$, we say g is *locally bounded* if $\sup_{t\in[0,T], x\in E} |g(t, x)| < \infty$ for every $T \geq 0$.

Let the *spatial motion* $\xi = \{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in E}\}$ be an E-valued Hunt process with its lifetime denoted by ζ and its transition semigroup denoted by $(P_t)_{t\geq 0}$. Let the *branching mechanism* ψ be defined as a function on $E \times [0,\infty)$ by

$$\psi(x,z) = -\beta(x)z + \alpha(x)z^2 + \int_0^\infty (e^{-zr} - 1 + zr)\pi(x,dr), \qquad x \in E, z \geq 0,$$

with $\beta \in b\mathscr{B}_E, \alpha \in bp\mathscr{B}_E$ and $\pi(x,dy)$ being a kernel from E to $(0,\infty)$ satisfying that

$$\sup_{x\in E} \int_{(0,\infty)} (y \wedge y^2)\pi(x,dy) < \infty.$$

Define an operator Ψ on $p\mathscr{B}_E$ by

$$(\Psi f)(x) := \psi(x, f(x)), \quad f \in p\mathscr{B}_E, x \in E.$$

Let $\mathcal{M}_f$ denote the space of all finite measures on E equipped with the weak topology. A (ξ,ψ)-*superprocess* is an $\mathcal{M}_f$-valued Hunt process $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ satisfying

$$\mathbf{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad t \geq 0, \mu \in \mathcal{M}_f, f \in bp\mathscr{B}_E, \tag{3.1.4}$$

where, for each $f \in bp\mathscr{B}_E$, the function $(t,x) \mapsto V_t f(x)$ on $[0,\infty) \times E$ is the unique locally bounded positive solution to the equation

$$V_t f(x) + \Pi_x\Big[\int_0^t (\Psi V_{t-s} f)(\xi_s)ds\Big] = \Pi_x[f(\xi_t)], \quad t \geq 0, x \in E. \tag{3.1.5}$$

We refer our readers to [16, 23] and [56, Section 2.3 & Theorem 5.11] for detailed discussions about the existence of such processes. Notice that we always have $\mathbf{P_0}(X_t = \mathbf{0}) = 1$ for each $t \geq 0$, i.e. the null measure $\mathbf{0}$ is an absorption state of the superprocess.

We will always assume that our superprocess is *non-persistent*:

Assumption 3.1. $\mathbf{P}_{\delta_x}(X_t = \mathbf{0}) > 0$ for each $x \in E$ and $t > 0$.

By a *size-biased transform of a measure* we mean the following: For a non-negative measurable function g on a measure space $(D, \mathscr{F}_D, \mathbf{D})$ with $\mathbf{D}(g) \in (0,\infty)$, we define the g-*transform* $\mathbf{D}^g$ *of the measure* $\mathbf{D}$ by

$$d\mathbf{D}^g := \frac{g}{\mathbf{D}(g)} d\mathbf{D}.$$

Note that, the measure $\mathbf{D}$ is not necessarily a probability measure, but after the g-transform, $\mathbf{D}^g$ is always a probability measure.

Our first result is about a decomposition theorem of the size-biased transforms of superprocesses. To state it, we need to introduce the Kuznetsov measures $(\mathbb{N}_x)_{x\in E}$ (also known as the excursion measures or $\mathbb{N}$-measures) of the superprocess X.

Lemma 3.1.1 ([56, Section 8.4 & Theorem 8.24])**.** *Under Assumption 3.1, there exists an unique family of σ-finite measures $(\mathbb{N}_x)_{x\in E}$ defined on the Skorokhod space of measure-valued paths*

$$\mathbb{W} := \{w = (w_t)_{t\geq 0} : w \textit{ is an } \mathcal{M}_f\textit{-valued càdlàg function on } [0,\infty) \textit{ having } \mathbf{0} \textit{ as a trap}\}$$

such that

1. $\mathbb{N}_x\{\forall t > 0, w_t = \mathbf{0}\} = 0$ *for each* $x \in E$*;*
2. $\mathbb{N}_x\{w_0 \neq \mathbf{0}\} = 0$ *for each* $x \in E$*;*
3. *for each* $\mu \in \mathcal{M}_f$*, if* $\mathcal{N}(dw)$ *is a Poisson random measure on* $\mathbb{W}$ *with mean measure*

$$\mathbb{N}_\mu(dw) := \int_E \mathbb{N}_x(dw)\mu(dx), \quad w \in \mathbb{W},$$

then the process defined by

$$\widetilde{X}_0 := \mu; \quad \widetilde{X}_t := \int_{\mathbb{W}} w_t\, \mathcal{N}(dw), \quad t > 0,$$

is a realization of the superprocess $\{X; \mathbf{P}_\mu\}$*.*

The measures $(\mathbb{N}_x)_{x\in E}$ are called the *Kuznetsov measures of the superprocess X*. Note that, the superprocess X itself can be considered as a $\mathbb{W}$-valued random element. Roughly speaking, the branching property of superprocess says that X can be considered as an "infinitely divisible" $\mathbb{W}$-valued random element. The Kuznetsov measure $\mathbb{N}_x$ can then be interpreted as the "Lévy measure" of X under $\mathbf{P}_{\delta_x}$. We refer our readers to [24] and [56, Section 8.4] for more details about such measures.

In the remainder of this chapter, we will always use $(\mathbb{N}_x)_{x\in E}$ to denote the Kuznetsov measures of our superprocess X. We will always use $w = (w_t)_{t\geq 0}$ to denote a generic element in $\mathbb{W}$. With a slight abuse of notation, we always assume that our superprocess X is given by

$$X_0 := \mu; \quad X_t := \int_{\mathbb{W}} w_t\, \mathcal{N}(dw), \quad t > 0,$$

where, for each $\mu \in \mathcal{M}_f$, $\{\mathcal{N}; \mathbf{P}_\mu\}$ is a Poisson random measure on $\mathbb{W}$ with mean measure $\mathbb{N}_\mu$. Recall that, for any $w \in \mathbb{W}$ and $t \geq 0$, w_t is a finite measure on E, and thus $w_t(f) = \int_E f(x)w_t(dx)$ for any $f \in p\mathscr{B}_E$.

Our first result is about the $\mathcal{N}(F)$-transform of the superprocess X, where F is a non-negative measurable function on $\mathbb{W}$ with $\mathbb{N}_\mu[F] \in (0,\infty)$ for a given $\mu \in \mathcal{M}_f$. In this case, according to Campbell's formula, we have

$$\mathbf{P}_\mu[\mathcal{N}(F)] = \mathbb{N}_\mu[F] \in (0,\infty).$$

Therefore, both $\mathbb{N}_\mu^F$ — the F-transform of $\mathbb{N}_\mu$, and $\mathbf{P}_\mu^{\mathcal{N}(F)}$ — the $\mathcal{N}(F)$-transform of $\mathbf{P}_\mu$, are well defined probability measures.

Theorem 3.1.2. *Suppose that Assumption 3.1 holds. Let $\mu \in \mathcal{M}_f$ and F be a non-negative measurable function on $\mathbb{W}$ with $\mathbb{N}_\mu(F) \in (0,\infty)$. Let $\{(Y_t)_{t\geq 0}; \mathbf{Q}_\mu\}$ be a $\mathbb{W}$-valued random element with law $\mathbb{N}_\mu^F$. Then we have $\{(X_t)_{t\geq 0}; \mathbf{P}_\mu^{\mathcal{N}(F)}\} \overset{f.d.d.}{=} \{(X_t+Y_t)_{t\geq 0}; \mathbf{P}_\mu \otimes \mathbf{Q}_\mu\}$.*

In order to prove Theorem 3.1.2, we develop a decomposition theorem for size-biased transforms of Poisson random measures which we think should be of independent interest:

Theorem 3.1.3. *Let $(S,\mathscr{S})$ be a measurable space with a σ-finite measure N. Let $\{\mathbf{N}; P\}$ be a Poisson random measure on $(S,\mathscr{S})$ with mean measure N. Let $g \in p\mathscr{S}$ satisfy $N(g) \in (0,\infty)$. Denote by N^g and $P^{\mathbf{N}(g)}$ the g-transform of N and the $\mathbf{N}(g)$-transform of P, respectively. Let $\{\vartheta; Q\}$ be an S-valued random element with law N^g. Then we have $\{\mathbf{N}; P^{\mathbf{N}(g)}\} \overset{law}{=} \{\mathbf{N}+\delta_\vartheta; P\otimes Q\}$.*

Define $(S_t)_{t\geq 0}$ the *mean semigroup* of the superprocess X by

$$S_t f(x) := \Pi_x[e^{\int_0^t \beta(\xi_s)ds} f(\xi_t)], \quad x\in E, t\geq 0, f\in p\mathscr{B}_E.$$

For each $\mu \in \mathcal{M}_f$, we define $(\mu\Pi)(\cdot) := \int_E \Pi_x(\cdot)\mu(dx)$. Note that $\mu\Pi$ is not necessarily a probability measure. It is well known (see [56, Proposition 2.27] for example) that for each $\mu \in \mathcal{M}_f, t\geq 0$ and $f \in p\mathscr{B}_E$,

$$\mathbf{P}_\mu[X_t(f)] = \mathbb{N}_\mu[w_t(f)] = (\mu\Pi)[e^{\int_0^T \beta(\xi_s)ds} f(\xi_T)\mathbf{1}_{T<\zeta}] = \mu(S_t f). \tag{3.1.6}$$

Thanks to Theorem 3.1.2, in order to study the size-biased transform of a superprocess we only have to study the corresponding size-biased transform of its Kuznetsov measures. We first consider the case when the function F in Theorem 3.1.2 takes the form of $F(w) = w_T(g)$ where $T>0$ and $g\in p\mathscr{B}_E$ with $\mu(S_T g)\in(0,\infty)$ for a given $\mu\in\mathcal{M}_f$. In this case, according to (3.1.6), we have

$$\mathbf{P}_\mu[X_T(g)] = \mathbb{N}_\mu[w_T(g)] = (\mu\Pi)[e^{\int_0^T \beta(\xi_s)ds} g(\xi_T)\mathbf{1}_{T<\zeta}] \in (0,\infty).$$

Therefore, $\mathbf{P}_\mu^{X_T(g)}$ — the $X_T(g)$-transform of $\mathbf{P}_\mu$, $\mathbb{N}_\mu^{w_T(g)}$ — the $w_T(g)$-transform of the Kuznetsov measure $\mathbb{N}_\mu$, and $\Pi_\mu^{(g,T)}$ — the $(e^{\int_0^T \beta(\xi_s)ds} g(\xi_T)\mathbf{1}_{T<\zeta})$-transform of the measure $\mu\Pi$, are all well defined probability measures. Also note that, in this case, we have $X_T(g) = \mathcal{N}(F)$, therefore $\mathbf{P}_\mu^{X_T(g)} = \mathbf{P}_\mu^{\mathcal{N}(F)}$. Recall that the superprocess X itself can be considered as a $\mathbb{W}$-valued random element. Denote by $\mathbf{P}_\mu(X\in dw)$ the push-forward of $\mathbf{P}_\mu$ under X, i.e., the distribution of X

under $\mathbf{P}_\mu$. Then, $\mathbf{P}_\mu(X \in dw)$ is a probability measure on $\mathbb{W}$. Recall that we always assume that Assumption 3.1 holds.

Definition 3.1.4. Suppose that $\mu \in \mathcal{M}_f$, $T > 0$ and $g \in p\mathscr{B}_E$ satisfy $\mu(S_T g) \in (0, \infty)$. We say $\{(\xi_t)_{0\le t\le T}, (Y_t)_{0\le t\le T}, \mathbf{n}_T; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is a *spine representation of* $\mathbb{N}_\mu^{w_T(g)}$ if the following are true:

1. the *spine process* $\{(\xi_t)_{0\le t\le T}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is a copy of $\{(\xi_t)_{0\le t\le T}; \Pi_\mu^{(g,T)}\}$;
2. conditioned on $\sigma(\xi_t : 0 \le t \le T)$, the *immigration process* $\{(Y_t)_{0\le t\le T}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is an $\mathcal{M}_f$-valued process given by

$$Y_t := \int_{(0,t]\times\mathbb{W}} w_{t-s}\mathbf{n}_T(ds, dw), \quad 0 \le t \le T, \tag{3.1.7}$$

where, $\mathbf{n}_T$ is a Poisson random measure on $[0, T] \times \mathbb{W}$ with mean measure

$$\mathbf{m}_T^\xi(ds, dw) := 2\alpha(\xi_s)\mathbb{N}_{\xi_s}(dw) \cdot ds + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}(X \in dw)\pi(\xi_s, dy) \cdot ds. \tag{3.1.8}$$

We are now ready to present our theorem on the spine decomposition of superprocesses:

Theorem 3.1.5. *Suppose that Assumption 3.1 holds. Suppose that $\mu \in \mathcal{M}_f$, $T > 0$ and $g \in p\mathscr{B}_E$ satisfy $\mu(S_T g) \in (0, \infty)$. Let $\{(\xi_t)_{0\le t\le T}, (Y_t)_{0\le t\le T}, \mathbf{n}_T; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ be a spine representation of $\mathbb{N}_\mu^{w_T(g)}$. Then, $\{(Y_t)_{t\le T}; \dot{\mathbf{P}}_\mu^{(g,T)}\} \overset{f.d.d.}{=} \{(w_t)_{t\le T}; \mathbb{N}_\mu^{w_T(g)}\}$.*

As a simple consequence of Theorems 3.1.2 and 3.1.5, we have the following:

Corollary 3.1.6. *Suppose that Assumption 3.1 holds. Suppose that $\mu \in \mathcal{M}_f$, $T > 0$ and $g \in p\mathscr{B}_E$ satisfy $\mu(S_T g) \in (0, \infty)$. Let $\{(\xi_t)_{0\le t\le T}, (Y_t)_{0\le t\le T}, \mathbf{n}_T; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ be a spine representation of $\mathbb{N}_\mu^{w_T(g)}$. Then, $\{(X_t)_{t\ge 0}; \mathbf{P}_\mu^{X_T(g)}\} \overset{f.d.d.}{=} \{(X_t + Y_t)_{t\ge 0}; \mathbf{P}_\mu \otimes \dot{\mathbf{P}}_\mu^{(g,T)}\}$.*

Corollary 3.1.6 can be considered as a generalization of the classical spine decomposition theorem for superprocesses developed in [25, 28, 57]. In these earlier papers, the testing function g is chosen specifically to be the principal eigenfunction ϕ of the mean semigroup of the superprocess (which will be introduced shortly). In the classical case (i.e. $g = \phi$), the four families of probability measures $(\mathbf{P}_\mu^{X_T(g)})_{T\ge 0}$, $(\Pi_\mu^{(g,T)})_{T\ge 0}$, $(\dot{\mathbf{P}}_\mu^{(g,T)})_{T>0}$ and $(\mathbb{N}_\mu^{w_T(g)})_{T>0}$ are all consistent, but in the general case (i.e. $g \neq \phi$), they are typically not consistent. More details about these consistencies will be provided in Lemma 3.3.4 and Remark 3.3.6.

In the papers mentioned in the paragraph above, the Kuznetsov measures have already been used to describe infinitesimal immigrations along the spine. However, our Theorem 3.1.5 provides another relation between immigration and the Kuznetsov measures: the total immigration $\{(Y_t)_{t\ge 0}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ actually has law of a size-biased transform of the Kuznetsov measures. It seems that this fact has not been exploited before, even in the classical case.

The study of the limit behavior of superprocesses X relies heavily on the spectral property of the mean semigroup. In this chapter, we assume the following:

Assumption 3.2. There exist a σ-finite Borel measure m with full support on E and a family of strictly positive, bounded continuous functions $\{p(t,\cdot,\cdot) : t > 0\}$ on $E \times E$ such that,

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy), \quad t > 0, x \in E, f \in b\mathscr{B}_E, \tag{3.1.9}$$

$$\int_E p(t,x,y) m(dx) \le 1, \quad t > 0, y \in E, \tag{3.1.10}$$

$$\int_E \int_E p(t,x,y)^2 m(dx) m(dy) < \infty, \quad t > 0, \tag{3.1.11}$$

and that $x \mapsto \int_E p(t,x,y)^2 m(dy)$ and $y \mapsto \int_E p(t,x,y)^2 m(dx)$ are both continuous on E.

In the reminder of this chapter, we will always use m to denote the reference measure in Assumption 3.2.

Assumption 3.2 is a pretty weak assumption. (3.1.10) implies that the adjoint operator P^*_t of P_t is also Markovian, and (3.1.11) implies that P_t and P^*_t are Hilbert-Schmidt operators. Under Assumption 3.2, it is proved in [68] and [67] that the semigroup $(P_t)_{t\ge 0}$ and its adjoint semigroup $(P^*_t)_{t\ge 0}$ are both strongly continuous semigroups of compact operators on $L^2(E,m)$. According to [68, Lemma 2.1], there exists a function $q(t,x,y)$ on $(0,\infty) \times E \times E$ which is continuous in (x,y) for each $t > 0$ such that

$$e^{-\|\beta\|_\infty t} p(t,x,y) \le q(t,x,y) \le e^{\|\beta\|_\infty t} p(t,x,y), \quad t > 0, x, y \in E,$$

and that for any $t > 0, x \in E$ and $f \in b\mathscr{B}_E$,

$$S_t f(x) = \int_E q(t,x,y) f(y) m(dy). \tag{3.1.12}$$

(From (3.1.6), we see that $q(t,x,y)m(dy)$ can be roughly interpreted as the density of the expected mass of X_t at position y, under probability $\mathbf{P}_{\delta_x}$.) Define a family of transition kernels $(S^*_t)_{t\ge 0}$ on E by

$$S^*_0 = I; \quad S^*_t f(y) := \int_E q(t,x,y) f(x) m(dx), \quad t > 0, y \in E, f \in b\mathscr{B}_E.$$

It is clear that $(S^*_t)_{t\ge 0}$ is the adjoint semigroup of $(S_t)_{t\ge 0}$ in $L^2(E,m)$. It is proved in [68] and [67] that $(S_t)_{t\ge 0}$ and $(S^*_t)_{t\ge 0}$ are also strongly continuous semigroups of compact operators in $L^2(E,m)$. Let L and L^* be the generators of the semigroups $(S_t)_{t\ge 0}$ and $(S^*_t)_{t\ge 0}$, respectively. Denote by $\sigma(L)$ and $\sigma(L^*)$ the spectra of L and L^*, respectively. According to [73, Theorem V.6.6.], $\lambda := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(L^*))$ is a common eigenvalue of multiplicity 1 for both

L and L^*. Using the argument in [68], the eigenfunctions ϕ of L and ϕ^* of L^* associated with the eigenvalue λ can be chosen to be strictly positive and continuous everywhere on E. We further normalize ϕ and ϕ^* so that $\langle \phi, \phi \rangle_m = \langle \phi, \phi^* \rangle_m = 1$. Moreover, for each $t \geq 0, x \in E$, we have $S_t\phi(x) = e^{\lambda t}\phi(x)$ and $S_t^*\phi^*(x) = e^{\lambda t}\phi^*(x)$. We call ϕ the *principal eigenfunction* of the mean semigroup $(S_t)_{t\geq 0}$.

Remark 3.1.7. Note that we do not require the operators $(P_t)_{t\geq 0}$ to be self-adjoint in $L^2(E, m)$, i.e., we do not assume $p(t, x, y) = p(t, y, x)$ for each $x, y \in E$ and $t > 0$. In other word, the spatial motion ξ considered in this chapter is not necessarily a symmetric Markov process with respect to the measure m. As a consequence, $(S_t)_{t\geq 0}$ are not necessarily self-adjoint either.

We will use the following function

$$A(x) := 2\alpha(x) + \int_{(0,\infty)} y^2 \pi(x, dy), \quad x \in E$$

in Assumption 3.3 below.

For all $t \geq 0$ and $x \in E$, it is now clear that $\mathbf{P}_{\delta_x}[X_t(\phi)] = S_t\phi(x) = e^{\lambda t}\phi(x)$. If $\lambda > 0$, the mean of $X_t(\phi)$ will increase exponentially; if $\lambda < 0$, the mean of $X_t(\phi)$ will decrease exponentially; and if $\lambda = 0$, the mean of $X_t(\phi)$ will be a constant. Because of this, we say X is *supercritical, critical* or *subcritical*, according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. In this chapter, we are mainly interested in critical superprocesses with finite second moments. So, for the remainder of this chapter, we always assume the following:

Assumption 3.3.
1. the superprocess X is critical, i.e., $\lambda = 0$;
2. the function $\phi A : x \mapsto \phi(x)A(x)$ is bounded on E.

Assumption 3.3.(2) is satisfied, for example, when ϕ and A are bounded on E. These conditions appeared in the literature and was used by [68] in the proof of the Kolmogorov type and the Yaglom type results for critical superprocesses.

Denote by $\mathcal{M}_f^\phi$ the collection of all the measures $\mu \in \mathcal{M}_f$ such that $\mu(\phi) \in (0, \infty)$. It will be proved in Proposition 3.4.2 that $\mathbf{P}_\mu[X_t(\phi)^2] < \infty$ for each $\mu \in \mathcal{M}_f^\phi$ and $t > 0$ provided the function $\phi A : x \mapsto \phi(x)A(x)$ is bounded on E.

Taking $\mu \in \mathcal{M}_f^\phi, T \geq 0$ and $g = \phi$ in Definition 3.1.4.(1), it will be proved in Lemma 3.3.4 that the family of probability measures $(\Pi_\mu^{(\phi,T)})_{T\geq 0}$ is consistent, i.e., there exists an E-valued process $\{(\xi_t)_{t\geq 0}; \dot{\Pi}_\mu\}$ such that

$$\{(\xi_t)_{0\leq t\leq T}; \Pi_\mu^{(\phi,T)}\} \overset{f.d.d}{=} \{(\xi_t)_{0\leq t\leq T}; \dot{\Pi}_\mu\}, \quad T \geq 0.$$

The process $\{(\xi_t)_{t\geq 0}; \dot{\Pi}_\mu\}$ is exactly the spine process in the classical spine decomposition.

It will also be proved in Proposition 3.4.2 that, under Assumptions 3.1, 3.2 and 3.3, for all $\mu \in \mathcal{M}_f^\phi$ and $T > 0$, we have

$$\mathbb{N}_\mu[w_T(\phi)^2] = \langle \mu, \phi\rangle \dot{\Pi}_\mu\Big[\int_0^T (A\phi)(\xi_s)ds\Big] \in (0,\infty).$$

As a consequence, $\mathbb{N}_\mu^{w_T(\phi)^2}$ — the $w_T(\phi)^2$-transform of $\mathbb{N}_\mu$, and $\ddot{\Pi}_\mu^{(T)}$ — the $(\int_0^T (A\phi)(\xi_s)ds)$-transform of $\dot{\Pi}_\mu$, are both well defined probability measures. Recall that we always assume that Assumptions 3.1, 3.2 and 3.3 hold.

Definition 3.1.8. Let $\mu \in \mathcal{M}_f^\phi$ and $T > 0$. We say

$$\{(\xi_t)_{0\leq t\leq T}, \kappa, (\xi'_t)_{\kappa\leq t\leq T}, (Y_t)_{0\leq t\leq T}, \mathbf{n}_T, (Y'_t)_{\kappa\leq t\leq T}, \mathbf{n}'_T, (X'_t)_{\kappa\leq t\leq T}, (Z_t)_{0\leq t\leq T}; \ddot{\mathbf{P}}_\mu^{(T)}\}$$

is a *2-spine representation* of $\mathbb{N}_\mu^{w_T(\phi)^2}$ if the following are true:

1. *the main spine* $\{(\xi_t)_{0\leq t\leq T}; \ddot{\mathbf{P}}_\mu^{(T)}\}$ is a copy of $\{(\xi_t)_{0\leq t\leq T}; \ddot{\Pi}_\mu^{(T)}\}$;
2. conditioned on $(\xi_t)_{0\leq t\leq T}$, the *splitting time* κ is a random variable taking values in $[0,T]$ with law
$$\ddot{\mathbf{P}}_\mu^{(T)}\big(\kappa \in ds \big| (\xi_t)_{0\leq t\leq T}\big) = \frac{\mathbf{1}_{0\leq s\leq T}(A\phi)(\xi_s)ds}{\int_0^T (A\phi)(\xi_r)dr};$$
3. conditioned on $(\xi_t)_{t\leq T}$ and κ, the *auxiliary spine* $(\xi'_t)_{\kappa\leq t\leq T}$ is defined such that
$$\{(\xi'_{\kappa+t})_{0\leq t\leq T-\kappa}; \ddot{\mathbf{P}}_\mu^{(T)}(\cdot|\xi,\kappa)\} \overset{law}{=} \{(\xi_t)_{0\leq t\leq T-\kappa}; \dot{\Pi}_{\xi_\kappa}\}; \qquad (3.1.13)$$
4. write $\mathscr{G} := \sigma\{(\xi_t)_{t\leq T}, \kappa, (\xi'_t)_{\kappa\leq t\leq T}\}$; conditioned on $\mathscr{G}$, *the main immigration* $(Y_t)_{0\leq t\leq T}$ is given by
$$Y_t := \int_{(0,t]\times\mathbb{W}} w_{t-s}\mathbf{n}_T(ds,dw), \quad t \in [0,T],$$
where $\mathbf{n}_T$ is a Poisson random measure on $[0,T]\times\mathbb{W}$ with mean measure
$$\mathbf{m}_T^\xi(ds,dw) := 2\alpha(\xi_s)\mathbb{N}_{\xi_s}(dw)\cdot ds + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}(X\in dw)\pi(\xi_s,dy)\cdot ds;$$
5. conditioned on $\mathscr{G}$, *the auxiliary immigration* $(Y'_t)_{\kappa\leq t\leq T}$ is given by
$$Y'_t := \int_{(\kappa,t]\times\mathbb{W}} w_{t-s}\mathbf{n}'_T(ds,dw), \quad t\in[\kappa,T],$$
where $\mathbf{n}'_T$ is a Poisson random measure on $[\kappa,T]\times\mathbb{W}$ with mean measure
$$\mathbf{m}_{\kappa,T}^{\xi'}(ds,dw) := 2\alpha(\xi'_s)\mathbb{N}_{\xi'_s}(dw)\cdot ds + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi'_s}}(X\in dw)\pi(\xi'_s,dy)\cdot ds;$$

6. conditioned on $\mathscr{G}$, *the splitting-time immigration* $(X'_t)_{\kappa\le t\le T}$ is defined by

$$\{(X'_{\kappa+t})_{0\le t\le T-\kappa};\ddot{\mathbf{P}}_\mu(\cdot|\mathscr{G})\} \overset{law}{=} \{(X_t)_{0\le t\le T-\kappa};\widetilde{\mathbf{P}}_{\xi_\kappa}\},$$

   where, for each $x\in E$, the probability measure $\widetilde{\mathbf{P}}_x$ is given by

$$\widetilde{\mathbf{P}}_x(\cdot) := \begin{cases} \frac{2\alpha(x)\mathbf{P_0}(\cdot)+\int_{(0,\infty)} y^2\mathbf{P}_{y\delta_x}(\cdot)\pi(x,dy)}{2\alpha(x)+\int_{(0,\infty)} y^2\pi(x,dy)}, & \text{if } A(x)>0, \\ \mathbf{P_0}(\cdot), & \text{if } A(x)=0. \end{cases} \tag{3.1.14}$$

7. Conditioned on $\mathscr{G}$, the main immigration $\{Y,\mathbf{n}_T\}$, the auxiliary immigration $\{Y',\mathbf{n}'_T\}$ and the splitting-time immigration X' are mutually independent. Setting $Y'_t=\mathbf{0}$ and $X'_t=\mathbf{0}$ for each $t\le\kappa$, the *total immigration* $(Z_t)_{0\le t\le T}$ is given by

$$Z_t := Y_t + Y'_t + X'_t,\quad 0\le t\le T.$$

We are now ready to state our 2-spine decomposition theorem for critical superprocesses:

Theorem 3.1.9. *Suppose that Assumptions 3.1, 3.2 and 3.3 hold. Let $\mu\in\mathcal{M}_f^\phi$ and $T>0$. Suppose that $\{(\xi_t)_{0\le t\le T},\kappa,(\xi'_t)_{\kappa\le t\le T},(Y_t)_{0\le t\le T},\mathbf{n}_T,(Y'_t)_{\kappa\le t\le T},\mathbf{n}'_T,(X'_t)_{\kappa\le t\le T},(Z_t)_{0\le t\le T};\ddot{\mathbf{P}}^{(T)}_\mu\}$ is a 2-spine representation of $\mathbb{N}_\mu^{w_T(\phi)^2}$. Then $\{(Z_t)_{t\le T};\ddot{\mathbf{P}}^{(T)}_\mu\}\overset{f.d.d.}{=}\{(w_t)_{t\le T};\mathbb{N}_\mu^{w_T(\phi)^2}\}$.*

As mentioned earlier in Subsection 3.1.1, this 2-spine decomposition theorem for superprocesses is an analog of the 2-spine decomposition theorem for Galton-Watson trees in [63], and is closely related to the multi-spine theory appeared in [37], [36], [45] and [1]. Of course, depend on the choice of F, there are many versions of Theorem 3.1.2. We only consider the cases when $F(w)$ takes the forms of $w_t(g)$ and $w_t(\phi)^2$, because they are sufficient for our purpose to give probabilistic proofs of the Kolmogorov type and Yaglom type results for critical superprocesses.

We now turn our attention to the limit behavior of critical superprocesses. First, we want to consider the asymptotic behavior of $v_t(x) := -\log\mathbf{P}_{\delta_x}(X_t=\mathbf{0})$, where $t>0$ and $x\in E$. (They are well defined thanks to Assumption 3.1.) From (3.1.4) and monotone convergence, we have

$$v_t(x) = \lim_{\theta\to\infty} V_t(\theta\mathbf{1}_E)(x),\quad t>0, x\in E, \tag{3.1.15}$$

and

$$\mathbf{P}_\mu(X_t=\mathbf{0}) = e^{-\mu(v_t)},\quad \mu\in\mathcal{M}_f, t\ge 0, \tag{3.1.16}$$

where the operators $(V_t)_{t\ge0}$ are given by (3.1.4). We call $(V_t)_{t\ge0}$ the *cumulant semigroup* of the superprocess X, because it satisfies the semigroup property in the sense that, for all $f\in p\mathscr{B}_E, t,s\ge 0$ and $x\in E$, it holds that $V_tV_sf(x)=V_{t+s}f(x)$ (see [56, Theorem 2.21]).

Let ψ_0 be a function on $E \times [0, \infty)$ defined by

$$\psi_0(x, z) := \psi(x, z) + \beta(x)z = \alpha(x)z^2 + \int_{(0,\infty)} (e^{-rz} - 1 + rz)\pi(x, dr), \quad x \in E, z \geq 0.$$

Let Ψ_0 be an operator on $p\mathscr{B}_E$ defined by

$$(\Psi_0 f)(x) := \psi_0(x, f(x)), \quad f \in p\mathscr{B}_E, x \in E.$$

It is known, see [56, Theorem 2.23] for example, that for each $f \in bp\mathscr{B}_E$, $(t, x) \mapsto V_t f(x)$ is the solution of the equation

$$V_t f(x) + \int_0^t (S_{t-s}\Psi_0 V_s f)(x)ds = S_t f(x), \quad t \geq 0, x \in E. \tag{3.1.17}$$

Indeed, (3.1.17) can be obtained from (3.1.5) using a Feynman−Kac type argument. It is also clear that

$$\begin{aligned} V_t v_s(x) &= -\log \mathbf{P}_{\delta_x}[e^{-\langle X_t, \lim_{\theta\to\infty} V_s(\theta \mathbf{1}_E)\rangle}] = -\lim_{\theta\to\infty} \log \mathbf{P}_{\delta_x}[e^{-\langle X_t, V_s(\theta \mathbf{1}_E)\rangle}] \qquad (3.1.18)\\ &= -\lim_{\theta\to\infty} V_t V_s(\theta \mathbf{1}_E)(x) = v_{t+s}(x), \quad s, t > 0, x \in E. \end{aligned}$$

So, if we allow extended values, it follows from (3.1.17) and (3.1.18) that we have the following equation for $(v_t)_{t\geq 0}$:

$$v_{t+s}(x) + \int_0^t (S_{t-r}\Psi_0 v_{r+s})(x)dr = S_t v_s(x), \quad x \in E, t \geq 0. \tag{3.1.19}$$

In order to study the asymptotic behavior of $(v_t)_{t\geq 0}$ using (3.1.19), we need to understand the asymptotic behavior of the mean semigroup $(S_t)_{t\geq 0}$. The following assumption is commonly used for this purpose:

Assumption 3.4. In addition to Assumption 3.2, we further assume that the mean semigroup $(S_t)_{t\geq 0}$ is *intrinsically ultracontractive*, that is, for each $t > 0$ there exists $c_t > 0$ such that for all $x, y \in E$, we have $q(t, x, y) \leq c_t \phi(x)\phi^*(y)$.

The concept of intrinsic ultracontractivity was first introduced by Davies and Simon [15] in the symmetric setting and was extended to the non-symmetric setting in [47]. Assumption 3.4 is a pretty strong condition on the mean semigroup $(S_t)_{t\geq 0}$. For instance, it excludes the case of super Brownian motions in the whole space. However, it is satisfied in a lot of cases. For a long list of (symmetric and non-symmetric) Markov processes satisfying Assumption 3.4, see [68].

A consequence of this assumption is that (see [47, Theorem 2.7]) there exist constants

$c > 0$ and $\gamma > 0$ such that

$$\Big|\frac{q(t,x,y)}{\phi(x)\phi^*(y)} - 1\Big| \le ce^{-\gamma t}, \quad x \in E, t > 1. \tag{3.1.20}$$

We will see in Subsection 3.3.2 that, under Assumption 3.2, the spine process $\{(\xi_t)_{t\ge 0}; (\dot{\Pi}_x)_{x\in E}\}$ in the classical spine decomposition is a time homogeneous Markov process with invariant measure $\phi(x)\phi^*(x)m(dx)$. It can be verified that its transition density with respect to measure $\phi(x)\phi^*(x)m(dx)$ is $\frac{q(t,x,y)}{\phi(x)\phi^*(y)}$. Therefore Assumption 3.4 implies that the spine process in classical spine decomposition is exponentially ergodic.

Define $\nu(dy) := \phi^*(y)m(dy)$. Under Assumption 3.4, $\nu(dy)$ is a finite measure on E. In fact, according to (3.1.20), for $t > 0$ large enough, there is a $c'_t > 0$ such that $\phi^*(y) \le q(t,x,y)(c'_t)^{-1}\phi^{-1}(x)$, and clearly, the right hand of this inequality is integrable in y with respect to measure m. Therefore, we can consider a superprocess X with initial configuration ν. Under Assumptions 3.1 and 3.4, it will be proved in Lemma 3.5.2 that the following statements are equivalent:

- $S_t v_s(x) < \infty$ for some $s > 0, t > 0$ and some $x \in E$;
- $\mathbf{P}_\nu(X_t = \mathbf{0}) > 0$ for some $t > 0$.

Note that, in order to take advantage of (3.1.19), we need $S_t v_s(x)$ to be finite at least for some large $s, t > 0$ and some $x \in E$. Therefore, we also need the following assumption:

Assumption 3.5. In addition to Assumption 3.1, we further assume that $\mathbf{P}_\nu(X_t = \mathbf{0}) > 0$ for some $t > 0$.

We are now ready to state our Kolmogorov type and Yaglom type limit results for superprocesses:

Theorem 3.1.10. *Suppose that Assumptions 3.5, 3.4 and 3.3 hold. Then,*

$$t\mathbf{P}_\mu(X_t \neq \mathbf{0}) \xrightarrow[t\to\infty]{} \frac{\langle \mu, \phi\rangle}{\frac{1}{2}\langle A\phi, \phi\phi^*\rangle_m}, \quad \mu \in \mathcal{M}_f^\phi,$$

where m is the reference measure appeared in Assumption 3.2.

Theorem 3.1.11. *Suppose that Assumptions 3.5, 3.4 and 3.3 hold. Let $f \in bp\mathscr{B}_E^\phi$ and $\mu \in \mathcal{M}_f^\phi$. Then,*

$$\left\{t^{-1}X_t(f); \mathbf{P}_\mu(\cdot|X_t \neq \mathbf{0})\right\} \xrightarrow[t\to\infty]{law} \frac{1}{2}\langle \phi^*, f\rangle_m \langle \phi A, \phi\phi^*\rangle_m \mathbf{e},$$

where $\mathbf{e}$ is an exponential random variable with mean 1, and m is the reference measure in Assumption 3.2.

As mentioned earlier, our Kolmogorov type and Yaglom type results for critical superprocesses are established under slightly weaker conditions than [68]. We now make this more precise. In [68], the authors considered a (ξ,ψ)-superprocess $\{(X_t)_{t\geq 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ which also satisfies Assumption 3.1, 3.2 and 3.3.(1) as the basic setting. In addition to that, [68] assumed the following

(a) the transition semigroup (P_t) of the spatial motion is intrinsically ultracontractive,

(b) the principal eigenfunction of (P_t) is bounded,

(c) the function A is bounded, and

(d) there exists $t_0 > 0$ such that $\inf_{x\in E}\mathbf{P}_{\delta_x}(X_{t_0}=\mathbf{0}) > 0$.

It is shown in [68] that, under conditions (a) and (b), the mean semigroup (S_t) is also intrinsically ultracontractive, and the principal eigenfunction ϕ of (S_t) is also bounded. Therefore, conditions (a), (b) and (c) combined together are stronger than our Assumption 3.5 and 3.3. Condition (d) is stronger than our Assumption 3.4 because according to (3.1.16), we always have the following:

$$\mathbf{P}_\nu(X_t=\mathbf{0}) = \exp\{-\langle v_t,\nu\rangle\} = \exp\{\langle \log \mathbf{P}_{\delta_\cdot}(X_t=\mathbf{0}),\nu\rangle\},\quad t>0.$$

3.2 Size-biased decomposition

3.2.1 Size-biased transform of Poisson random measures

In this subsection, we digress briefly from superprocesses and prove the size-biased decomposition theorem for Poisson random measures, i.e., Theorem 3.1.3. Let $(S,\mathscr{S})$ be a measurable space with a σ-finite measure N. Let $\{\mathbf{N};P\}$ be a Poisson random measure on $(S,\mathscr{S})$ with mean measure N. Campbell's theorem, see [49, Proof of Theorem 2.7] for example, characterizes the law of $\{\mathbf{N};P\}$ by its Laplace functionals:

$$P[e^{-\mathbf{N}(g)}] = e^{-N(1-e^{-g})},\quad g\in p\mathscr{S}.$$

According to [49, Theorem 2.7], we also have that $P[\mathbf{N}(g)] = N(g)$ for each $g\in\mathscr{S}$ with $N(|g|)<\infty$. By monotonicity, one can verify that

$$P[\mathbf{N}(g)] = N(g),\quad g\in p\mathscr{S}.$$

Lemma 3.2.1. *If $g\in L^1(N)$ and $f\in p\mathscr{S}$, then $\mathbf{N}(g)e^{-\mathbf{N}(f)}$ is integrable and*

$$P[\mathbf{N}(g)e^{-\mathbf{N}(f)}] = P[e^{-\mathbf{N}(f)}]N[ge^{-f}]. \tag{3.2.1}$$

Furthermore, (3.2.1) *is true for each $g,f\in p\mathscr{S}$ if we allow extended values.*

Proof. Since N is a σ-finite measure on $(S, \mathscr{S})$, there exists a strictly positive measurable function h on S such that $N(h) < \infty$. According to [49, Theorem 2.7.], $\mathbf{N}(h)$ has finite mean. For any $g \in bp\mathscr{S}^h := \{g \in p\mathscr{S} : \|h^{-1}g\|_\infty < \infty\}$ and $f \in p\mathscr{S}$, it is clear that $\mathbf{N}(g)$ and $\mathbf{N}(g)e^{-\mathbf{N}(f)}$ are integrable. Therefore, by the dominated convergence theorem, we deduce that

$$
\begin{aligned}
P[\mathbf{N}(g)e^{-\mathbf{N}(f)}] &= P[-\partial_\theta|_{\theta=0} e^{-\mathbf{N}(f+\theta g)}] = -\partial_\theta|_{\theta=0} P[e^{-\mathbf{N}(f+\theta g)}] \\
&= -\partial_\theta|_{\theta=0} e^{-N(1-e^{-(f+\theta g)})} = e^{-N(1-e^{-f})} \partial_\theta|_{\theta=0} N(1 - e^{-(f+\theta g)}) \\
&= P[e^{-\mathbf{N}(f)}] N[g e^{-f}].
\end{aligned}
$$

For any $g \in p\mathscr{S}$ and $s \in S$, define $g^{(n)}(s) := h(s)\min\{h(s)^{-1}g(s), n\}$. Then $(g^{(n)})_{n\in\mathbb{N}}$ is a $bp\mathscr{S}^h$-sequence which increasingly converges to g pointwise. Note that (3.2.1) is true for each $g^{(n)}$ and f. Letting $n \to \infty$, by monotonicity, we see that if we allow extended values, then (3.2.1) is true for each $g, f \in p\mathscr{S}$. In the case when $g \in L^1(N)$, we simply consider its positive and negative parts. □

Proof of Theorem 3.1.3. By Lemma 3.2.1, it is easy to see that, for any $f \in p\mathscr{S}$,

$$
\begin{aligned}
P^{\mathbf{N}(g)}[e^{-\mathbf{N}(f)}] &= N(g)^{-1} P[\mathbf{N}(g) e^{-\mathbf{N}(f)}] = N(g)^{-1} P[e^{-\mathbf{N}(f)}] N[g e^{-f}] \\
&= P[e^{-\mathbf{N}(f)}] N^g[e^{-f}] = (P \otimes Q)[e^{-\mathbf{N}(f) - f(\vartheta)}] = (P \otimes Q)[e^{-(\mathbf{N}+\delta_\vartheta)(f)}],
\end{aligned}
$$

which completes the proof. □

Lemma 3.2.2. *For all $g, f \in L^1(N) \cap L^2(N)$, $\mathbf{N}(g)\mathbf{N}(f)$ is integrable and*

$$
P[\mathbf{N}(g)\mathbf{N}(f)] = N(g)N(f) + N(gf). \tag{3.2.2}
$$

Furthermore, (3.2.2) *is true for all $g, f \in p\mathscr{S}$ if we allow extended values.*

Proof. Since N is a σ-finite measure on $(S, \mathscr{S})$, there exists a strictly positive measurable function $\tilde{h}$ on S such that $N(\tilde{h}) < \infty$. Define $h(s) := \min\{\tilde{h}(s), \tilde{h}(s)^{1/2}\}$ for each $s \in S$. It is clear that h is a strictly positive measurable function on S such that $N(h) < \infty$ and $N(h^2) < \infty$. According to [49, Theorem 2.7], $\mathbf{N}(h)$ has finite 1st and 2nd moments. For any $g, f \in bp\mathscr{S}^h := \{g \in p\mathscr{S} : \|h^{-1}g\|_\infty < \infty\}$, it is easy to see that $\mathbf{N}(g), \mathbf{N}(f), \mathbf{N}(f)\mathbf{N}(g)$ are integrable. Thus, using Lemma 3.2.1 and the dominated convergence theorem, we have

$$
\begin{aligned}
P[\mathbf{N}(g)\mathbf{N}(f)] &= -P[\partial_\theta|_{\theta=0} \mathbf{N}(g) e^{-\mathbf{N}(\theta f)}] = -\partial_\theta|_{\theta=0} P[\mathbf{N}(g) e^{-\mathbf{N}(\theta f)}] \\
&= -\partial_\theta|_{\theta=0} P[e^{-\mathbf{N}(\theta f)}] N(g e^{-\theta f}) \\
&= -N[g] \partial_\theta|_{\theta=0} P[e^{-\mathbf{N}(\theta f)}] - \partial_\theta|_{\theta=0} N(g e^{-\theta f}) \\
&= -N(g) P[\partial_\theta|_{\theta=0} e^{-\mathbf{N}(\theta f)}] - N(\partial_\theta|_{\theta=0} g e^{-\theta f})
\end{aligned}
$$

$$= N(g)N(f) + N(gf).$$

For any $g, f \in p\mathscr{S}$ and $s \in S$, define $g^{(n)}(s) := h(s)\min\{h(s)^{-1}g(s), n\}$. Then $(g^{(n)})_{n\in\mathbb{N}}$ is a $bp\mathscr{S}^h$-sequence which increasingly converges to g pointwise. Define $f^{(n)}$ similarly. Then from what we have proved, (3.2.2) is true for $g^{(n)}$ and $f^{(n)}$. Letting $n \to \infty$, by monotonicity, (3.2.2) is true for each $g, f \in p\mathscr{S}$ if we allow extended values. In the case when $g, f \in L^1(N) \cap L^2(N)$ we simply consider their positive and negative parts. □

3.2.2 Size-biased transform of the superprocesses

Let $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ, ψ)-superprocess introduced in Subsection 3.1.2 which satisfies Assumption 3.1. In this subsection, we will give a proof of Theorem 3.1.2. Recall that, for any $\mu \in \mathcal{M}_f$, $\{\mathcal{N}; \mathbf{P}_\mu\}$ is a Poisson random measure with mean measure $\mathbb{N}_\mu$, and our (ξ, ψ)-superprocess $(X_t)_{t\geq 0}$ is given by

$$X_0 := \mu; \quad X_t(\cdot) := \mathcal{N}[w_t(\cdot)], \quad t > 0.$$

For any $T > 0$, we write $(K, f) \in \mathcal{K}_T$ if $f : (s, x) \mapsto f_s(x)$ is a bounded non-negative Borel function on $(0, T] \times E$ and K is an atomic measure on $(0, T]$ with finitely many atoms. For any $(K, f) \in \mathcal{K}_T$ and any $\mathcal{M}_f$-valued process $(Y_t)_{t>0}$, we define the random variable

$$K^f_{(s,T]}(Y) := \int_{(s,T]} Y_{r-s}(f_r)K(dr), \quad s \in [0, T].$$

It is clear that the two $\mathcal{M}_f$-valued processes $(Y_t)_{t>0}$ and $(X_t)_{t>0}$ have same finite-dimensional distributions if and only if

$$\mathbf{E}[e^{-K^f_{(0,T]}(X)}] = \mathbf{E}[e^{-K^f_{(0,T]}(Y)}], \quad (K, f) \in \mathcal{K}_T, T > 0.$$

Proof of Theorem 3.1.2. Since $\mathbb{N}_\mu(F) \in (0, \infty)$, it follows from Campbell's formula that $\mathbf{P}_\mu[\mathcal{N}(F)] = \mathbb{N}_\mu(F) \in (0, \infty)$. Therefore, $\mathbf{P}^{\mathcal{N}(F)}_\mu$ – the $\mathcal{N}(F)$-transform of Π_μ, and $\mathbb{N}^F_\mu$ — the F-transform of $\mathbb{N}_\mu$, are both well defined probability measures. Notice that, under $\mathbf{P}^{\mathcal{N}(F)}_\mu$, $X_0 \overset{\text{a.s.}}{=} \mu$ is deterministic, and so is $X_0 + Y_0$ under $\mathbf{P}_\mu \otimes \mathbf{Q}_\mu$ since $X_0 + Y_0 \overset{\text{a.s.}}{=} \mu$. Therefore, we only have to show that,

$$\{(X_t)_{t>0}; \mathbf{P}^{\mathcal{N}(F)}_\mu\} \overset{f.d.d.}{=} \{(X_t + Y_t)_{t>0}; \mathbf{P}_\mu \otimes \mathbf{Q}_\mu\}.$$

It then immediately follows from Theorem 3.1.3 that

$$\{\mathcal{N}; \mathbf{P}^{\mathcal{N}(F)}_\mu\} \overset{law}{=} \{\mathcal{N} + \delta_Y; \mathbf{P}_\mu \otimes \mathbf{Q}_\mu\}.$$

This completes the proof since for any $T > 0$ and $(K, f) \in \mathcal{K}_T$,

$$\mathbf{P}_\mu^{\mathcal{N}(F)}[e^{-K^f_{(0,T]}(X)}] = \mathbf{P}_\mu^{\mathcal{N}(F)}[e^{-\mathcal{N}[K^f_{(0,T]}(w)]}] = (\mathbf{P}_\mu \otimes \mathbf{Q}_\mu)[e^{-(\mathcal{N}+\delta_Y)[K^f_{(0,T]}(w)]}]$$
$$= (\mathbf{P}_\mu \otimes \mathbf{Q}_\mu)[e^{-K^f_{(0,T]}(X+Y)}]. \qquad \square$$

3.3 Spine decomposition of superprocesses

The classical spine decomposition theorem characterizes the superprocess X after a martingale change of measure, and has been investigated in the literature in different situations, see [25, 28, 57] for example. The martingale that is used for the change of measure is defined by $M_t := e^{-\lambda t} X_t(\phi)$, where ϕ is the principal eigenfunction of the generator of the mean semigroup of X with λ being the corresponding eigenvalue. After this martingale change of measure, the transformed process preserves the Markov property, and thus, to prove the spine decomposition theorem, one only needs to focus on the one-dimensional distribution of the transformed process.

In this section, we generalize this classical result by considering the $X_T(g)$-transform of the superprocess X, where g is a non-negative Borel function on E. If g is not equal to ϕ, the $X_T(g)$-transformed process is typically not a Markov process. So we have to use a different method to develop the theorem. Thanks to Theorem 3.1.2, we only have to consider the $w_T(g)$-transform of the Kuznetsov measures.

3.3.1 Spine decomposition theorem

Let $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ, ψ)-superprocess introduced in Subsection 3.1.2 which satisfies Assumption 3.1. In this subsection, we will give a proof of Theorem 3.1.5. Recall that $(\mathbb{N}_x)_{x\in E}$ are the Kuznetsov measures defined in Lemma 3.1.1. We now recall a result from [56] which is useful for calculations related to $(\mathbb{N}_x)_{x\in E}$.

Lemma 3.3.1 ([56, Theorems 5.15 and 8.23])**.** *Under Assumption 3.1, for all $T > 0$ and $(K, f) \in \mathcal{K}_T$, we have*

$$\mathbb{N}_\mu\big[1 - e^{-K^f_{(s,T]}(w)}\big] = \mu(u_s) = -\log \mathbf{P}_\mu\big[e^{-K^f_{(s,T]}(X)}\big], \quad s \in [0, T], \mu \in \mathcal{M}_f,$$

where the function $u : (s, x) \mapsto u_s(x)$ on $[0, T] \times E$ is the unique bounded positive solution to the following integral equation:

$$u_s(x) = \Pi_x\Big[\int_{(s,T]} f_r(\xi_{r-s}) K(dr) - \int_s^T (\Psi u_r)(\xi_{r-s}) dr\Big], \quad s \in [0, T], x \in E.$$

We now prove the following lemmas:

Lemma 3.3.2. *For all $x \in E, T > 0, (K, f) \in \mathcal{K}_T$ and $g \in p\mathscr{B}_E$, we have*

$$\mathbb{N}_x[w_T(g)e^{-K^f_{(0,T]}(w)}] = \Pi_x[g(\xi_T)e^{-\int_0^T \psi'(\xi_s, u_s(\xi_s))ds}], \tag{3.3.1}$$

where

$$\psi'(x,z) := \partial_z \psi(x,z) = -\beta(x) + 2\alpha(x)z + \int_{(0,\infty)} (1-e^{-yz})y\pi(x,dy), \quad x \in E, z \geq 0,$$

and $u : (s,x) \mapsto u_s(x)$ on $[0,T] \times E$ is defined in Lemma 3.3.1.

Proof. We first prove assertion (3.3.1) in the case when $g \in bp\mathscr{B}_E$. Throughout this proof, we fix $(K, f) \in \mathcal{K}_T$ and consider $0 \leq \theta \leq 1$. Define

$$u^\theta_s(x) := \mathbb{N}_x\big[1 - e^{-K^f_{(s,T]}(w) - w_{T-s}(\theta g)}\big], \quad s \geq 0, x \in E. \tag{3.3.2}$$

Let

$$\tilde{K}(dr) := \mathbf{1}_{0\leq r<T}K(dr) + \delta_T(dr),$$
$$\tilde{f}_r := \mathbf{1}_{0\leq r<T} f_r + \mathbf{1}_{r=T}\big(K(\{T\})f_T + \theta g\big).$$

Then $(\tilde{K}, \tilde{f}) \in \mathcal{K}_T$ and (3.3.2) can be rewritten as

$$u^\theta_s(x) := \mathbb{N}_x\big[1 - e^{-\tilde{K}^{\tilde{f}}_{(s,T]}(w)}\big], \quad s \geq 0, x \in E.$$

It follows from Lemma 3.3.1 that, for any $\theta \geq 0$, $(s,x) \mapsto u^\theta_s(x)$ is the unique bounded positive solution to the equation

$$u^\theta_s(x) = \Pi_x\Big[\int_{(s,T]} \tilde{f}_r(\xi_{r-s})\tilde{K}(dr) - \int_s^T (\Psi u^\theta_r)(\xi_{r-s})dr\Big], \quad s \in [0,T], x \in E,$$

which is equivalent to

$$u^\theta_s(x) = \Pi_x\Big[\int_{(s,T]} f_r(\xi_{r-s})K(dr) + \theta g(\xi_{T-s}) - \int_s^T (\Psi u^\theta_r)(\xi_{r-s})dr\Big]. \tag{3.3.3}$$

We claim that $u^\theta_s(x)$ is differentiable in θ at $\theta = 0$. In fact, since

$$\frac{|e^{-K^f_{(s,T]}(w) - w_{T-s}(\theta g)} - e^{-K^f_{(s,T]}(w)}|}{\theta} \leq w_{T-s}(g), \quad 0 < \theta \leq 1, \tag{3.3.4}$$

and

$$\mathbb{N}_x[w_{T-s}(g)] = S_{T-s}g(x) = \Pi_x[e^{\int_0^{T-s}\beta(\xi_r)dr} g(\xi_{T-s})] \leq e^{T\|\beta\|_\infty}\|g\|_\infty, \tag{3.3.5}$$

it follows from (3.3.2) and the dominated convergence theorem that

$$\dot{u}_s(x) := \partial_\theta|_{\theta=0} u^\theta_s(x) = \mathbb{N}_x[w_{T-s}(g)e^{-K^f_{(s,T]}(w)}] \leq e^{T\|\beta\|_\infty}\|g\|_\infty. \tag{3.3.6}$$

From (3.3.2), we also have the following upper bound for $u^\theta_s(x)$ with $0 \leq \theta \leq 1$:

$$\begin{aligned} u^\theta_s(x) &\leq \mathbb{N}_x\Big[\int_{(s,T]} w_{r-s}(f_r)K(dr) + w_{T-s}(\theta g)\Big] \\ &= \int_{(s,T]} \mathbb{N}_x[w_{r-s}(f_r)]K(dr) + \mathbb{N}_x[w_{T-s}(\theta g)] \\ &\leq e^{T\|\beta\|_\infty}\big(\|f\|_\infty K((0,T]) + \|g\|_\infty\big) =: L_0. \end{aligned} \tag{3.3.7}$$

By elementary analysis, one can verify that, for each $L > 0$, there exists a constant $C_{\psi,L} > 0$ such that for each $x \in E$ and $0 \leq z, z_0 \leq L$,

$$|\psi(x,z_0) - \psi(x,z)| \leq C_{\psi,L}|z - z_0|. \tag{3.3.8}$$

In fact, one can choose $C_{\psi,L} := \|\beta\|_\infty + 2L\|\alpha\|_\infty + \max\{L,1\}\sup_{x\in E}\int_{(0,\infty)}(y \wedge y^2)\pi(x,dy)$. This upper bound also implies that

$$|\psi'(x,z)| \leq C_{\psi,L}, \quad x \in E, 0 \leq z \leq L.$$

Therefore, we can verify that $\Pi_x[\int_s^T (\Psi u^\theta_r)(\xi_{r-s})dr]$ is differentiable in θ at $\theta = 0$. In fact, by (3.3.8), (3.3.7), (3.3.2), (3.3.4) and (3.3.5), we have

$$\begin{aligned} \frac{|(\Psi u^\theta_r)(x) - (\Psi u^0_r)(x)|}{\theta} &\leq C_{\psi,L_0}\frac{|u^\theta_r(x) - u^0_r(x)|}{\theta} \\ &\leq C_{\psi,L_0}\cdot e^{T\|\beta\|_\infty}\|g\|_\infty, \quad 0 \leq \theta \leq 1. \end{aligned}$$

Therefore, by the bounded convergence theorem, we have

$$\partial_\theta|_{\theta=0}\Pi_x\Big[\int_s^T (\Psi u^\theta_r)(\xi_{r-s})dr\Big] = \Pi_x\Big[\int_s^T \psi'\big(\xi_{r-s}, u^0_r(\xi_{r-s})\big)\dot{u}_r(\xi_{r-s})\,dr\Big]. \tag{3.3.9}$$

Now, taking $\partial_\theta|_{\theta=0}$ on the both sides of (3.3.3), we obtain from (3.3.9) that

$$\dot{u}_s(x) = \Pi_x\Big[g(\xi_{T-s}) - \int_s^T \psi'\big(\xi_{r-s}, u^0_r(\xi_{r-s})\big)\dot{u}_r(\xi_{r-s})\,dr\Big], \quad s \in [0,T], x \in E. \tag{3.3.10}$$

Notice that the function $\dot{u} : (s,x) \mapsto \dot{u}_s(x)$ is bounded on $[0,T] \times E$ by $e^{T\|\beta\|_\infty}\|g\|_\infty$; g is bounded on E by $\|g\|_\infty$; and $\psi'(x, u^0_r(x))$ is bounded on E by C_{ψ,L_0}. These bounds allow us to apply the classical Feynman-Kac formula, see [23, Lemma A.1.5] for example, to equation (3.3.10) and get that

$$\dot{u}_0(x) = \Pi_x[g(\xi_T)e^{-\int_0^T \psi'(\xi_s, u_s(\xi_s))ds}]. \tag{3.3.11}$$

The desired result when $g \in bp\mathscr{B}_E$ then follows from (3.3.6) and (3.3.11).

In the case when $g \in p\mathscr{B}_E$, we write $g^{(n)}(x) := \min\{g(x), n\}$ for $x \in E$ and $n \in \mathbb{N}$. Then, from what we have proved, we know that

$$\mathbb{N}_x[w_T(g^{(n)})e^{-K^f_{(0,T]}(w)}] = \Pi_x[g^{(n)}(\xi_T)e^{-\int_0^T \psi'(\xi_s,u_s(\xi_s))ds}], \quad n \in \mathbb{N}.$$

Letting $n \to \infty$ we complete the proof. □

Lemma 3.3.3. *Let $T > 0, k \in [0,T]$ and $(K,f) \in \mathcal{K}_T$. Let $\mu \in \mathcal{M}_f$ and $g \in p\mathscr{B}_E$ satisfy that $\mu(S_T g) \in (0,\infty)$. Suppose that $\{(\xi_t)_{0\le t\le T}, (Y_t)_{0\le t\le T}, \mathbf{n}_T; \dot{\mathbf{P}}^{(g,T)}_\mu\}$ is a spine representation of $\mathbb{N}^{w_T(g)}_\mu$. Then, we have*

$$-\log \dot{\mathbf{P}}^{(g,T)}_\mu[e^{-K^f_{(k,T]}(Y)}|\xi] = \int_k^T \psi_0'(\xi_{s-k}, u_s(\xi_{s-k}))ds, \tag{3.3.12}$$

where the function u is defined in Lemma 3.3.1.

Proof. Throughout this proof, we denote by $\mathbf{n}_{T-k}$ and $\mathbf{m}^\xi_{T-k}$ the restriction of $\mathbf{n}_T$ and $\mathbf{m}^\xi_T$ on $[0,T-k]\times\mathbb{W}$ respectively. It follows from properties of Poisson random measures that, conditioned on ξ, $\mathbf{n}_{T-k}$ is a Poisson random measure with mean measure $\mathbf{m}^\xi_{T-k}$.

It follows from (3.1.7) and Fubini's theorem that

$$\begin{aligned}
K^f_{(k,T]}(Y) &= \int_{(k,T]} Y_{r-k}(f_r)K(dr) \\
&= \int_{(k,T]} K(dr)\int_{(0,r-k]\times\mathcal{M}_f} w_{(r-k)-s}(f_r)\mathbf{n}_T(ds,dw) \\
&= \int_{(0,T-k]\times\mathcal{M}_f} \mathbf{n}_T(ds,dw)\int_{(k+s,T]} w_{r-(k+s)}(f_r)K(dr) \\
&= \int K^f_{(k+s,T]}(w)\mathbf{n}_{T-k}(ds,dw).
\end{aligned} \tag{3.3.13}$$

Conditioned on ξ, it follows from Campbell's formula and Lemma 3.3.1 that

$$\begin{aligned}
&-\log \dot{\mathbf{P}}^{(g,T)}_\mu[e^{-K^f_{(k,T]}(Y)}|\xi] = -\log \dot{\mathbf{P}}^{(g,T)}_\mu\left[e^{-\int K^f_{(k+s,T]}(w)\mathbf{n}_{T-k}(ds,dw)}\Big|\xi\right] \\
&= \int (1-e^{-K^f_{(k+s,T]}(w)})\mathbf{m}^\xi_{T-k}(ds,dw) \\
&= \int_0^{T-k}\Big(2\alpha(\xi_s)\mathbb{N}_{\xi_s}[1-e^{-K^f_{(k+s,T]}(w)}] \\
&\qquad + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}[1-e^{-K^f_{(k+s,T]}(X)}]\pi(\xi_s,dy)\Big)ds \\
&= \int_0^{T-k}\Big(2\alpha(\xi_s)u_{k+s}(\xi_s) + \int_{(0,\infty)}(1-e^{-yu_{k+s}(\xi_s)})y\pi(\xi_s,dy)\Big)ds \\
&= \int_0^{T-k}\psi_0'\big(\xi_s, u_{s+k}(\xi_s)\big)ds = \int_k^T \psi_0'\big(\xi_{s-k}, u_s(\xi_{s-k})\big)ds,
\end{aligned}$$

as desired. □

Proof of Theorem 3.1.5. We only need to prove that

$$\{(Y_t)_{0<t\le T};\dot{\mathbf{P}}_\mu^{(g,T)}\} \overset{f.d.d.}{=} \{(w_t)_{0<t\le T};\mathbb{N}_\mu^{w_T(g)}\},$$

since both $\{Y_0;\dot{\mathbf{P}}_\mu^{(g,T)}\}$ and $\{w_0;\mathbb{N}_\mu^{w_T(g)}\}$ are deterministic with common value $\mathbf{0}$. By Lemma 3.3.2 and 3.3.3, we have

$$\begin{aligned}
\mathbb{N}_\mu^{w_T(g)}\big[e^{-K^f_{(0,T]}(w)}\big] &= \mathbb{N}_\mu[w_T(g)]^{-1}\mathbb{N}_\mu\big[w_T(g)e^{-K^f_{(0,T]}(w)}\big]\\
&= \mu(S_T g)^{-1}\Pi_\mu\big[g(\xi_T)e^{-\int_0^T \psi'(\xi_s,u_s(\xi_s))ds}\big]\\
&= \Pi_\mu^{(g,T)}[e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))ds}] = \dot{\mathbf{P}}_\mu^{(g,T)}\big[\dot{\mathbf{P}}_\mu^{(g,T)}[e^{-K^f_{(0,T]}(Y)}|\xi]\big]\\
&= \dot{\mathbf{P}}_\mu^{(g,T)}[e^{-K^f_{(0,T]}(Y)}].
\end{aligned}$$

The proof is complete. □

3.3.2 Classical spine decomposition theorem

Let $X=\{(X_t)_{t\ge 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ,ψ)-superprocess introduced in Subsection 3.1.2 which satisfies Assumptions 3.1 and 3.2. In this subsection, we will recover the classical spine decomposition theorem for X which is developed previously in [25, 28, 57].

It is clear that $\{(e^{-\lambda t}\phi(\xi_t)e^{\int_0^t\beta(\xi_s)ds}\mathbf{1}_{t<\zeta})_{t\ge 0};(\Pi_x)_{x\in E}\}$ is a non-negative martingale. Denote by $\{(\xi_t)_{t\ge 0};(\dot{\Pi}_x)_{x\in E}\}$ the martingale transform (also known as Doob's h-transform) of $\{(\xi_t)_{t\ge 0};(\Pi_x)_{x\in E}\}$ via this martingale in the sense that

$$\frac{d\dot{\Pi}_x|_{\mathscr{F}^\xi_t}}{d\Pi_x|_{\mathscr{F}^\xi_t}} := e^{-\lambda t}\frac{\phi(\xi_t)}{\phi(x)}e^{\int_0^t\beta(\xi_s)ds}\mathbf{1}_{t<\zeta},\quad x\in E, t\ge 0,$$

where $(\mathscr{F}^\xi_t)_{t\ge 0}$ is the natural filtration of the spatial motion ξ. It can be shown that (see [47] for example) $\{(\xi_t)_{t\ge 0};(\dot{\Pi}_x)_{x\in E}\}$ is a time homogeneous Markov process. Its semigroup is Doob's h-transform of $(S_t)_{t\ge 0}$ with $h=\phi$ and its transition density with respect to the measure m is

$$\dot{q}(t,x,y) := e^{-\lambda t}\frac{\phi(y)}{\phi(x)}q(t,x,y),\quad x,y\in E, t>0.$$

It can also be verified that $\phi(x)\phi^*(x)m(dx)$ is an invariant measure for $\{(\xi_t)_{t\ge 0};(\dot{\Pi}_x)_{x\in E}\}$.

Recall that, for each $T>0$, $\Pi_\mu^{(\phi,T)}$ is defined as the $(e^{\int_0^T\beta(\xi_s)ds}\phi(\xi_T)\mathbf{1}_{\zeta<T})$-transform of the measure $\mu\Pi(\cdot) := \int_E \Pi_x(\cdot)\mu(dx)$.

Lemma 3.3.4. *Let $\mu\in\mathcal{M}_f^\phi$. Define a probability measure $\dot{\Pi}_\mu(\cdot) := \mu(\phi)^{-1}\int_E\phi(x)\dot{\Pi}_x(\cdot)\mu(dx)$. Then, for each $T>0$, we have $\{(\xi_t)_{0\le t\le T};\Pi_\mu^{(\phi,T)}\} \overset{law}{=} \{(\xi_t)_{0\le t\le T};\dot{\Pi}_\mu\}$.*

Proof. Let $A \in \mathscr{F}_T^\xi$. Then we have

$$\begin{aligned}\Pi_\mu^{(\phi,T)}(A) &= \frac{(\mu\Pi)[\mathbf{1}_A e^{\int_0^T \beta(\xi_s)ds}\phi(\xi_T)\mathbf{1}_{T<\zeta}]}{(\mu\Pi)[e^{\int_0^T \beta(\xi_s)ds}\phi(\xi_T)\mathbf{1}_{T<\zeta}]} \\ &= \mu(\phi)^{-1}(\mu\Pi)[\mathbf{1}_A e^{-\lambda T} e^{\int_0^T \beta(\xi_s)ds}\phi(\xi_T)\mathbf{1}_{T<\zeta}] \\ &= \mu(\phi)^{-1}\int_E \Pi_x[\mathbf{1}_A e^{-\lambda T} e^{\int_0^T \beta(\xi_s)ds}\phi(\xi_T)\mathbf{1}_{T<\zeta}]\,\mu(dx) \\ &= \mu(\phi)^{-1}\int_E \phi(x)\dot\Pi_x(A)\,\mu(dx) = \dot\Pi_\mu(A).\end{aligned}$$

□

Fix a measure $\mu \in \mathcal{M}_f^\phi$. Define $M_t := e^{-\lambda t}X_t(\phi)$ for each $t \geq 0$. It is clear that $\{(M_t)_{t\geq 0}; \mathbf{P}_\mu\}$ is a non-negative martingale. Let $\{(X_t)_{t\geq 0}; \mathbf{P}_\mu^M\}$ be the martingale transform of $\{(X_t)_{t\geq 0}; \mathbf{P}_\mu\}$ via this martingale in the sense that

$$\frac{d\mathbf{P}_\mu^M|_{\mathscr{F}_t^X}}{d\mathbf{P}_\mu|_{\mathscr{F}_t^X}} := \frac{M_t}{\mu(\phi)}, \quad t \geq 0.$$

We now give the classical spine decomposition theorem:

Theorem 3.3.5 (Spine decomposition, [25, 28, 57])**.** *Suppose that Assumptions 3.1 and 3.2 hold. Let $\mu \in \mathcal{M}_f^\phi$. Let the spine immigration $\{(\xi_t)_{t\geq 0}, (Y_t)_{t\geq 0}, \mathbf{n}; \dot{\mathbf{P}}_\mu\}$ be defined as follows:*

1. *the* spine process *$\{(\xi_t)_{t\geq 0}; \dot{\mathbf{P}}_\mu\}$ is a copy of $\{(\xi_t)_{t\geq 0}; \dot\Pi_\mu\}$;*
2. *the* immigration process *$\{(Y_t)_{t\geq 0}; \dot{\mathbf{P}}_\mu\}$ is an $\mathcal{M}_f$-valued process given by*

$$Y_t := \int_{(0,t]\times\mathbb{W}} w_{t-s}\mathbf{n}(ds, dw), \quad t \geq 0,$$

where, conditioned on ξ, $\mathbf{n}$ is a Poisson random measure on $[0,\infty)\times\mathbb{W}$ with mean measure

$$\mathbf{m}^\xi(ds, dw) := 2\alpha(\xi_s)\mathbb{N}_{\xi_s}(dw)\cdot ds + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}(X \in dw)\pi(\xi_s, dy)\cdot ds.$$

Then, $\{(X_t)_{t\geq 0}; \mathbf{P}_\mu^M\} \overset{f.d.d.}{=} \{(X_t + Y_t)_{t\geq 0}; \mathbf{P}_\mu \otimes \dot{\mathbf{P}}_\mu\}$.

Proof. Fix $T > 0$. We only need to show that

$$\{(X_t)_{t\leq T}; \mathbf{P}_\mu^M\} \overset{f.d.d.}{=} \{(X_t + Y_t)_{t\leq T}; \mathbf{P}_\mu \otimes \dot{\mathbf{P}}_\mu\}.$$

From Lemma 3.3.4, we can verify that

$$\{(Y_t)_{t\leq T}; \dot{\mathbf{P}}_\mu\} \overset{f.d.d.}{=} \{(Y_t)_{t\leq T}; \dot{\mathbf{P}}_\mu^{(\phi,T)}\}. \tag{3.3.14}$$

Also it follows easily from the definitions of $\mathbf{P}_\mu^M$ and $\mathbf{P}_\mu^{X_T(\phi)}$ that

$$\{(X_t)_{t\leq T}; \mathbf{P}_\mu^M\} \overset{f.d.d.}{=} \{(X_t)_{t\leq T}; \mathbf{P}_\mu^{X_T(\phi)}\}. \tag{3.3.15}$$

The desired result then follows from Corollary 3.1.6. □

Remark 3.3.6. Lemma 3.3.4 indicates that $\{(\xi_t)_{0\le t\le T};\Pi_\mu^{(\phi,T)}\}$ are consistent. From (3.3.15) we have that $\{(X_s)_{0\le s\le T};\mathbf{P}_\mu^{X_T(\phi)}\}$ are consistent. From (3.3.14) we have that $\{(Y_t)_{t\le T};\dot{\mathbf{P}}_\mu^{(\phi,T)}\}$ are consistent. According to Theorem 3.1.5, we have $\{(w_t)_{t\le T};\mathbb{N}_\mu^{w_T(\phi)}\}\overset{f.d.d}{=}\{(Y_t)_{t\le T};\dot{\mathbf{P}}_\mu^{(\phi,T)}\}$ which implies that $\{(w_t)_{t\le T};\mathbb{N}_\mu^{w_T(\phi)}\}$ are also consistent.

3.4 2-spine decomposition of critical superprocesses

3.4.1 Second moment formula

Let $X=\{(X_t)_{t\ge 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ,ψ)-superprocess introduced in Subsection 3.1.2 which satisfies Assumptions 3.1, 3.2 and 3.3. In this subsection, we give a second moment formula for superprocesses.

Lemma 3.4.1. *Suppose that Assumptions 3.1, 3.2 and 3.3 hold. Let $g,f\in bp\mathscr{B}_E^\phi,\mu\in\mathcal{M}_f^\phi$ and $t\ge 0$. Suppose that $\{(\xi_s)_{0\le s\le t},(Y_s)_{0\le s\le t},\mathbf{n}_t;\dot{\mathbf{P}}_\mu^{(g,t)}\}$ is the spine representation of $\mathbb{N}_\mu^{w_t(g)}$. Then,*

$$\dot{\mathbf{P}}_\mu^{(g,t)}[Y_t(f)|\xi]=\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds\le t\|A\phi\|_\infty\|\phi^{-1}f\|_\infty,\quad \dot{\mathbf{P}}_\mu^{(g,t)}\text{-a.s.}.$$

Proof. Define $G(s,w):=\mathbf{1}_{s\le t}w_{t-s}(f)$ for all $s\ge 0$ and $w\in\mathbb{W}$. Under Assumption 3.3, it is clear from (3.1.8) that

$$\begin{aligned}\mathbf{m}_t^\xi(G)&=\int_0^t 2\alpha(\xi_s)\mathbb{N}_{\xi_s}[w_{t-s}(f)]ds+\int_0^t ds\int_{(0,\infty)}y\mathbf{P}_{y\delta_{\xi_s}}[X_{t-s}(f)]\pi(\xi_s,dy)\\&=\int_0^t 2\alpha(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds+\int_0^t ds\int_{(0,\infty)}y^2\cdot(S_{t-s}f)(\xi_s)\pi(\xi_s,dy)\\&=\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds.\end{aligned}$$

Since, conditioned on ξ, $\{\mathbf{n}_t;\dot{\mathbf{P}}_\mu^{(g,t)}\}$ is a Poisson random measure on $[0,t]\times\mathbb{W}$ with mean measure $\mathbf{m}_t^\xi$, we conclude from Campbell's theorem that

$$\dot{\mathbf{P}}_\mu^{(g,t)}[Y_t(f)|\xi]=\dot{\mathbf{P}}_\mu^{(g,t)}[\mathbf{n}_t(G)|\xi]=\mathbf{m}_t^\xi(G)=\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds,\quad \dot{\mathbf{P}}_\mu^{(g,t)}\text{-a.s..}$$

Noticing that

$$\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds=\int_0^t[(A\phi)\phi^{-1}S_{t-s}(\phi\cdot\phi^{-1}f)](\xi_s)ds\le t\|A\phi\|_\infty\|\phi^{-1}f\|_\infty,$$

we have our result as desired. □

Proposition 3.4.2. *Under Assumptions 3.1, 3.2 and 3.3, for all $g, f \in b\mathscr{B}_E^\phi$, $\mu \in \mathcal{M}_f^\phi$ and $t \geq 0$, we have that $X_t(g)X_t(f)$ is integrable with respect to $\mathbf{P}_\mu$ and*

$$\mathbf{P}_\mu[X_t(g)X_t(f)] = \langle \mu, S_t g\rangle\langle \mu, S_t f\rangle + \langle \mu, \phi\rangle \dot{\Pi}_\mu\Big[(\phi^{-1}g)(\xi_t)\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds\Big]. \quad (3.4.1)$$

Proof. We first consider the case when $g, f \in bp\mathscr{B}_E^\phi$. In this case, the right hand of (3.4.1) is finite. Actually, by Lemma 3.4.1, the right side of (3.4.1) is less than or equal to

$$\begin{aligned}&\langle \mu, S_t g\rangle\langle \mu, S_t f\rangle + \langle \mu, \phi\rangle \dot{\Pi}_\mu\left[(\phi^{-1}g)(\xi_t)\right]t\|A\phi\|_\infty\|\phi^{-1}f\|_\infty\\ &\leq \langle \mu, \phi\rangle^2 + \langle \mu, \phi\rangle t\|A\phi\|_\infty\|\phi^{-1}g\|_\infty\|\phi^{-1}f\|_\infty < \infty.\end{aligned}$$

We can also assume that $m(g) > 0$. Since if $g \in bp\mathscr{B}_E$ with $m(g) = 0$, then according to (3.1.12), (3.1.6) and Lemma 3.3.4, we have

$$S_t g(x) = \int_E q(t,x,y)g(y)m(dy) = 0, \quad t > 0, x \in E,$$

$$\mathbf{P}_\mu[X_t(g)] = \mu(S_t g) = 0, \quad \mu \in \mathcal{M}_f, t > 0,$$

$$\dot{\Pi}_\mu[\phi^{-1}g(\xi_t)] = \Pi_\mu^{(\phi,t)}[\phi^{-1}g(\xi_t)] = \frac{\mu(S_t g)}{\mu(\phi)} = 0, \quad \mu \in \mathcal{M}_f, t > 0.$$

These imply that the both sides of (3.4.1) are 0.

Now in the case when $g, f \in bp\mathscr{B}_E^\phi$ and $m(g) > 0$, from Theorem 3.1.5 and Lemma 3.4.1 we know that, for each $x \in E$,

$$\begin{aligned}\mathbb{N}_x^{w_t(g)}[w_t(f)] &= \dot{\mathbf{P}}_{\delta_x}^{(g,t)}[Y_t(f)] = \dot{\mathbf{P}}_{\delta_x}^{(g,t)}\left[\dot{\mathbf{P}}_{\delta_x}^{(g,t)}[Y_t(f)|\xi]\right]\\ &= \dot{\mathbf{P}}_{\delta_x}^{(g,t)}\Big[\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds\Big] = \Pi_x^{(g,t)}\Big[\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds\Big]\\ &= S_t g(x)^{-1}\Pi_x\Big[g(\xi_t)e^{\int_0^t \beta(\xi_s)ds}\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{N}_x[w_t(g)w_t(f)] &= \mathbb{N}_x[w_t(g)]\mathbb{N}_x^{w_t(g)}[w_t(f)]\\ &= \Pi_x\Big[g(\xi_t)e^{\int_0^t \beta(\xi_s)ds}\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds\Big]\\ &= \phi(x)\dot{\Pi}_x\Big[(\phi^{-1}g)(\xi_t)\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds\Big].\end{aligned}$$

Integrating with $\mu \in \mathcal{M}_f^\phi$, we have

$$\mathbb{N}_\mu[w_t(g)w_t(f)] = \langle \mu, \phi\rangle \dot{\Pi}_\mu\Big[(\phi^{-1}g)(\xi_t)\int_0^t A(\xi_s)\cdot(S_{t-s}f)(\xi_s)ds\Big]. \quad (3.4.2)$$

It then follows from Lemmas 3.1.1 and 3.2.2 that

$$\begin{aligned}\mathbf{P}_\mu[X_t(g)X_t(f)] &= \mathbb{N}_\mu[w_t(g)]\mathbb{N}_\mu[w_t(f)] + \mathbb{N}_\mu[w_t(g)w_t(f)] \\ &= \langle \mu, S_t g\rangle\langle \mu, S_t f\rangle + \langle \mu, \phi\rangle \dot{\Pi}_\mu\Big[(\phi^{-1}g)(\xi_t)\int_0^t (AS_{t-s}f)(\xi_s)ds\Big]\end{aligned}$$

as desired. For the more general case when $g, f \in b\mathscr{B}_E^\phi$, we only need to consider their positive and negative parts. □

3.4.2 2-Spine decomposition theorem

Let $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ,ψ)-superprocess introduced in Subsection 3.1.2 which satisfies Assumptions 3.1, 3.2 and 3.3. In this subsection, we will prove the 2-spine decomposition theorem for superprocesses, i.e., Theorem 3.1.9.

First, we give a lemma which says that $\mathbb{N}_\mu^{w_T(\phi)^2}$ — the $w_T(\phi)^2$-transform of $\mathbb{N}_\mu$, and $\ddot{\Pi}_\mu^{(T)}$ — the $(\int_0^T (A\phi)(\xi_s)ds)$-transform of $\dot{\Pi}_\mu$, are both well defined probability measures.

Lemma 3.4.3. $\mathbb{N}_\mu[w_T(\phi)^2] = \mu(\phi)\dot{\Pi}_\mu[\int_0^T (A\phi)(\xi_s)ds] \in (0,\infty)$ *for all* $\mu \in \mathcal{M}_f^\phi$ *and* $T > 0$.

Proof. According to (3.4.2), we have

$$\mathbb{N}_\mu[w_T(\phi)^2] = \mu(\phi)\dot{\Pi}_\mu\Big[\int_0^T (A\phi)(\xi_s)ds\Big] \leq \mu(\phi)T\|A\phi\|_\infty < \infty.$$

According to $\mathbb{N}_\mu[w_T(\phi)] = \mu(\phi) > 0$, we must have $\mathbb{N}_\mu[w_T(\phi)^2] > 0$. □

Remark 3.4.4. Note that $\mathbb{N}_\mu^{w_T(\phi)^2}$ is also the $w_T(\phi)$-transform of $\mathbb{N}_\mu^{w_T(\phi)}$. In fact, the size-biased transforms satisfy the following chain rule: If g, f are non-negative measurable functions on some measure space $(D, \mathscr{F}_D, \mathbf{D})$ with $\mathbf{D}(g) \in (0,\infty)$ and $\mathbf{D}(gf) \in (0,\infty)$. Denoted by $\mathbf{D}^g$ the g-transform of $\mathbf{D}$, then $(\mathbf{D}^g)^f = \mathbf{D}^{gf}$, i.e., the f-transform of $\mathbf{D}^g$ is the gf-transform of $\mathbf{D}$. This is true because it is easy to see that

$$\mathbf{D}^{gf}(ds) := \frac{g(s)f(s)\mathbf{D}(ds)}{\mathbf{D}[gf]} = \frac{f(s)\mathbf{D}^g(ds)}{\mathbf{D}^g[f]} = (\mathbf{D}^g)^f(ds), \quad s \in S.$$

For each $\mu \in \mathcal{M}_f^\phi$, let the spine immigration $\{(\xi_t)_{t\geq 0}, (Y_t)_{t\geq 0}, \mathbf{n}; \dot{\mathbf{P}}_\mu\}$ be given by Theorem 3.3.5. We first state a property of $\{Y; \dot{\mathbf{P}}_\mu\}$, which is needed later.

Lemma 3.4.5. $\dot{\mathbf{P}}_\mu(Y_t = \mathbf{0}) = 0$ *for all* $\mu \in \mathcal{M}_f^\phi$ *and* $t > 0$.

Proof. According to Theorem 3.1.5, we have

$$\dot{\mathbf{P}}_\mu(Y_t = 0) = \mathbb{N}_\mu^{w_t(\phi)}(w_t(\phi) = 0) = \langle \mu, \phi\rangle^{-1}\mathbb{N}_\mu[w_t(\phi)\mathbf{1}_{w_t(\phi)=0}] = 0.$$ □

The proof of Theorem 3.1.9 relies on the following lemma:

Lemma 3.4.6. *For any $\mu \in \mathcal{M}_f^\phi$, $T > 0$ and $(K, f) \in \mathcal{K}_T$, we have*

$$\begin{aligned}
&\dot{\mathbf{P}}_\mu[Y_T(\phi)e^{-K^f_{(0,T]}(Y)}|\xi] \\
&\quad= \dot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Y)}|\xi] \int_0^T (A\phi)(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}[e^{-K^f_{(s,T]}(Y)}]\widetilde{\mathbf{P}}_{\xi_s}[e^{-K^f_{(s,T]}(X)}]ds,
\end{aligned}$$

where $\widetilde{\mathbf{P}}_x$ is defined by (3.1.14) *for each $x \in E$.*

Proof. Define $G(s, w) := \mathbf{1}_{s\le T} w_{T-s}(\phi)$ for all $s \ge 0$ and $w \in \mathbb{W}$. Notice that from (3.3.13), under the probability $\dot{\mathbf{P}}_\mu$, we have $Y_T(\phi) = \mathbf{n}(G)$ and $K^f_{(0,T]}(Y) = \mathbf{n}(K^f_{(s,T]}(w))$. From Lemmas 3.4.1 and 3.4.5 we know that

$$0 < \dot{\mathbf{P}}_\mu[Y_T(\phi)|\xi] < \infty, \quad \dot{\mathbf{P}}_\mu\text{-a.s..}$$

Therefore, we can apply Lemma 3.2.1 to the conditioned Poisson random measure $\mathbf{n}$, and get

$$\dot{\mathbf{P}}_\mu[\mathbf{n}(G)e^{-\mathbf{n}(K^f_{(s,T]}(w))}|\xi] = \dot{\mathbf{P}}_\mu[e^{-\mathbf{n}(K^f_{(s,T]}(w))}|\xi]\mathbf{m}^\xi[Ge^{-K^f_{(s,T]}(w)}]. \tag{3.4.3}$$

It is clear from the definitions of $\mathbf{m}^\xi$, $\mathbb{N}^{w_t(\phi)}$ and $\mathbf{P}^M$ that

$$\begin{aligned}
\mathbf{m}^\xi[Ge^{-K^f_{(s,T]}(w)}] &= \int_0^T \Big(2\alpha(\xi_s)\mathbb{N}_{\xi_s}[w_{T-s}(\phi)e^{-K^f_{(s,T]}(w)}] \\
&\qquad + \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}[X_{T-s}(\phi)e^{-K^f_{(s,T]}(X)}]\pi(\xi_s, dy)\Big)ds \\
&= \int_0^T \Big(2(\alpha\phi)(\xi_s)\mathbb{N}^{w_{T-s}(\phi)}_{\xi_s}[e^{-K^f_{(s,T]}(w)}] \\
&\qquad + \int_{(0,\infty)} y^2\phi(\xi_s)\mathbf{P}^M_{y\delta_{\xi_s}}[e^{-K^f_{(s,T]}(X)}]\pi(\xi_s, dy)\Big)ds.
\end{aligned} \tag{3.4.4}$$

According to Theorem 3.1.5, we have

$$\mathbb{N}^{w_{T-s}(\phi)}_x[e^{-K^f_{(s,T]}(w)}] = \dot{\mathbf{P}}_{\delta_x}[e^{-K^f_{(s,T]}(Y)}] = \dot{\mathbf{P}}_{\delta_x}[e^{-K^f_{(s,T]}(Y)}]\mathbf{P}_\mathbf{0}[e^{-K^f_{(s,T]}(X)}], \tag{3.4.5}$$

where we used the fact that $\mathbf{P}_\mathbf{0}(X_t = \mathbf{0},$ for any $t \ge 0) = 1$. It follows from Theorem 3.3.5 that for any $s \in [0, T], x \in E$ and $y \in (0, \infty)$,

$$\mathbf{P}^M_{y\delta_x}[e^{-K^f_{(s,T]}(X)}] = \dot{\mathbf{P}}_{y\delta_x}[e^{-K^f_{(s,T]}(X+Y)}] = \dot{\mathbf{P}}_{\delta_x}[e^{-K^f_{(s,T]}(Y)}]\mathbf{P}_{y\delta_x}[e^{-K^f_{(s,T]}(X)}]. \tag{3.4.6}$$

Plugging (3.4.5) and (3.4.6) back into (3.4.4) and rearranging terms, we have that

$$\begin{aligned}
&\mathbf{m}^\xi[Ge^{-K^f_{(s,T]}}(w)] \\
&\quad= \int_0^T \Big(2(\alpha\phi)(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}[e^{-K^f_{(s,T]}(Y)}]\mathbf{P}_\mathbf{0}[e^{-K^f_{(s,T]}(X)}]
\end{aligned} \tag{3.4.7}$$

$$+\int_{(0,\infty)} y^2\phi(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}[e^{-K_s^f(Y)}]\mathbf{P}_{y\delta_{\xi_s}}[e^{-K^f_{(s,T]}(X)}]\pi(\xi_s,dy)\Big)ds.$$

$$= \int_0^T \phi(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}[e^{-K^f_{(s,T]}(Y)}] \times \Big(2\alpha(\xi_s)\mathbf{P_0}[e^{-K^f_{(s,T]}(X)}] + \int_{(0,\infty)} y^2\mathbf{P}_{y\delta_{\xi_s}}[e^{-K^f_{(s,T]}(X)}]\pi(\xi_s,dy)\Big)ds$$

$$= \int_0^T (A\phi)(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}[e^{-K^f_{(s,T]}(Y)}]\widetilde{\mathbf{P}}_{\xi_s}[e^{-K^f_{(s,T]}(X)}]ds.$$

Plugging (3.4.7) back into (3.4.3), we get the desired result. □

Proof of Theorem 3.1.9. Note that $\{Z_0;\ddot{\mathbf{P}}^{(T)}_\mu\}$ and $\{w_0;\mathbb{N}^{w_T(\phi)^2}_\mu\}$ are both deterministic with common value $\mathbf{0}$. So we only have to prove $\{(Z_t)_{0<t\leq T};\ddot{\mathbf{P}}^{(T)}_\mu\} \overset{f.d.d.}{=} \{(w_t)_{0<t\leq T};\mathbb{N}^{w_T(\phi)^2}_\mu\}$. In order to show this, according to Theorem 3.1.5 and Remark 3.4.4, we only need to show that $\{(Z_t)_{0<t\leq T};\ddot{\mathbf{P}}^{(T)}_\mu\}$ is the $Y_T(\phi)$-transform of process $\{(Y_t)_{0<t\leq T};\dot{\mathbf{P}}_\mu\}$.

Let $(K,f)\in\mathcal{K}_T$. Similar to (3.3.13), we have $K^f_{(r,T]}(Y) = \mathbf{n}_T[K^f_{(r+\cdot,T]}]$ and $K^f_{(r,T]}(Y') = \mathbf{n}'_T[K^f_{(r+\cdot,T]}]$ for each $r\leq T$. Therefore, using Campbell's theorem and an argument similar to that used in the proof of Lemma 3.3.3, one can verify that

$$-\log\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Y)}|\mathscr{G}] = \int_0^T \psi_0'\big(\xi_s,u_s(\xi_s)\big)ds \tag{3.4.8}$$

and

$$-\log\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Y')}|\mathscr{G}] = \int_\kappa^T \psi_0'\big(\xi'_s,u_s(\xi'_s)\big)ds, \tag{3.4.9}$$

where $u:(s,x)\mapsto u_s(x)$ is the function on $[0,T]\times E$ defined in Lemma 3.3.1. It is then clear from (3.4.9), (3.1.13) and Lemma 3.3.3 that

$$\begin{aligned}\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Y')}|\xi,\kappa] &= \ddot{\mathbf{P}}_\mu[e^{-\int_\kappa^T \psi_0'(\xi'_s,u_s(\xi'_s))ds}|\xi,\kappa]\\ &= \dot{\Pi}_{\xi_r}[e^{-\int_r^T \psi_0'(\xi_{s-r},u_s(\xi_{s-r}))ds}]|_{r=\kappa} = \dot{\mathbf{P}}_{\delta_{\xi_r}}[e^{-K^f_{(r,T]}(Y)}]|_{r=\kappa}.\end{aligned} \tag{3.4.10}$$

By the construction of the splitting immigration X' at time κ, we also have

$$\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(X')}|\mathscr{G}] = \widetilde{\mathbf{P}}_{\xi_r}[e^{-K^f_{(r,T]}(X)}]|_{r=\kappa}. \tag{3.4.11}$$

Using (3.4.8), (3.4.10), (3.4.11) and the construction of the 2-spine immigration, we deduce that

$$\begin{aligned}\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Z)}|\xi,\kappa] &= \ddot{\mathbf{P}}_\mu\big[\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Z)}|\mathscr{G}]\big|\xi,\kappa\big]\\ &= \ddot{\mathbf{P}}_\mu\Big[\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Y)}|\mathscr{G}]\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Y')}|\mathscr{G}]\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(X')}|\mathscr{G}]\Big|\xi,\kappa\Big]\\ &= e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))ds}\dot{\mathbf{P}}_{\delta_{\xi_r}}[e^{-K^f_{(r,T]}(Y)}]\widetilde{\mathbf{P}}_{\xi_r}[e^{-K^f_{(r,T]}(X)}]\Big|_{r=\kappa}.\end{aligned}$$

Therefore, from the conditioned law of κ given ξ, we have

$$\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Z)}|\xi] \tag{3.4.12}$$
$$= \frac{e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))ds}}{\int_0^T (A\phi)(\xi_r)dr}\int_0^T (A\phi)(\xi_r)\dot{\mathbf{P}}_{\delta_{\xi_r}}[e^{-K^f_{(r,T]}(Y)}]\widetilde{\mathbf{P}}_{\xi_r}[e^{-K^f_{(r,T]}(X)}]dr.$$

Taking expectation, we get that

$$\begin{aligned}
&\ddot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Z)}]\\
&\overset{(3.4.12)}{=} \ddot{\Pi}^{(T)}_\mu\Big\{\frac{e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))ds}}{\int_0^T (A\phi)(\xi_r)dr}\int_0^T (A\phi)(\xi_r)\dot{\mathbf{P}}_{\delta_{\xi_r}}[e^{-K^f_{(r,T]}(Y)}]\widetilde{\mathbf{P}}_{\xi_r}[e^{-K^f_{(r,T]}(X)}]dr\Big\}\\
&= \dot{\Pi}_\mu\Big\{\frac{e^{-\int_0^T \psi_0'(\xi_s,u_s(\xi_s))ds}}{\dot{\Pi}_\mu[\int_0^T (A\phi)(\xi_r)dr]}\int_0^T (A\phi)(\xi_r)\dot{\mathbf{P}}_{\delta_{\xi_r}}[e^{K^f_{(r,T]}(Y)}]\widetilde{\mathbf{P}}_{\xi_r}[e^{-K^f_{(r,T]}(X)}]dr\Big\}\\
&\overset{(3.3.12)}{=} \dot{\mathbf{P}}_\mu\Big\{\frac{\dot{\mathbf{P}}_\mu[e^{-K^f_{(0,T]}(Y)}|\xi]}{\dot{\mathbf{P}}_\mu[\int_0^T (A\phi)(\xi_r)dr]}\int_0^T (A\phi)(\xi_r)\dot{\mathbf{P}}_{\delta_{\xi_r}}[e^{-K^f_{(r,T]}(Y)}]\widetilde{\mathbf{P}}_{\xi_r}[e^{-K^f_{(r,T]}(X)}]dr\Big\}\\
&\overset{\text{Lemma 3.4.6}}{=} \dot{\mathbf{P}}_\mu\Big\{\frac{\dot{\mathbf{P}}_\mu[Y_T(\phi)e^{-K^f_{(0,T]}(Y)}|\xi]}{\dot{\mathbf{P}}_\mu[Y_T(\phi)]}\Big\} = \frac{\dot{\mathbf{P}}_\mu[Y_T(\phi)e^{-K^f_{(0,T]}(Y)}]}{\dot{\mathbf{P}}_\mu[Y_T(\phi)]},
\end{aligned}$$

where in the second equality we used the definition of $\ddot{\Pi}^{(T)}_\mu$. The display above says that $(Z_t)_{0<t\le T}$ is the $Y_T(\phi)$-transform of the process $\{(Y_t)_{0<t\le T};\dot{\mathbf{P}}_\mu\}$, as desired. □

3.5 The asymptotic behavior of critical superprocesses

3.5.1 Intrinsic ultracontractivity

Let $\{(X_t)_{t\ge 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ,ψ)-superprocess introduced in Subsection 3.1.2 which satisfies Assumptions 3.1 and 3.4. In this subsection, we give some more results related to intrinsic ultracontractivity.

Lemma 3.5.1. *Suppose that $F(x,u,t)$ is a bounded Borel function on $E\times[0,1]\times[0,\infty)$ such that $F(x,u):=\lim_{t\to\infty}F(x,u,t)$ exists for all $x\in E$ and $u\in[0,1]$. Then we have,*

$$\int_0^1 F(\xi_{ut},u,t)du \xrightarrow[t\to\infty]{L^2(\dot{\Pi}_x)} \int_0^1 \langle F(\cdot,u),\phi\phi^*\rangle_m du,\quad x\in E.$$

Proof. We first show that

$$\dot{\Pi}_x[F(\xi_{ut},u,t)] \xrightarrow[t\to\infty]{} \langle F(\cdot,u),\phi\phi^*\rangle_m,\quad x\in E, u\in(0,1). \tag{3.5.1}$$

In fact,

$$\dot{\Pi}_x[F(\xi_{ut},u,t)] = \int_E \frac{\dot{q}(ut,x,y)}{(\phi\phi^*)(y)} F(y,u,t)(\phi\phi^*)(y)m(dy).$$

Note that $\int_\cdot (\phi\phi^*)(y)m(dy)$ is a finite measure, $(y,t) \mapsto \frac{\dot{q}(ut,x,y)}{(\phi\phi^*)(y)}F(y,u,t)$ is bounded by $(1 + ce^{-\gamma ut})\|F\|_\infty$ for $t > u^{-1}$, and $\frac{\dot{q}(ut,x,y)}{(\phi\phi^*)(y)}F(y,u,t) \xrightarrow[t\to\infty]{} F(y,u)$. Using the bounded convergence theorem, we get (3.5.1). By Fubini's theorem,

$$\dot{\Pi}_x\Big[\int_0^1 F(\xi_{ut},u,t)du\Big] = \int_0^1 \dot{\Pi}_x[F(\xi_{ut},u,t)]du, \quad x \in E.$$

Since $\dot{\Pi}_x[F(\xi_{ut},u,t)]$ is bounded by $\|F\|_\infty$ and $\dot{\Pi}_x[F(\xi_{ut},u,t)] \xrightarrow[t\to\infty]{} \langle F(\cdot,u),\phi\phi^*\rangle_m$, by the bounded convergence theorem, we get

$$\dot{\Pi}_x\Big[\int_0^1 F(\xi_{ut},u,t)du\Big] \xrightarrow[t\to\infty]{} c_F := \int_0^1 \langle F(\cdot,u),\phi\phi^*\rangle_m du.$$

Using (3.1.20) and a similar argument, one can verify that for any $0 < u < v \le 1$,

$$\begin{aligned}
&\dot{\Pi}_x[F(\xi_{ut},u,t)F(\xi_{vt},v,t)] \\
&= \int_E\int_E \dot{q}(ut,x,y)\dot{q}((v-u)t,y,z)F(y,u,t)F(z,v,t)m(dy)m(dz) \\
&\xrightarrow[t\to\infty]{} \langle F(\cdot,u),\phi\phi^*\rangle_m \langle F(\cdot,v),\phi\phi^*\rangle_m.
\end{aligned}$$

The above convergence is also true for $0 < v < u \le 1$ since the limit is symmetric in u and v. We have again, by Fubini's theorem and the bounded convergence theorem,

$$\dot{\Pi}_x\Big[\Big(\int_0^1 F(\xi_{ut},u,t)du\Big)^2\Big] = \int_0^1 du\int_0^1 \dot{\Pi}_x[F(\xi_{ut},u,t)F(\xi_{vt},v,t)]dv \xrightarrow[t\to\infty]{} c_F^2.$$

Finally, we have

$$\begin{aligned}
&\dot{\Pi}_x\Big[\Big(\int_0^1 F(\xi_{ut},u,t)du - c_F\Big)^2\Big] \\
&= \dot{\Pi}_x\Big[\Big(\int_0^1 F(\xi_{ut},u,t)du\Big)^2\Big] - 2c_F\dot{\Pi}_x\Big[\int_0^1 F(\xi_{ut},u,t)du\Big] + c_F^2 \\
&\xrightarrow[t\to\infty]{} 0,
\end{aligned}$$

as desired. □

As mentioned earlier in Subsection 3.1.2, in order to study the asymptotic behavior of $(v_t)_{t\ge0}$ and take advantage of (3.1.19), we need $S_t v_s(x)$ to be finite at least for some large $s,t > 0$ and for some $x \in E$. The following lemma addresses this need.

Lemma 3.5.2. *Under Assumption 3.1 and 3.4, the following statements are equivalent.*

(1) $S_t v_s(x) < \infty$ *for some* $s > 0, t > 0$ *and* $x \in E$.

(1′) *There is an* $s_0 > 0$ *such that for any* $s \geq s_0$, $t > 0$ *and* $x \in E$, *we have* $S_t v_s(x) < \infty$.

(2) $\langle v_s, \phi^* \rangle_m < \infty$ *for some* $s > 0$.

(2′) *There is an* $s_0 > 0$ *such that for any* $s \geq s_0$, *we have* $\langle v_s, \phi^* \rangle_m < \infty$.

(3) *There is an* $s_0 > 0$ *such that for any* $s \geq s_0$, *we have* $v_s \in bp\mathscr{B}_E^{\phi}$.

(4) $\mathbf{P}_\nu(X_t = \mathbf{0}) > 0$ *for some* $t > 0$.

(5) $\phi^{-1} v_t$ *converges to* 0 *uniformly when* $t \to \infty$.

(6) *For any* $\mu \in \mathcal{M}_f^{\phi}$, $\mathbf{P}_\mu(\exists t > 0,\ s.t.\ X_t = \mathbf{0}) = 1$.

Proof. We first give some estimates. In this proof, we allow the extended value $+\infty$. According to (3.1.16) and the fact that $\mathbf{0}$ is an absorption state of the superprocess X, we have

$$\begin{aligned} \langle v_{s_0}, \phi^* \rangle_m &= -\log \mathbf{P}_\nu(X_{s_0} = \mathbf{0}) \\ &\geq -\log \mathbf{P}_\nu(X_s = \mathbf{0}) = \langle v_s, \phi^* \rangle_m, \quad 0 < s_0 \leq s. \end{aligned} \tag{3.5.2}$$

According to Assumption 3.4, we have for each $t \geq 0$, there is a $c_t > 0$ such that $q(t, x, y) \leq c_t \phi(x) \phi^*(y)$. Using an argument similar to that of [47, Proposition 2.5], we have for each $t \geq 0$, there is a $c_t' < 0$ such that $q(t, x, y) \geq c_t' \phi(x) \phi^*(y)$. Therefore, we have

$$\phi(x) \langle v_s, \phi^* \rangle_m c_t' \leq S_t v_s(x) \leq \phi(x) \langle v_s, \phi^* \rangle_m c_t, \quad s > 0, t > 0, x \in E. \tag{3.5.3}$$

Let $c, \gamma > 0$ be the constants in (3.1.20). Notice that ϕ is strictly positive, using (3.1.17), one can verify that

$$\frac{V_t f(x)}{\phi(x)} \leq \frac{S_t f(x)}{\phi(x)} \leq (1 + c e^{-\gamma t}) \langle f, \phi^* \rangle, \quad f \in bp\mathscr{B}_E, x \in E, t > 1. \tag{3.5.4}$$

Taking $f = V_s(\theta \mathbf{1}_E)$ in (3.5.4) and letting $\theta \to \infty$, by (3.1.15) and (3.1.18), we have that,

$$\frac{v_{t+s}(x)}{\phi(x)} \leq (1 + c e^{-\gamma t}) \langle v_s, \phi^* \rangle_m, \quad x \in E, s > 0, t > 1. \tag{3.5.5}$$

We can also verify that

$$S_t v_s(x) \leq \|\phi^{-1} v_s\|_\infty S_t \phi(x) = \|\phi^{-1} v_s\|_\infty \phi(x) \quad s, t > 0, x \in E. \tag{3.5.6}$$

Now, we are ready to give the proof of this lemma using the following steps: (1′) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (2′) $\Rightarrow$ (3) $\Rightarrow$ (1′) and (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2). In fact, it is obvious that (1′) $\Rightarrow$ (1). For (1) $\Rightarrow$ (2) we use (3.5.3). For (2) $\Rightarrow$ (2′) we use (3.5.2). For (2′) $\Rightarrow$ (3) we use (3.5.5). For (3) $\Rightarrow$ (1′) we use (3.5.6).

For (2) $\Rightarrow$ (5), we follow the argument in [68, Lemma 3.3]. Note that, from what we have proved, (2) is equivalent to (1), (1′), (2′) and (3). Integrating (3.1.17) with respect to the measure ν, by Fubini's theorem and monotonicity, we have that, for any $f \in p\mathscr{B}_E$ and $t \geq 0$,

$$\begin{aligned}\langle f, \phi^* \rangle_m &= \langle f, S_t^* \phi^* \rangle_m = \langle S_t f, \phi^* \rangle_m \\ &= \langle V_t f, \phi^* \rangle_m + \int_0^t \langle S_{t-r} \Psi_0 V_r f, \phi^* \rangle_m dr \\ &= \langle V_t f, \phi^* \rangle_m + \int_0^t \langle \Psi_0 V_r f, \phi^* \rangle_m dr.\end{aligned} \tag{3.5.7}$$

Define

$$v(x) := \lim_{t \to \infty} v_t(x) = \lim_{t \to \infty} (-\log \mathbf{P}_{\delta_x}(X_t = \mathbf{0})) = -\log \mathbf{P}_{\delta_x}(\exists t > 0, \text{ s.t. } X_t = \mathbf{0}).$$

Since $v_t(x) = -\log \mathbf{P}_{\delta_x}(X_t = \mathbf{0})$ is non-increasing in t, and by (3), we know that $v_t \in bp\mathscr{B}_E^\phi$ for t large enough. Therefore, we have $v \in bp\mathscr{B}_E^\phi \subset L^2(E, m)$. Taking $f = V_s(\theta \mathbf{1}_E)$ in (3.5.7) and letting $\theta \to \infty$, by monotonicity and (2′), we have that, there is an $s_0 > 0$ such that

$$\int_0^t \langle \Psi_0 v_{r+s}, \phi^* \rangle_m dr = \langle v_s, \phi^* \rangle_m - \langle v_{t+s}, \phi^* \rangle_m, \quad s \geq s_0, t \geq 0. \tag{3.5.8}$$

Letting $s \to \infty$, by monotonicity, we have

$$\int_0^t \langle \Psi_0 v, \phi^* \rangle_m dr = t \langle \Psi_0 v, \phi^* \rangle_m = \langle v, \phi^* \rangle_m - \langle v, \phi^* \rangle_m = 0.$$

Since ϕ^* is strictly positive on E, we must have $\Psi_0(v) = 0, m$-a.e.. This, with (3.1.9), implies that $S_t \Psi_0(v) \equiv 0$ for any $t > 0$. By (1′), we know that $S_t v_s(x)$ take finite value for s large enough. Letting $s \to \infty$ in the (3.1.19), by monotonicity, we have

$$v(x) = S_t v(x) - \int_0^t S_{t-r} \Psi_0(v)(x) dr = S_t v(x), \quad x \in E, t \geq 0,$$

which says that the non-negative function v, if not identically 0, is an eigenfunction of L corresponding to $\lambda = 0$, where L is the generator of the semigroups $(S_t)_{t \geq 0}$. Since $v \in L^2(E, m)$, by the uniqueness of the eigenfunction in $L^2(E, m)$ corresponding to $\lambda = 0$, there is a constant $c \in \mathbb{R}$, such that $v(x) = c\phi(x)$ for all $x \in E$. So with $\Psi_0(v) \equiv 0, m$-a.e., we must have $v \equiv 0$. Using the fact that $v_t(x)$ converges to 0 pointwise, by monotonicity and (3.5.5), we can verify the desired result (5).

For (5) $\Rightarrow$ (6), note that, by the definition of v_t, for any $\mu \in \mathcal{M}_f^\phi$, we have

$$-\log \mathbf{P}_\mu\{\exists t > 0, \text{ s.t. } X_t = 0\} = \lim_{t \to \infty} (-\log \mathbf{P}_\mu(X_t = \mathbf{0})) = \lim_{t \to \infty} \langle \mu, v_t \rangle = 0.$$

Finally, note that (6) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2) are obvious. □

3.5.2 Kolmogorov type result

Let $\{(X_t)_{t\geq 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ,ψ)-superprocess introduced in Subsection 3.1.2 which satisfies Assumptions 3.5 and 3.4 and 3.3. In this subsection, we will give a proof of Theorem 3.1.10. Thanks to Lemma 3.5.2, we know that each of the statements in 3.5.2 is true. In particular, $v_t(x)/\phi(x)$ converges to 0 uniformly in $x\in E$.

Lemma 3.5.3. *Under Assumptions 3.5, 3.4 and 3.3, we have*

$$\sup_{x\in E}\Big|\frac{v_t(x)}{\langle v_t,\phi^*\rangle_m\phi(x)}-1\Big|\xrightarrow[t\to\infty]{}0.$$

Proof. We use an argument similar to that used in [62] for critical branching diffusions. Fix a non-trivial $\mu\in\mathcal{M}_f^\phi$, and let the spine immigration $\{(\xi_t)_{t\geq0},(Y_t)_{t\geq0},\mathbf{n};\dot{\mathbf{P}}_\mu\}$ be given by Theorem 3.3.5. For any $t>0$, we have

$$\begin{aligned}\langle\mu,\phi\rangle\dot{\mathbf{P}}_\mu[(Y_t(\phi))^{-1}] &\overset{(3.3.14)}{=} \langle\mu,\phi\rangle\mathbf{P}_\mu^{(\phi,T)}[(Y_t(\phi))^{-1}] \qquad (3.5.9)\\ &\overset{\text{Theorem 3.1.5}}{=} \langle\mu,\phi\rangle\mathbb{N}_\mu^{w_t(\phi)}[(w_t(\phi))^{-1}]=\mathbb{N}_\mu\{w_t(\phi)>0\}=\lim_{\lambda\to\infty}\mathbb{N}_\mu[1-e^{-\lambda w_t(\phi)}]\\ &\overset{\text{Campbell's formula}}{=} \lim_{\lambda\to\infty}(-\log\mathbf{P}_\mu[e^{-\lambda X_t(\phi)}])=-\log\mathbf{P}_\mu\{X_t=\mathbf{0}\}\\ &\overset{(3.1.16)}{=}\langle\mu,v_t\rangle.\end{aligned}$$

Taking $\mu=\delta_x$ in (3.5.9), we get $v_t(x)/\phi(x)=\dot{\mathbf{P}}_{\delta_x}[(Y_t(\phi))^{-1}]$. Taking $\mu=\nu$, we get $\langle v_t,\phi^*\rangle_m=\dot{\mathbf{P}}_\nu[(Y_t(\phi))^{-1}]$. Therefore, to complete the proof, we only need to show that

$$\sup_{x\in E}\Big|\frac{\dot{\mathbf{P}}_{\delta_x}[(Y_t(\phi))^{-1}]}{\dot{\mathbf{P}}_\nu[(Y_t(\phi))^{-1}]}-1\Big|\xrightarrow[t\to\infty]{}0.$$

For any Borel subset $G\subset(0,t]$, define

$$Y_t^G:=\int_{G\times\mathbb{W}}w_{t-s}\mathbf{n}(ds,dw).$$

Then we have the following decomposition of Y:

$$Y_t=Y_t^{(0,t_0]}+Y_t^{(t_0,t]},\quad 0<t_0<t<\infty. \tag{3.5.10}$$

It is easy to see, from the construction and the Markov property of the spine immigration $\{Y,\xi;\dot{\mathbf{P}}\}$, that for any $0<t_0<t<\infty$,

$$\dot{\mathbf{P}}[(Y_t^{(t_0,t]}(\phi))^{-1}|\mathscr{F}_{t_0}^\xi]=\dot{\mathbf{P}}_{\delta_{\xi_{t_0}}}[(Y_{t-t_0}(\phi))^{-1}]=(\phi^{-1}v_{t-t_0})(\xi_{t_0}).$$

Therefore, we have

$$\dot{\mathbf{P}}_\nu[(Y_t^{(t_0,t]}(\phi))^{-1}]=\dot{\Pi}_\nu[(\phi^{-1}v_{t-t_0})(\xi_{t_0})]=\langle v_{t-t_0},\phi^*\rangle_m$$

and

$$\dot{\mathbf{P}}_{\delta_x}[(Y_t^{(t_0,t]}(\phi))^{-1}] = \dot{\Pi}_x[(\phi^{-1}v_{t-t_0})(\xi_{t_0})] = \int_E \dot{q}(t_0,x,y)(\phi^{-1}v_{t-t_0})(y)m(dy). \tag{3.5.11}$$

By the decomposition (3.5.10), we have

$$\begin{aligned}\phi^{-1}v_t(x) &= \dot{\mathbf{P}}_{\delta_x}[(Y_t(\phi))^{-1}] \\ &= \dot{\mathbf{P}}_\nu[(Y_t^{(t_0,t]}(\phi))^{-1}] + \big(\dot{\mathbf{P}}_{\delta_x}[(Y_t^{(t_0,t]}(\phi))^{-1}] - \dot{\mathbf{P}}_\nu[(Y_t^{(t_0,t]}(\phi))^{-1}]\big) \\ &\quad + \big(\dot{\mathbf{P}}_{\delta_x}[(Y_t(\phi))^{-1} - (Y_t^{(t_0,t]}(\phi))^{-1}]\big) \\ &=: \langle v_{t-t_0},\phi^*\rangle_m + \epsilon_x^1(t_0,t) + \epsilon_x^2(t_0,t).\end{aligned} \tag{3.5.12}$$

Suppose that $t_0 > 1$, and let $c,\gamma > 0$ be the constants in (3.1.20), we have

$$\begin{aligned}|\epsilon_x^1(t_0,t)| &= \big|\dot{\mathbf{P}}_{\delta_x}[(Y_t^{(t_0,t]}(\phi))^{-1}] - \dot{\mathbf{P}}_\nu[(Y_t^{(t_0,t]}(\phi))^{-1}]\big| \\ &= \Big|\int_E \dot{q}(t_0,x,y)(\phi^{-1}v_{t-t_0})(y)m(dy) - \langle v_{t-t_0},\phi^*\rangle_m\Big| \\ &\leq \int_{y\in E}\big|\dot{q}(t_0,x,y) - (\phi\phi^*)(y)\big|(\phi^{-1}v_{t-t_0})(y)m(dy) \\ &\leq ce^{-\gamma t_0}\langle v_{t-t_0},\phi^*\rangle_m.\end{aligned} \tag{3.5.13}$$

We also have

$$\begin{aligned}|\epsilon_x^2(t_0,t)| &= \big|\dot{\mathbf{P}}_{\delta_x}[(Y_t(\phi))^{-1} - (Y_t^{(t_0,t]}(\phi))^{-1}]\big| \\ &= \dot{\mathbf{P}}_{\delta_x}[Y_t^{(0,t_0]}(\phi)\cdot(Y_t(\phi))^{-1}\cdot(Y_t^{(t_0,t]}(\phi))^{-1}] \\ &\leq \dot{\mathbf{P}}_{\delta_x}[\mathbf{1}_{Y_t^{(0,t_0]}(\phi)>0}\cdot(Y_t^{(t_0,t]}(\phi))^{-1}] \\ &= \dot{\mathbf{P}}_{\delta_x}\big[\dot{\mathbf{P}}_{\delta_x}[\mathbf{1}_{Y_t^{(0,t_0]}(\phi)>0}|\mathscr{F}_{t_0}^\xi]\cdot\dot{\mathbf{P}}_{\delta_x}[(Y_t^{(t_0,t]}(\phi))^{-1}|\mathscr{F}_{t_0}^\xi]\big].\end{aligned} \tag{3.5.14}$$

Notice that, by Campbell's formula, one can verify that

$$\dot{\mathbf{P}}_{\delta_x}[e^{-\langle Y_t^{(0,t_0]},\theta\mathbf{1}_E\rangle}|\mathscr{F}_{t_0}^\xi] = e^{-\int_0^{t_0}\psi_0'(\xi_s,V_{t-s}(\theta\mathbf{1}_E)(\xi_s))ds}.$$

Letting $\theta\to\infty$ we have

$$\dot{\mathbf{P}}_{\delta_x}[\mathbf{1}_{Y_t^{(0,t_0]}=\mathbf{0}}|\mathscr{F}_{t_0}^\xi] = e^{-\int_0^{t_0}\psi_0'(\xi_s,v_{t-s}(\xi_s))ds}.$$

We also have

$$\begin{aligned}\psi_0'(x,v_{t-s}(x)) &= 2\alpha(x)v_{t-s}(x) + \int_{(0,\infty)}(1-e^{-yv_{t-s}(x)})y\pi(x,dy) \\ &\leq \big(2\alpha(x) + \int_{(0,\infty)}y^2\pi(x,dy)\big)v_{t-s}(x)\end{aligned}$$

$$= A(x)v_{t-s}(x) \leq \|A\phi\|_\infty \|\phi^{-1}v_{t-s}\|_\infty.$$

Therefore

$$\dot{\mathbf{P}}_{\delta_x}[\mathbf{1}_{Y_t^{(0,t_0]}\neq \mathbf{0}}|\mathscr{F}^\xi_{t_0}] = 1 - e^{-\int_0^{t_0}\psi_0'(\xi_s, v_{t-s}(\xi_s))ds} \leq t_0\|A\phi\|_\infty\|\phi^{-1}v_{t-t_0}\|_\infty. \tag{3.5.15}$$

Plugging (3.5.15) into (3.5.14), using (3.5.11) and letting $c,\gamma > 0$ be the constants in (3.1.20), we have that

$$\begin{aligned}|\epsilon^2_x(t_0,t)| &\leq t_0\|A\phi\|_\infty\|(\phi^{-1}v_{t-t_0})\|_\infty \dot{\mathbf{P}}_{\delta_x}[(Y_t^{(t_0,t]}(\phi))^{-1}|\mathscr{F}^\xi_{t_0}] \\ &\leq t_0\|A\phi\|_\infty\|(\phi^{-1}v_{t-t_0})\|_\infty \int_E \dot{q}(t_0,x,y)(\phi^{-1}v_{t-t_0})(y)m(dy) \\ &\leq t_0\|A\phi\|_\infty\|\phi^{-1}v_{t-t_0}\|_\infty(1+ce^{-\gamma t_0})\langle v_{t-t_0},\phi^*\rangle_m.\end{aligned} \tag{3.5.16}$$

Combining (3.5.12), (3.5.13) and (3.5.16), we have that

$$\begin{aligned}\Big|\frac{\phi^{-1}v_t(x)}{\langle v_{t-t_0},\phi^*\rangle_m} - 1\Big| &\leq \frac{|\epsilon^1_x(t_0,t)|}{\langle v_{t-t_0},\phi^*\rangle_m} + \frac{|\epsilon^2_x(t_0,t)|}{\langle v_{t-t_0},\phi^*\rangle_m} \\ &\leq ce^{-\gamma t_0} + t_0\|A\phi\|_\infty\|\phi^{-1}v_{t-t_0}\|_\infty(1+ce^{-\gamma t_0}).\end{aligned} \tag{3.5.17}$$

Since we know from Lemma 3.5.2(5) that $\|\phi^{-1}v_t\|_\infty \to 0$ when $t\to\infty$, there exists a map $t\mapsto t_0(t)$ such that,

$$t_0(t)\xrightarrow[t\to\infty]{}\infty; \quad t_0(t)\|\phi^{-1}v_{t-t_0(t)}\|_\infty \xrightarrow[t\to\infty]{} 0.$$

Plugging this choice of $t_0(t)$ back into (3.5.17), we have that

$$\sup_{x\in E}\Big|\frac{\phi^{-1}v_t(x)}{\langle v_{t-t_0(t)},\phi^*\rangle_m} - 1\Big| \xrightarrow[t\to\infty]{} 0. \tag{3.5.18}$$

Now notice that

$$\begin{aligned}\Big|\frac{\langle v_t,\phi^*\rangle_m}{\langle v_{t-t_0(t)},\phi^*\rangle_m} - 1\Big| &\leq \int \Big|\frac{\phi^{-1}v_t(x)}{\langle v_{t-t_0(t)},\phi^*\rangle} - 1\Big|\phi\phi^*(x)m(dx) \\ &\leq \sup_{x\in E}\Big|\frac{\phi^{-1}v_t(x)}{\langle v_{t-t_0(t)},\phi^*\rangle_m} - 1\Big| \xrightarrow[t\to\infty]{} 0.\end{aligned} \tag{3.5.19}$$

Finally, by (3.5.18), (3.5.19) and the property of uniform convergence,

$$\sup_{x\in E}\Big|\frac{\phi^{-1}v_t(x)}{\langle v_t,\phi^*\rangle_m} - 1\Big| \xrightarrow[t\to\infty]{} 0,$$

as desired. □

Lemma 3.5.4. *Under Assumptions 3.5, 3.4 and 3.3, we have*

$$\frac{1}{t\langle v_t,\phi^*\rangle_m} \xrightarrow[t\to\infty]{} \frac{1}{2}\langle A\phi,\phi\phi^*\rangle_m.$$

Proof. We use an argument similar to that used in [62] for critical branching diffusions.

$$R(x,z) := \psi_0(x,z) - \frac{1}{2}A(x)z^2$$

where

$$e(x,z) := \int_{(0,\infty)} y^2\big(1 \wedge \frac{1}{6}yz\big)\pi(x,dy) \le A(x).$$

By monotonicity, we have that

$$e(x,z) \xrightarrow[z\to 0]{} 0, \quad x \in E. \tag{3.5.20}$$

Taking $b(t) := \langle v_t, \phi^* \rangle_m$ and writing $l_t(x) := v_t(x) - b(t)\phi(x)$, Lemma 3.5.3 says that,

$$\sup_{x\in E}\Big|\frac{l_t(x)}{b(t)\phi(x)}\Big| \xrightarrow[t\to\infty]{} 0. \tag{3.5.21}$$

Now, taking $s_0 > 0$ as in (3.5.8), we have that $t \mapsto b(t)$ is differentiable on the set

$$C = \{t > s_0 : \text{the function } t \mapsto \langle \Psi_0(v_t), \phi^* \rangle_m \text{ is continuous at } t\}$$

and that

$$\begin{aligned}\frac{d}{dt}b(t) &= -\langle \Psi_0(v_t), \phi^*\rangle_m = -\Big\langle \frac{1}{2}A \cdot v_t^2 + R(\cdot, v_t(\cdot)), \phi^* \Big\rangle_m \\ &= -\Big\langle \frac{1}{2}A \cdot \big(b(t)\phi + l_t\big)^2 + R(\cdot, v_t(\cdot)), \phi^* \Big\rangle_m \\ &= -b(t)^2\Big[\frac{1}{2}\langle A\phi, \phi\phi^*\rangle_m + g(t)\Big], \quad t \in C,\end{aligned} \tag{3.5.22}$$

where

$$\begin{aligned}g(t) &= \Big\langle \frac{l_t}{b(t)\phi}, A\phi^2\phi^* \Big\rangle_m + \frac{1}{2}\Big\langle \Big(\frac{l_t}{b(t)\phi}\Big)^2, A\phi^2\phi^* \Big\rangle_m + \Big\langle \frac{R(\cdot, v_t(\cdot))}{b(t)^2\phi^2}, \phi^2\phi^* \Big\rangle_m \\ &=: g_1(t) + g_2(t) + g_3(t).\end{aligned}$$

From (3.5.21), we have $g_1(t) \to 0$ and $g_2(t) \to 0$ as $t \to \infty$. From

$$\frac{R(x, v_t(x))}{b(t)^2\phi(x)^2} \le \frac{e(x, v_t(x)) \cdot v_t(x)^2}{b(t)^2\phi(x)^2} = e(x, v_t(x))\Big(1 + \frac{l_t(x)}{b(t)\phi(x)}\Big)^2,$$

using (3.5.21), (3.5.20), Lemma 3.5.2 (5) and the dominated convergence theorem ($e(x, v_t(x))$ is dominated by $A(x)$), we conclude that $g_3(t) \to 0$ as $t \to \infty$.

Finally, from (3.5.22) we can write

$$\frac{d}{dt}\Big(\frac{1}{b(t)}\Big) = -\frac{db(t)}{b(t)^2 dt} = \frac{1}{2}\langle A\phi, \phi\phi^*\rangle_m + g(t), \quad t \in C. \tag{3.5.23}$$

Notice that, since the function $t \mapsto \langle \Psi_0(v_t), \phi^*\rangle_m$ is non-increasing in t, the complement of C

has at most countably many elements. Therefore, using (3.5.8) and (3.5.23), one can verify that $t \mapsto \frac{1}{b(t)}$ is absolutely continuous on the interval $[s_0, t_0]$ as long as s_0 and t_0 are large enough. This allows us to integrate (3.5.23) on the interval $[s_0, t_0]$ with respect to the Lebesgue measure, and get that

$$\frac{1}{b(t_0)} = \frac{1}{b(s_0)} + \frac{1}{2}\langle A\phi, \phi\phi^*\rangle_m (t_0 - s_0) + \int_{s_0}^{t_0} g(s)ds, \quad \text{for } 0 \leq s_0 \leq t_0 \text{ large enough.}$$

Dividing by t_0 and letting $t_0 \to \infty$ in the above equation, we have

$$\frac{1}{b(t)t} \xrightarrow[t\to\infty]{} \frac{1}{2}\langle A\phi, \phi\phi^*\rangle_m$$

as desired. □

Proof of Theorem 3.1.10. For $\mu \in \mathcal{M}_f^\phi$, from Lemma 3.5.2.(5) we know that

$$\langle \mu, v_t\rangle = \int_E v_t(x)\mu(dx) = \int_E \frac{v_t(x)}{\phi(x)}\phi(x)\mu(dx) \xrightarrow[t\to\infty]{} 0. \tag{3.5.24}$$

From Lemma 3.5.3 we know that

$$\frac{\langle \mu, v_t\rangle}{\langle v_t, \phi^*\rangle_m} = \int_E \frac{v_t(x)}{\langle v_t, \phi^*\rangle_m \phi(x)}\phi(x)\mu(dx) \xrightarrow[t\to\infty]{} \langle \mu, \phi\rangle. \tag{3.5.25}$$

It then follows from (3.5.24), (3.5.25) and Lemma 3.5.4 that

$$\begin{aligned} t\mathbf{P}_\mu(X_t \neq \mathbf{0}) = t(1 - e^{-\langle \mu, v_t\rangle}) &= t\langle v_t, \phi^*\rangle \frac{\langle \mu, v_t\rangle}{\langle v_t, \phi^*\rangle_m}\frac{1 - e^{-\langle \mu, v_t\rangle}}{\langle \mu, v_t\rangle} \\ &\xrightarrow[t\to\infty]{} \frac{\langle \mu, \phi\rangle}{\frac{1}{2}\langle A\phi, \phi\phi^*\rangle_m}, \quad x \in E. \end{aligned}$$

□

3.5.3 Yaglom type result

Let $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ be the (ξ, ψ)-superprocess introduced in Subsection 3.1.2 which satisfies Assumptions 3.5 and 3.4 and 3.3. In this subsection, we will give a proof of Theorem 3.1.11.

Slutsky's theorem is used quite often to prove convergence in law of two components, in which one contributes to the limit, and the other one is negligible. The following proposition says that under $\dot{\mathbf{P}}_\mu$, the weighted mass $Y_t(\phi)$ coming off spine, normalized by t, converges to a Gamma distribution as $t \to \infty$.

Proposition 3.5.5. *Suppose that Assumptions 3.5, 3.4 and 3.3 hold. Suppose that* $\mu \in \mathcal{M}_f^\phi$*. Let* $\{(\xi_t)_{t\geq 0}, (Y_t)_{t\geq 0}, \mathbf{n}; \dot{\mathbf{P}}_\mu\}$ *be the spine immigration given by Theorem 3.3.5. Then* $W_t := \frac{Y_t(\phi)}{t}$ *converges weakly to a Gamma distribution* $\Gamma(2, c_0^{-1})$ *with* $c_0 := \frac{1}{2}\langle \phi A, \phi\phi^*\rangle_m$.

Proof. We only have to prove that

$$\dot{\mathbf{P}}_\mu[e^{-\theta W_t}] \xrightarrow[t\to\infty]{} \frac{1}{(1+c_0\theta)^2}, \quad \theta \geq 0, \mu \in \mathcal{M}_f^\phi.$$

First we consider the case when $\mu = \delta_x$ for an arbitrary $x \in E$. To simplify notation, for all $x \in E, \theta \geq 0$ and $t \geq 0$, we write

$$\begin{aligned} J(x,\theta,t) &:= (\phi A)(x)\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}]\widetilde{\mathbf{P}}_x[e^{-X_t(\frac{\theta\phi}{t})}], \\ J_0(x,\theta,t) &:= (\phi A)(x)\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}] \end{aligned}$$

and

$$M(x,\theta,t) := \Big|\frac{1}{(1+c_0\theta)^2} - \dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}]\Big|.$$

Step 1. We will show that

$$\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}] = \dot{\mathbf{P}}_{\delta_x}[e^{-\int_0^1 du \int_0^\theta d\rho \cdot J(\xi_{ut},\rho(1-u),t(1-u))}]. \tag{3.5.26}$$

In fact, we have

$$\frac{\partial}{\partial\theta}\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}|\xi] = -\dot{\mathbf{P}}_{\delta_x}[W_t e^{-\theta W_t}|\xi], \quad t \geq 0, \theta \geq 0.$$

Applying Lemma 3.4.6 with $K(dr) = \delta_t(dr)$ and $f_t = \frac{\theta\phi}{t}$, for each $\theta \geq 0$, we have

$$\begin{aligned} -\frac{\partial}{\partial\theta}\log\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}|\xi] &= \frac{\dot{\mathbf{P}}_{\delta_x}[W_t e^{-\theta W_t}|\xi]}{\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}|\xi]} \\ &= \frac{1}{t}\int_0^t (A\phi)(\xi_s)\dot{\mathbf{P}}_{\delta_{\xi_s}}[e^{-(\theta\frac{t-s}{t})W_{t-s}}]\widetilde{\mathbf{P}}_{\xi_s}[e^{-X_{t-s}(\frac{\theta\phi}{t})}]ds \end{aligned}$$

Integrating both sides of the above equation yields that

$$-\log\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}|\xi] = \int_0^1 du \int_0^\theta J(\xi_{ut},\rho(1-u),t(1-u))d\rho,$$

which implies (3.5.26).

Step 2. We will show that

$$\int_0^1 du \int_0^\theta (J_0 - J)(\xi_{ut},\rho(1-u),t(1-u))d\rho \xrightarrow[t\to\infty]{L^2(\dot{\mathbf{P}}_{\delta_x})} 0, \quad \theta \geq 0. \tag{3.5.27}$$

To get this result, we will apply Lemma 3.5.1 with

$$\begin{aligned} F(x,u,t) &:= \int_0^\theta d\rho \cdot (J_0 - J)(x,\rho(1-u),t(1-u)) \\ &= \int_0^\theta d\rho \cdot (A\phi)(x)\dot{\mathbf{P}}_{\delta_x}[e^{-\rho(1-u)W_{t(1-u)}}]\widetilde{\mathbf{P}}_x[1-e^{-X_{t(1-u)}(\frac{\rho\phi}{t})}]. \end{aligned} \tag{3.5.28}$$

Firstly note that $F(x,u,t)$ is bounded by $\theta\|\phi A\|_\infty$ on $E\times[0,1]\times[0,\infty)$. Secondly note that $F(x,u,t)\xrightarrow[t\to\infty]{}0$ for each $x\in E$ and $u\in[0,1]$, since $|J_0-J|$ is bounded by $\|\phi A\|_\infty$ and

$$\begin{aligned}\left|(J_0-J)(x,\theta,t)\right| &= (A\phi)(x)\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}]\widetilde{\mathbf{P}}_x[1-e^{-X_t(\frac{\theta\phi}{t})}]\\ &\le (A\phi)(x)\widetilde{\mathbf{P}}_x(X_t\neq\mathbf{0})\\ &= (A\phi)(x)\frac{2\alpha(x)\mathbf{P}_{\mathbf{0}}(X_t\neq\mathbf{0})+\int_{(0,\infty)}y^2\mathbf{P}_{y\delta_x}(X_t\neq\mathbf{0})\pi(x,dy)}{2\alpha(x)+\int_{(0,\infty)}y^2\pi(x,dy)}\\ &\xrightarrow[t\to\infty]{} 0,\quad x\in E,\theta\ge 0.\end{aligned}$$

Therefore, we can apply Lemma 3.5.1 with $F(x,u,t)$ given by (3.5.28), and get (3.5.27).

Step 3. We will show that

$$\frac{1}{(1+c_0\theta)^2}=\lim_{t\to\infty}\dot{\mathbf{P}}_{\delta_x}\left[e^{-\int_0^1du\int_0^\theta d\rho\frac{(A\phi)(\xi_{ut})}{(1+c_0\rho(1-u))^2}}\right],\quad \theta\ge 0. \tag{3.5.29}$$

By elementary calculus, the following map

$$(x,u)\mapsto\int_0^\theta\frac{(A\phi)(x)}{(1+c_0\rho(1-u))^2}d\rho=\frac{(A\phi)(x)\theta}{1+c_0\theta(1-u)}$$

is bounded by $\theta\|A\phi\|_\infty$ on $E\times[0,1]$. According to Lemma 3.5.1, we have that

$$\begin{aligned}\int_0^1du\int_0^\theta\frac{(A\phi)(\xi_{ut})}{\big(1+c_0\rho(1-u)\big)^2}d\rho &\xrightarrow[t\to\infty]{L^2(\dot{\mathbf{P}}_{\delta_x})}\int_0^1\Big\langle\frac{\theta A\phi}{1+c_0\theta(1-u)},\phi\phi^*\Big\rangle_m du\\ &=\langle A\phi,\phi\phi^*\rangle_m\int_0^1\frac{\theta}{1+c_0\theta(1-u)}du\\ &=2\log(1+c_0\theta).\end{aligned}$$

Therefore, by the bounded convergence theorem, we get (3.5.29).

Step 4. We will show that

$$M(x,\theta):=\limsup_{t\to\infty}M(x,\theta,t)=0,\quad x\in E,\theta\ge 0. \tag{3.5.30}$$

In fact,

$$M(x,\theta,t)\le I_1+I_2+I_3, \tag{3.5.31}$$

where

$$I_1:=\left|\frac{1}{(1+c_0\theta)^2}-\dot{\mathbf{P}}_{\delta_x}\left[e^{-\int_0^1du\int_0^\theta\frac{(A\phi)(\xi_{ut})}{[1+c_0\rho(1-u)]^2}d\rho}\right]\right|\xrightarrow[t\to\infty]{\text{by (3.5.29)}}0,$$

$$I_2:=\left|\dot{\mathbf{P}}_{\delta_x}[e^{-\int_0^1du\int_0^\theta\frac{(A\phi)(\xi_{ut})}{(1+c_0\rho(1-u))^2}d\rho}]-\dot{\mathbf{P}}_{\delta_x}[e^{-\int_0^1du\int_0^\theta J_0(\xi_{ut},\rho(1-u),t(1-u))d\rho}]\right|$$

$$\leq \dot{\mathbf{P}}_{\delta_x}\Big[\int_0^1 du\int_0^\theta (A\phi)(\xi_{ut})M(\xi_{ut},\rho(1-u),t(1-u))d\rho\Big]$$
$$= \int_0^1 du\int_0^\theta d\rho\int_E \dot{q}(ut,x,y)(A\phi)(y)M(y,\rho(1-u),t(1-u))m(dy),$$

and by (3.5.26) and (3.5.27),

$$I_3 := \big|\dot{\mathbf{P}}_{\delta_x}[e^{-\int_0^1 du\int_0^\theta J_0(\xi_{ut},\rho(1-u),t(1-u))d\rho}] - \dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}]\big|$$
$$= \big|\dot{\mathbf{P}}_{\delta_x}[e^{-\int_0^1 du\int_0^\theta J_0(\xi_{ut},\rho(1-u),t(1-u))d\rho}] - \dot{\mathbf{P}}_{\delta_x}[e^{-\int_0^1 du\int_0^\theta J(\xi_{ut},\rho(1-u),t(1-u))d\rho}]\big|$$
$$\leq \dot{\mathbf{P}}_{\delta_x}\Big[\Big|\int_0^1 du\int_0^\theta (J_0-J)(\xi_{ut},\rho(1-u),t(1-u))d\rho\Big|\Big] \xrightarrow[t\to\infty]{} 0.$$

Therefore, taking $\limsup_{t\to\infty}$ in (3.5.31), by the reverse Fatou's lemma, we get

$$M(x,\theta) \leq \int_0^1 du\int_0^\theta \langle A\phi M(\cdot,\rho(1-u)),\phi\phi^*\rangle_m d\rho, \quad x\in E,\theta\geq 0. \tag{3.5.32}$$

Integrating with respect to the finite measure $(A\phi\phi\phi^*)(x)m(dx)$ yields that

$$\langle A\phi M(\cdot,\theta),\phi\phi^*\rangle_m \leq \langle A\phi,\phi\phi^*\rangle_m\int_0^1 du\int_0^\theta \langle A\phi M(\cdot,\rho(1-u)),\phi\phi^*\rangle_m d\rho, \quad \theta\geq 0.$$

According to [63, Lemma 3.1], this inequality implies that $\langle A\phi M(\cdot,\theta),\phi\phi^*\rangle_m = 0$ for each $\theta\geq 0$. This and (3.5.32) imply (3.5.30), which completes the proof when $\mu=\delta_x$.

Finally, for any $\mu\in\mathcal{M}_f^\phi$, since

$$\langle\mu,\phi\rangle\dot{\mathbf{P}}_\mu[e^{-\theta W_t}] = \langle\mu,\phi\rangle\mathbb{N}_\mu^{w_t(\phi)}[e^{-\theta\frac{w_t(\phi)}{t}}] = \mathbb{N}_\mu[w_t(\phi)e^{-\theta\frac{w_t(\phi)}{t}}]$$
$$= \int_E \mu(dx)\mathbb{N}_x[w_t(\phi)e^{-\theta\frac{w_t(\phi)}{t}}] = \int_E \mu(dx)\phi(x)\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}],$$

we have that, by the bounded convergence theorem,

$$\Big|\dot{\mathbf{P}}_\mu[e^{-\theta W_t}] - \frac{1}{(1+c_0\theta)^2}\Big| \leq \int_E \Big|\dot{\mathbf{P}}_{\delta_x}[e^{-\theta W_t}] - \frac{1}{(1+c_0\theta)^2}\Big|\frac{\phi(x)\mu(dx)}{\langle\mu,\phi\rangle} \xrightarrow[t\to\infty]{} 0,$$

as desired. □

The following lemma says that, conditional on survival up to time t, the weighted and normalized mass $t^{-1}X_t(\phi)$ (weighted by ϕ, and normalized by t) has a limit distribution which is exponential with explicit parameter. Later we will consider limit of $t^{-1}X_t(f)$ with a general $f\in bp\mathscr{B}_E^\phi$.

Lemma 3.5.6. *Suppose that Assumptions 3.5, 3.4 and 3.3 hold. Let $\mu\in\mathcal{M}_f^\phi$. Then it holds that $\{t^{-1}X_t(\phi);\mathbf{P}_\mu(\cdot|X_t\neq\mathbf{0})\}$ converges weakly to an exponential distribution* $\mathrm{Exp}(c_0^{-1})$ *with $c_0 := \frac{1}{2}\langle\phi A,\phi\phi^*\rangle_m$.*

Proof. We only have to show that

$$\mathbf{P}_\mu[e^{-\theta t^{-1}X_t(\phi)}|X_t \neq \mathbf{0}] \xrightarrow[t\to\infty]{} \frac{1}{1+c_0\theta}, \quad \theta \geq 0, \mu \in \mathcal{M}_f^\phi.$$

Notice that, by Lemma 3.5.2(6), we have

$$\{t^{-1}X_t(\phi); \mathbf{P}_\mu\} \xrightarrow[t\to\infty]{law} 0.$$

Therefore, by Theorem 3.3.5 and Proposition 3.5.5, we have

$$\mathbf{P}_\mu^M[e^{-\theta t^{-1}X_t(\phi)}] = (\mathbf{P}_\mu \otimes \dot{\mathbf{P}}_\mu)[e^{-\theta t^{-1}(X_t+Y_t)(\phi)}] \xrightarrow[t\to\infty]{} \frac{1}{(1+c_0\theta)^2}.$$

Also notice that, by elementary calculus

$$\frac{1-e^{-\theta u}}{u} = \int_0^\theta e^{-\rho u} d\rho, \quad u > 0.$$

From Theorem 3.3.5 and Lemma 3.4.5 we know that $\mathbf{P}_\mu^M(X_t = \mathbf{0}) = 0$. Therefore by the bounded convergence theorem, we have

$$\begin{aligned}\mathbf{P}_\mu^M\Big[\frac{1-e^{-\theta t^{-1}X_t(\phi)}}{t^{-1}X_t(\phi)}\Big] &= \mathbf{P}_\mu^M\Big[\int_0^\theta e^{-\rho t^{-1}X_t(\phi)} d\rho\Big] = \int_0^\theta \mathbf{P}_\mu^M[e^{-\rho t^{-1}X_t(\phi)}]d\rho \\ &\xrightarrow[t\to\infty]{} \int_0^\theta \frac{1}{(1+c_0\rho)^2} d\rho = c_0^{-1}(1 - \frac{1}{1+c_0\theta}).\end{aligned}$$

Hence, by Theorem 3.1.10 we have

$$\begin{aligned}\mathbf{P}_\mu[1-e^{-\theta t^{-1}X_t(\phi)}|X_t \neq \mathbf{0}] &= \mathbf{P}_\mu(X_t \neq \mathbf{0})^{-1}\mathbf{P}_\mu[(1-e^{-\theta t^{-1}X_t(\phi)})\mathbf{1}_{X\neq\mathbf{0}}] \\ &= \mathbf{P}_\mu(X_t \neq \mathbf{0})^{-1}\mathbf{P}_\mu\Big[(1-e^{-\theta t^{-1}X_t(\phi)})\frac{X_t(\phi)}{X_t(\phi)}\Big] \\ &= (t\mathbf{P}_\mu(X_t \neq \mathbf{0}))^{-1}\langle\mu,\phi\rangle\mathbf{P}_\mu^M\Big[\frac{1-e^{-\theta t^{-1}X_t(\phi)}}{t^{-1}X_t(\phi)}\Big] \\ &\xrightarrow[t\to\infty]{} 1 - \frac{1}{1+c_0\theta},\end{aligned}$$

which completes the proof. □

Now we consider limit of $t^{-1}X_t(f)$ with general weight $f \in bp\mathscr{B}_E^\phi$. The main idea is to use the following decomposition for f: $f(x) = \langle\phi^*, f\rangle_m\phi(x) + \tilde{f}(x), x \in E$. The following lemma says that $\tilde{f}$ has no contribution to the limit, and then we can easily get that the conditional limit of $t^{-1}X_t(f)$ as $t \to \infty$ is the contribution of $\langle\phi^*, f\rangle_m t^{-1}X_t(\phi)$, which is known from Lemma 3.5.6.

Lemma 3.5.7. *Suppose that Assumptions 3.3, 3.4 and 3.5 hold. If $\tilde{f} \in b\mathscr{B}_E^\phi$ satisfies $\langle\tilde{f}, \phi^*\rangle =$*

0, *then we have, for any* $\mu \in \mathcal{M}_f^{\phi}$,

$$\left\{t^{-1}X_t(\tilde{f});\mathbf{P}_\mu(\cdot|X_t\neq\mathbf{0})\right\}\xrightarrow[t\to\infty]{}0,\quad \textit{in probability}.$$

Proof. If we can show that $\mathbf{P}_\mu\left[\left(t^{-1}X_t(\tilde{f})\right)^2|X_t\neq\mathbf{0}\right]\xrightarrow[t\to\infty]{}0$, then the desired result follows by the Chebyshev's inequality

$$\mathbf{P}_\mu\left(|t^{-1}X_t(\tilde{f})|\geq\epsilon\middle|X_t\neq\mathbf{0}\right)\leq\epsilon^{-2}\mathbf{P}_\mu\left[\left(t^{-1}X_t(\tilde{f})\right)^2\middle|X_t\neq\mathbf{0}\right].$$

By Proposition 3.4.2 we have that

$$\begin{aligned}\mathbf{P}_\mu\left[\left(t^{-1}X_t(\tilde{f})\right)^2\middle|X_t\neq\mathbf{0}\right]&=t^{-2}\mathbf{P}_\mu(X_t\neq\mathbf{0})^{-1}\mathbf{P}_\mu\left[X_t(\tilde{f})^2\mathbf{1}_{X_t\neq\mathbf{0}}\right]\\&=t^{-1}\mathbf{P}_\mu(X_t\neq\mathbf{0})^{-1}\Big(\frac{\langle\mu,S_t\tilde{f}\rangle^2}{t}+\langle\mu,\phi\rangle\dot{\Pi}_\mu\left[(\phi^{-1}\tilde{f})(\xi_t)\frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde{f})(\xi_s)ds\right]\Big).\end{aligned}\tag{3.5.33}$$

Letting $c,\gamma>0$ be the constants in (3.1.20), we know that

$$\begin{aligned}|S_t\tilde{f}(x)-\langle\phi^*,\tilde{f}\rangle_m\phi(x)|&=\Big|\int_E\left(q(t,x,y)-\phi(x)\phi^*(y)\right)\tilde{f}(y)m(dy)\Big|\\&\leq\int_E\Big|\frac{q(t,x,y)}{\phi(x)\phi^*(y)}-1\Big|\cdot|\phi(x)\phi^*(y)\tilde{f}(y)|m(dy)\\&\leq ce^{-\gamma t}\phi(x)\|\phi^{-1}\tilde{f}\|_\infty\int_E(\phi\phi^*)(y)m(dy)\\&\xrightarrow[t\to\infty]{}0,\quad x\in E.\end{aligned}\tag{3.5.34}$$

Therefore, by the dominated convergence theorem,

$$\langle\mu,S_t\tilde{f}\rangle\xrightarrow[t\to\infty]{}\langle\phi^*,\tilde{f}\rangle_m\langle\mu,\phi\rangle=0.$$

Hence,

$$\frac{\langle\mu,S_t\tilde{f}\rangle}{t}\xrightarrow[t\to\infty]{}0,\quad x\in E.\tag{3.5.35}$$

By (3.5.34) and Lemma 3.5.1, we know that

$$\begin{aligned}\frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde{f})(\xi_s)ds&=\int_0^1 A(\xi_{ut})\cdot(S_{t-ut}\tilde{f})(\xi_{ut})du\\&\xrightarrow[t\to\infty]{L^2(\dot{\Pi}_x)}\int_0^1\langle A\phi,\phi\phi^*\rangle_m\langle\phi^*,\tilde{f}\rangle_m du=0.\end{aligned}$$

Hence, by Lemma 3.4.1 and the bounded convergence theorem we have that

$$\begin{aligned}&\Big|\langle\mu,\phi\rangle\dot{\Pi}_\mu\left[(\phi^{-1}\tilde{f})(\xi_t)\frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde{f})(\xi_s)ds\right]\Big|\\&\leq\int\mu(dx)\phi(x)\Big|\dot{\Pi}_x\left[(\phi^{-1}\tilde{f})(\xi_t)\frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde{f})(\xi_s)ds\right]\Big|\end{aligned}\tag{3.5.36}$$

$$\leq \|\phi^{-1}\tilde{f}\|_\infty \cdot \int \mu(dx)\phi(x)\dot{\Pi}_x\Big[\Big|\frac{1}{t}\int_0^t A(\xi_s)\cdot(S_{t-s}\tilde{f})(\xi_s)ds\Big|^2\Big]^{\frac{1}{2}}$$

$$\xrightarrow[t\to\infty]{} 0.$$

Finally, using Theorem 3.1.10 and combining (3.5.33), (3.5.35) and (3.5.36), we have that

$$\mathbf{P}_\mu\big[\big(t^{-1}X_t(\tilde{f})\big)^2\big|X_t \neq \mathbf{0}\big] \xrightarrow[t\to\infty]{} 0$$

as required. □

Proof of Theorem 3.1.11. Define a function $\tilde{f}$ by

$$\tilde{f}(x) := f(x) - \langle\phi^*, f\rangle_m \phi(x), \quad x \in E. \tag{3.5.37}$$

It is easy to see that $\tilde{f} \in b\mathscr{B}_E^\phi$ and $\langle\tilde{f}, \phi^*\rangle_m = 0$. It then follows from Lemma 3.5.6 that

$$\big\{t^{-1}X_t(\langle\phi^*, f\rangle_m\phi); \mathbf{P}_\mu(\cdot|X_t \neq \mathbf{0})\big\} \xrightarrow[t\to\infty]{law} \frac{1}{2}\langle\phi^*, f\rangle_m\langle\phi A, \phi\phi^*\rangle_m\mathbf{e}, \tag{3.5.38}$$

and from Lemma 3.5.7 that

$$\big\{t^{-1}X_t(\tilde{f}); \mathbf{P}_\mu(\cdot|X_t \neq \mathbf{0})\big\} \xrightarrow[t\to\infty]{\text{in probability}} 0. \tag{3.5.39}$$

The desired result then follows from (3.5.37), (3.5.38), (3.5.39) and Slutsky's theorem. □

Remark 3.5.8. In the symmetric case, i.e. when (S_t) are self-adjoint operators, (3.5.37) is exactly an L^2-orthogonal decomposition.

Chapter 4 Spine decompositions of non-persistent superprocesses: characteristic functions

4.1 Introduction

4.1.1 Motivation

Consider a general (ξ,ψ)-superprocess $\{(X_t)_{t\geq 0},\mathbf{P}_\mu\}$ in a locally compact separable metric space E. Note that, in the previous chapter, we always take a non-negative testing function f and study the property of $\langle X_t, f\rangle$. In this case, $\langle X_t, f\rangle$ is also non-negative, and therefore its distribution property can be captured by its Laplace transform

$$\mathbf{P}_{\delta_x}[e^{-\theta\langle X_t,f\rangle}],\quad t\geq 0,\theta\geq 0,x\in E.$$

The definition of the superprocess says that the map $(t,x)\mapsto \mathbf{P}_{\delta_x}[e^{-\langle X_t,f\rangle}]$ is a mild solution to a non-linear partial differential equation, see (1.2.2). Therefore, several distributional properties of $\langle X_t, f\rangle$ can be obtained by taking advantage of that equations.

A natural question arises in studying the limiting theory for superprocesses is to consider the property of $\langle X_t, f\rangle$ where f is a Borel measurable function on E which may take both positive and negative values. Note that, in this case, when f is bounded, $\langle X_t, f\rangle$ is a well defined random variable whose Laplace transform may not exists. So we can't use the equation (1.2.2) anymore. Instead, we consider the characteristic function of $\langle X_t, f\rangle$:

$$\mathbf{P}_{\delta_x}[e^{-i\theta\langle X_t,f\rangle}],\quad t\geq 0,x\in E,\theta\geq 0,$$

and ask the question: whether map $(t,x)\mapsto \mathbf{P}_{\delta_x}[e^{i\langle X_t,f\rangle}]$ also satisfies some complex-valued non-linear partial differential equation.

In this chapter, we give a positive answer to this question under a non-persistent assumption. The precise statements of the results and the assumptions are presented in the next subsection. We mention here that our key tool is the general spine decomposition theorem for the superprocesses developed in Chapter 3. This is one of the evidence that the spine theory really captures the distributional properties of the superprocesses.

4.1.2 Main result

Let E be a locally compact separable metric space. Denote by $\mathcal{M}(E)$ the collection of all the finite measures on E equipped with the topology of weak convergence. For each function $F(x,z)$ on $E\times\mathbb{R}_+$ and each $\mathbb{R}_+$-valued function f on E, we use the following convention:

$$F(x,f):=F(x,f(x)),\quad x\in E.$$

A process $X=\{(X_t)_{t\geq 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}(E)}\}$ is said to be a (ξ,ψ)-superprocess if

- the spatial motion $\xi=\{(\xi_t)_{t\geq 0};(\Pi_x)_{x\in E}\}$ is an E-valued Hunt process with its lifetime denoted by ζ;
- the branching mechanism $\psi:E\times[0,\infty)\to\mathbb{R}$ is given by

  $$\psi(x,z)=-\beta(x)z+\alpha(x)z^2+\int_{(0,\infty)}(e^{-zy}-1+zy)\pi(x,dy),$$

  where $\beta\in\mathcal{B}_b(E)$, $\alpha\in\mathcal{B}_b(E,\mathbb{R}_+)$ and $\pi(x,dy)$ is a kernel from E to $(0,\infty)$ such that $\sup_{x\in E}\int_{(0,\infty)}(y\wedge y^2)\pi(x,dy)<\infty$;
- $X=\{(X_t)_{t\geq 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}(E)}\}$ is an $\mathcal{M}(E)$-valued Hunt process with transition probability determined by

  $$\mathbf{P}_\mu[e^{-X_t(f)}]=e^{-\mu(V_tf)},\quad t\geq 0,\mu\in\mathcal{M}(E),f\in\mathcal{B}_b^+(E),$$

  where for each $f\in\mathcal{B}_b(E)$, the function $(t,x)\mapsto V_tf(x)$ on $[0,\infty)\times E$ is the unique locally bounded positive solution to the equation

  $$V_tf(x)+\Pi_x\Big[\int_0^{t\wedge\zeta}\psi(\xi_s,V_{t-s}f)ds\Big]=\Pi_x[f(\xi_t)\mathbf{1}_{t<\zeta}],\quad t\geq 0,x\in E.$$

We refer our readers to [56] for more discussions about the definition and the existence of superprocesses. To avoid triviality, we assume that $\psi(x,z)$ is not identically equal to $-\beta(x)z$.

We say X is non-persistent if $\mathbf{P}_{\delta_x}(\|X_t\|=0)>0$ for all $x\in E$ and $t>0$. In this chapter, we will always assume that our superprocess X is non-persistent.

Let $\mathbb{C}_+:=\{x+iy:x\geq 0,y\in\mathbb{R}\}$ and $\mathbb{C}_+^0:=\{x+iy:x>0,y\in\mathbb{R}\}$. The branching mechanism ψ can be extended into a map from $E\times\mathbb{C}_+$ to $\mathbb{C}$ using Lemma 4.2.2 below in the sense that for each $x\in E$, $z\mapsto\psi(x,z)$ is a holomorphic function on $\mathbb{C}_+^0$ and continuous on $\mathbb{C}_+$. Define

$$\psi'(x,z):=-\beta(x)+2\alpha(x)z+\int_{(0,\infty)}(1-e^{-zy})y\pi(x,dy),\quad x\in E,z\in\mathbb{C}_+.$$

It will be proved in Lemma 4.2.2 below that for each $x\in E$, $z\mapsto\psi(x,z)$ is a holomorphic function on $\mathbb{C}_+^0$ with derivative $z \mapsto \psi'(x,z)$. Write $\psi_0(x,z) := \psi(x,z) + \beta(x)z$ and $\psi_0'(x,z) := \psi'(x,z) + \beta(x)$.

Define

$$\begin{aligned} L_1(\xi) &:= \{f \in \mathcal{B}(E) : \forall x \in E, t \geq 0, \quad \Pi_x[|f(\xi_t)|] < \infty\}, \\ L_2(\xi) &:= \{f \in \mathcal{B}(E) : |f|^2 \in L_1(\xi)\}. \end{aligned}$$

The mean behavior of the superprocess is well known:

$$\mathbf{P}_{\delta_x}[\langle X_t, f\rangle] = P_t^\beta f(x) := \Pi_x[e^{\int_0^t \beta(\xi_s)ds} f(\xi_t)\mathbf{1}_{t<\zeta}] \in \mathbb{R}, \quad f \in L_1(\xi), t \geq 0, x \in E.$$

This also says that the random variable $\langle X_t, f\rangle$ is well defined under probability $\mathbf{P}_{\delta_x}$ provided $f \in L_1(\xi)$. By the branching property of the superprocess, $\langle X_t, f\rangle$ is an infinitely divisible random variable. Therefore, for fixed $x \in E, t \geq 0$ and $f \in L_1(\xi)$, there exists a unique continuous map $\theta \mapsto U_t(\theta f)(x)$ from $\mathbb{R}$ to $\mathbb{C}$ such that $\mapsto U_t(0 \cdot f)(x) = 0$ and

$$e^{U_t(\theta f)(x)} = \mathbf{P}_{\delta_x}[e^{i\theta\langle X_t, f\rangle}].$$

This map is known as the characteristic exponent of the infinitely divisible random variable $\langle X_t, f\rangle$ under probability $\mathbf{P}_{\delta_x}$. See the paragraph immediately after [72, Lemma 7.6].

The main result in this chapter is the following:

Proposition 4.1.1. *If $f \in L_2(\xi)$, then for all $t \geq 0$ and $x \in E$,*

$$U_t f(x) - \Pi_x\Big[\int_0^t \psi\big(\xi_s, -U_{t-s}f\big)ds\Big] = i\Pi_x[f(\xi_t)] \tag{4.1.1}$$

and

$$U_t f(x) - \int_0^t P_{t-s}^\beta \psi_0\big(\cdot, -U_s f\big)(x)\, ds = iP_t^\beta f(x). \tag{4.1.2}$$

4.1.3 Some words before the proofs

We can consider decomposing the general testing function f into its positive and negative parts:

$$f = f^+ - f^-$$

and prove Proposition 4.1.1 for each $f^\pm$ using (1.2.2). However, this strategy will not work while proving Proposition 4.1.1 for f, because the dependence between $\langle X_t, f^+\rangle$ and $\langle X_t, f^-\rangle$ is not clear. (Note in particular, they are not independent.) so we don't know the relation between $(U_t f^+, U_t f^-)$ and $U_t f$.

Instead, our strategy is to use the general spine decomposition theory developed in

Chapter 3. The underlying idea is very simple: since the spine decomposition theory gives a decomposition about the superprocess X_t, it also gives a decomposition of $\langle X_t, f\rangle$. Translating this decomposition for random variable $\langle X_t, f\rangle$ in the language of characteristic functions will give us a complex valued identity. Using that identity we can give a proof of the desired result.

4.2 Preliminary

4.2.1 Some analytic facts

In this subsection, we collect some useful analytic facts.

Lemma 4.2.1. *For $z \in \mathbb{C}_+$, we have*

$$\Big|e^{-z} - \sum_{k=0}^{n} \frac{(-z)^k}{k!}\Big| \leq \frac{|z|^{n+1}}{(n+1)!} \wedge \frac{2|z|^n}{n!}, \quad n \in \mathbb{Z}_+. \tag{4.2.1}$$

Proof. Notice that $|e^{-z}| = e^{-\operatorname{Re} z} \leq 1$. Therefore,

$$|e^{-z} - 1| = \Big|\int_0^1 e^{-\theta z} z d\theta\Big| \leq |z|.$$

Also, notice that $|e^{-z} - 1| \leq |e^{-z}| + 1 \leq 2$. Thus (4.2.1) is true when $n = 0$. Now, suppose that (4.2.1) is true when $n = m$ for some $m \in \mathbb{Z}_+$. Then

$$\begin{aligned} \Big|e^{-z} - \sum_{k=0}^{m+1} \frac{(-z)^k}{k!}\Big| &= \Big|\int_0^1 \Big(e^{-\theta z} - \sum_{k=0}^{m} \frac{(-\theta z)^k}{k!}\Big) z d\theta\Big| \\ &\leq \Big(\int_0^1 \frac{|\theta z|^{m+1}}{(m+1)!} |z| d\theta\Big) \wedge \Big(\int_0^1 \frac{2|\theta z|^m}{m!} |z| d\theta\Big) = \frac{|z|^{m+2}}{(m+2)!} \wedge \frac{2|z|^{m+1}}{(m+1)!}, \end{aligned}$$

which says that (4.2.1) is true for $n = m + 1$. □

Lemma 4.2.2. *Suppose that π is a measure on $(0, \infty)$ with $\int_{(0,\infty)} (y \wedge y^2) \pi(dy) < \infty$. Then the functions*

$$h(z) = \int_{(0,\infty)} (e^{-zy} - 1 + zy) \pi(dy), \quad z \in \mathbb{C}_+$$

and

$$h'(z) = \int_{(0,\infty)} (1 - e^{-zy}) y \pi(dy), \quad z \in \mathbb{C}_+$$

are well defined, continuous on $\mathbb{C}_+$ and holomorphic on $\mathbb{C}_+^0$. Moreover,

$$\frac{h(z) - h(z_0)}{z - z_0} \xrightarrow[\mathbb{C}_+ \ni z \to z_0]{} h'(z_0), \quad z_0 \in \mathbb{C}_+.$$

Proof. It follows from Lemma 4.2.1 that h and h' are well defined on $\mathbb{C}_+$. According to [74, Theorems 3.2. & Proposition 3.6], h' is continuous on $\mathbb{C}_+$ and holomorphic on $\mathbb{C}_+^0$.

It follows from Lemma 4.2.1 that, for each $z_0 \in \mathbb{C}_+$, there exists $C > 0$ such that for $z \in \mathbb{C}_+$ close enough to z_0 and any $y > 0$,

$$\begin{aligned}&\Big|\frac{e^{-zy} - e^{-z_0 y} + (z - z_0)y}{z - z_0}\Big| = \frac{1}{|z - z_0|}\Big|\int_0^1 \big(- ye^{-(\theta z+(1-\theta)z_0)y} + y\big)(z - z_0)d\theta\Big|\\ &\leq y\int_0^1 |1 - e^{-(\theta z+(1-\theta)z_0)y}|d\theta \leq (2y) \wedge \Big(y^2 \int_0^1 |\theta z + (1 - \theta)z_0|d\theta\Big) \leq C(y \wedge y^2).\end{aligned}$$

Using this and the dominated convergence theorem, we have

$$\begin{aligned}\frac{h(z) - h(z_0)}{z - z_0} &= \int_{(0,\infty)} \frac{e^{-zy} + zy - (e^{-z_0 y} + z_0 y)}{z - z_0}\pi(dy)\\ &\xrightarrow[\mathbb{C}_+\ni z\to z_0]{} \int_{(0,\infty)} (1 - e^{-z_0 y})y\pi(dy) = h'(z_0),\end{aligned}$$

which says that h is continuous on $\mathbb{C}_+$ and holomorphic on $\mathbb{C}_+^0$. □

For each $z \in \mathbb{C} \setminus (-\infty,0]$, we define $\log z := \log|z| + i\arg z$ where $\arg z \in (-\pi,\pi)$ is uniquely determined by $z = |z|e^{i\arg z}$. For all $z \in \mathbb{C} \setminus (-\infty,0]$ and $\gamma \in \mathbb{C}$, we define $z^\gamma := e^{\gamma \log z}$. Then it is known, see [77, Theorem 6.1] for example, that $z \mapsto \log z$ is holomorphic in $\mathbb{C} \setminus (-\infty,0]$. Therefore, for each $\gamma \in \mathbb{C}$, $z \mapsto z^\gamma$ is holomorphic in $\mathbb{C} \setminus (-\infty,0]$. (We use the convention that $0^\gamma := \mathbf{1}_{\gamma=0}$.) Using the definition above we can easily show that $(z_1 z_0)^\gamma = z_1^\gamma z_0^\gamma$ provided $\arg(z_1 z_0) = \arg(z_1) + \arg(z_0)$.

Recall that the Gamma function Γ is defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.$$

It is known, see, for instance, [77, Theorem 6.1.3] and the remark following it, that the function Γ has an unique analytic extension in $\mathbb{C} \setminus \{0,-1,-2,\dots\}$ and that

$$\Gamma(z + 1) = z\Gamma(z), \quad z \in \mathbb{C} \setminus \{0,-1,-2,\dots\}.$$

Using this recursively, one gets that

$$\Gamma(x) := \int_0^\infty t^{x-1}\Big(e^{-t} - \sum_{k=0}^{n-1} \frac{(-t)^k}{k!}\Big)dt, \quad -n < x < -n + 1, n \in \mathbb{N}.$$

Fix a $\beta \in (0,1)$. Using the uniqueness of holomorphic extension and Lemma 4.2.2, we get that

$$z^\beta = \int_0^\infty (e^{-zy} - 1)\frac{dy}{\Gamma(-\beta)y^{1+\beta}}, \quad z \in \mathbb{C}_+,$$

by showing that the both sides

- are extension of the real function $x \mapsto x^\beta$ defined on $[0,\infty)$;

- are holomorphic on $\mathbb{C}_+^0$;
- are continuous on $\mathbb{C}_+$.

Similarly, we get that

$$z^{1+\beta} = \int_0^\infty (e^{-zy} - 1 + zy)\frac{dy}{\Gamma(-1-\beta)y^{2+\beta}}, \quad z \in \mathbb{C}_+.$$

Lemma 4.2.2 also says that the derivative of $z^{1+\beta}$ is $(1+\beta)z^\beta$ on $\mathbb{C}_+^0$.

Lemma 4.2.3. *For all* $z_0, z_1 \in \mathbb{C}_+$*, we have*

$$|z_0^{1+\beta} - z_1^{1+\beta}| \le (1+\beta)(|z_0|^\beta + |z_1|^\beta)|z_0 - z_1|.$$

Proof. Since $z^{1+\beta}$ is continuous on $\mathbb{C}_+$, we only need to prove the lemma assuming $z_0, z_1 \in \mathbb{C}_+^0$. Notice that

$$|z^\beta| = |e^{\beta \log|z| + i\beta \arg z}| = e^{\beta \log|z|} = |z|^\beta, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

Define a path $\gamma : [0,1] \to \mathbb{C}_+^0$ such that

$$\gamma(\theta) = z_0(1-\theta) + \theta z_1, \quad \theta \in [0,1].$$

Then, we have

$$\begin{aligned}|z_0^{1+\beta} - z_1^{1+\beta}| &\le (1+\beta)\int_0^1 |\gamma(\theta)^\beta| \cdot |\gamma'(\theta)| d\theta \le (1+\beta) \sup_{\theta\in[0,1]} |\gamma(\theta)|^\beta \cdot |z_1 - z_0| \\ &\le (1+\beta)(|z_1|^\beta + |z_0|^\beta)|z_1 - z_0|.\end{aligned}$$

□

Suppose that $\varphi(\theta)$ is a continuous function from $\mathbb{R}$ into $\mathbb{C}$ such that $\varphi(0) = 1$ and $\varphi(\theta) \neq 0$ for all $\theta \in \mathbb{R}$. Then according to [72, Lemma 7.6], there is a unique continuous function $f(\theta)$ from $\mathbb{R}$ into $\mathbb{C}$ such that $f(0) = 0$ and $e^{f(\theta)} = \varphi(\theta)$. Such a function f is called the distinguished logarithm of the function φ and is denoted as $\operatorname{Log}\varphi(\theta)$. In particular, when φ is the characteristic function of an infinitely divisible random variable Y, $\operatorname{Log}\varphi(\theta)$ is called the Lévy exponent of Y. This distinguished logarithm should not be confused with the log function defined on $\mathbb{C} \setminus (-\infty, 0]$. See the paragraph immediately after [72, Lemma 7.6].

4.2.2 Feynman-Kac formula with complex valued functions

In this subsection we give a version of the Feynman-Kac formula with complex valued functions. Suppose that $\{(\xi_t)_{t\in[r,\infty)}; (\Pi_{r,x})_{r\in[0,\infty), x\in E}\}$ is a (possibly non-homogeneous) Hunt process in a locally compact separable metric space E. We write

$$H^{(h)}_{(s,t)} := \exp\Big\{\int_s^t h(u,\xi_u)du\Big\},\quad 0\le s\le t, h\in\mathcal{B}_b([0,t]\times E,\mathbb{C}).$$

Lemma 4.2.4. *Let $t\ge 0$. Suppose that $\beta,\alpha\in\mathcal{B}_b([0,t]\times E,\mathbb{C})$ and $f\in\mathcal{B}_b(E,\mathbb{C})$. Then*

$$g(r,x) := \Pi_{r,x}[H^{(\beta+\alpha)}_{(r,t)} f(\xi_t)],\quad r\in[0,t], x\in E, \tag{4.2.2}$$

is the unique locally bounded solution to the equation

$$g(r,x) = \Pi_{r,x}[H^{(\beta)}_{(r,t)} f(\xi_t)] + \Pi_{r,x}\Big[\int_r^t H^{(\beta)}_{(r,s)}\alpha(s,\xi_s)g(s,\xi_s)\,ds\Big],\quad r\in[0,t], x\in E.$$

Proof. The proof is similar to that of [23, Lemma A.1.5]. We include it here for the sake of completeness. We first verify that (4.2.2) is a solution. Notice that

$$\Pi_{r,x}\Big[\int_r^t |H^{(\beta)}_{(r,t)}\alpha(s,\xi_s)H^{(\alpha)}_{(s,t)}f(\xi_t)|\,ds\Big] \le \int_r^t e^{(t-r)\|\beta\|_\infty}e^{(t-s)\|\alpha\|_\infty}\|\alpha\|_\infty\|f\|_\infty\,ds < \infty.$$

Also notice that

$$\frac{\partial}{\partial s}H^{(\alpha)}_{(s,t)} = -H^{(\alpha)}_{(s,t)}\alpha(s,\xi_s),\quad s\in(0,t).$$

Therefore, from the Markov property of ξ and Fubini's theorem we get that

$$\begin{aligned}
&\Pi_{r,x}\Big[\int_r^t H^{(\beta)}_{(r,s)}\,(\alpha g)(s,\xi_s)\,ds\Big] = \Pi_{r,x}\Big[\int_r^t H^{(\beta)}_{(r,s)}\alpha(s,\xi_s)\Pi_{s,\xi_s}[H^{(\beta+\alpha)}_{(s,t)}f(\xi_t)]\,ds\Big]\\
&= \Pi_{r,x}\Big[\int_r^t H^{(\beta)}_{(r,t)}\alpha(s,\xi_s)H^{(\alpha)}_{(s,t)}f(\xi_t)\,ds\Big] = \Pi_{r,x}[H^{(\beta)}_{(r,t)}f(\xi_t)(H^{(\alpha)}_{(r,t)}-1)]\\
&= g(r,x) - \Pi_{r,x}[H^{(\alpha)}_{(r,t)}f(\xi_t)].
\end{aligned}$$

For uniqueness, suppose $\tilde g$ is another solution. Put $h(r)=\sup_{x\in E}|g(r,x)-\tilde g(r,x)|$. Then

$$h(r)\le e^{t\|\beta\|_\infty}\|\alpha\|_\infty\int_r^t h(s)ds,\quad r\le t.$$

Applying Gronwall's inequality, we get that $h(r)=0$ for $r\in[0,t]$. □

4.2.3 Kuznetsov measure

Denote by $\mathbb{W}$ the space of $\mathcal{M}(E)$-valued càdlàg paths with its canonical path denoted by $(W_t)_{t\ge0}$. We say our superprocess X is *non-persistent* if $\mathbf{P}_{\delta_x}(\|X_t\|=0)>0$ for all $x\in E$ and $t>0$. Suppose that $(X_t)_{t\ge0}$ is non-persistent, then according to [56, Section 8.4], there is a unique family of measures $(\mathbb{N}_x)_{x\in E}$ on $\mathbb{W}$ such that

- $\mathbb{N}_x(\forall t>0, \|W_t\|=0)=0$;
- $\mathbb{N}_x(\|W_0\|\neq 0)=0$;

- for any $\mu \in \mathcal{M}(E)$, if $\mathcal{N}$ is a Poisson random measure defined on some probability space with intensity $\mathbb{N}_\mu(\cdot) := \int_E \mathbb{N}_x(\cdot)\mu(dx)$, then the superprocess $\{X; \mathbf{P}_\mu\}$ can be realized by $\widetilde{X}_0 := \mu$ and $\widetilde{X}_t(\cdot) := \mathcal{N}[W_t(\cdot)]$ for each $t > 0$.

We refer to $(\mathbb{N}_x)_{x\in E}$ as the *Kuznetsov measures* of X.

4.2.4 Semigroups for superprocesses

Let X be a non-persistent superprocess with its Kuznetsov measure denoted by $(\mathbb{N}_x)_{x\in E}$. We define the mean semigroup

$$P_t^\beta f(x) := \Pi_x[e^{\int_0^t \beta(\xi_s)ds} f(\xi_t)\mathbf{1}_{t<\zeta}], \quad t \geq 0, x \in E, f \in \mathcal{B}_b(E, \mathbb{R}_+).$$

It is known from [56, Proposition 2.27] and [49, Theorem 2.7] that for all $t > 0$, $\mu \in \mathcal{M}(E)$ and $f \in \mathcal{B}_b(E, \mathbb{R}_+)$,

$$\mathbb{N}_\mu[\langle W_t, f\rangle] = \mathbf{P}_\mu[\langle X_t, f\rangle] = \mu(P_t^\beta f). \tag{4.2.3}$$

Define

$$L_1(\xi) := \{f \in \mathcal{B}(E) : \forall x \in E, t \geq 0, \quad \Pi_x[|f(\xi_t)|] < \infty\},$$
$$L_2(\xi) := \{f \in \mathcal{B}(E) : |f|^2 \in L_1(\xi)\}.$$

Using monotonicity and linearity, we get from (4.2.3) that

$$\mathbb{N}_x[\langle W_t, f\rangle] = \mathbf{P}_{\delta_x}[\langle X_t, f\rangle] = P_t^\beta f(x) \in \mathbb{R}, \quad f \in L_1(\xi), t > 0, x \in E.$$

This says that the random variable $\langle X_t, f\rangle$ is well defined under probability $\mathbf{P}_{\delta_x}$ provided $f \in L_1(\xi)$. By the branching property of the superprocess, $\langle X_t, f\rangle$ is an infinitely divisible random variable. Therefore, we can write

$$U_t(\theta f)(x) := \operatorname{Log} \mathbf{P}_{\delta_x}[e^{i\theta\langle X_t, f\rangle}], \quad t \geq 0, f \in L_1(\xi), \theta \in \mathbb{R}, x \in E,$$

as its characteristic exponent. According to Campbell's formula, see [49, Theorem 2.7] for example, we have

$$\mathbf{P}_{\delta_x}[e^{i\theta\langle X_t, f\rangle}] = \exp(\mathbb{N}_x[e^{i\theta\langle W_t, f\rangle} - 1]), \quad t > 0, f \in L_1(\xi), \theta \in \mathbb{R}, x \in E.$$

Noticing that $\theta \mapsto \mathbb{N}_x[e^{i\theta W_t(f)} - 1]$ is a continuous function on $\mathbb{R}$ and that $\mathbb{N}_x[e^{i\theta\langle W_t, f\rangle} - 1] = 0$ if $\theta = 0$, according to [72, Lemma 7.6], we have

$$U_t(\theta f)(x) = \mathbb{N}_x[e^{i\langle W_t, \theta f\rangle} - 1], \quad t > 0, f \in L_1(\xi), \theta \in \mathbb{R}, x \in E. \tag{4.2.4}$$

Lemma 4.2.5. *There exists constants $C \geq 0$ such that for all $f \in L_1(\xi), x \in E$ and $t \geq 0$, we* *have*

$$\left|\psi\left(x, -U_t f\right)\right| \le C P_t^\beta |f|(x) + C(P_t^\beta |f|(x))^2. \tag{4.2.5}$$

Proof. Noticing that

$$e^{\operatorname{Re} U_t f(x)} = |e^{U_t f(x)}| = |\mathbf{P}_{\delta_x}[e^{i\langle X_t, f\rangle}]| \le 1,$$

we have

$$\operatorname{Re} U_t f(x) \le 0. \tag{4.2.6}$$

Therefore, we can speak of $\psi(x, -U_t f)$ since $z \mapsto \psi(x, z)$ is well defined on $\mathbb{C}_+$. According to Lemma 4.2.1, we have that

$$|U_t f(x)| \le \mathbb{N}_x[|e^{i\langle W_t, f\rangle} - 1|] \le \mathbb{N}_x[|i\langle W_t, f\rangle|] \le (P_t^\beta |f|)(x).$$

Notice that, for any compact $K \subset \mathbb{R}$,

$$\mathbb{N}_x\Big[\sup_{\theta\in K}\Big|\frac{\partial}{\partial\theta}(e^{i\theta\langle W_t, f\rangle} - 1)\Big|\Big] \le \mathbb{N}_x[|\langle W_t, f\rangle|]\sup_{\theta\in K}|\theta| \le (P_t^\beta|f|)(x)\sup_{\theta\in K}|\theta| < \infty.$$

Therefore, according to [20, Theorem A.5.2] and (4.2.4), $U_t(\theta f)(x)$ is differentiable in $\theta \in \mathbb{R}$ with

$$\frac{\partial}{\partial\theta}U_t(\theta f)(x) = i\mathbb{N}_x[\langle W_t, f\rangle e^{i\theta\langle W_t, f\rangle}], \quad \theta \in \mathbb{R}.$$

Moreover, from the above, it is clear that

$$\sup_{\theta\in\mathbb{R}}\Big|\frac{\partial}{\partial\theta}U_t(\theta f)(x)\Big| \le (P_t^\beta |f|)(x). \tag{4.2.7}$$

It follows from the dominated convergence theorem that $(\partial/\partial\theta)U_t(\theta f)(x)$ is continuous in θ. In other words, $\theta \mapsto -U_t(\theta f)(x)$ is a C^1 map from $\mathbb{R}$ to $\mathbb{C}_+$. Thus,

$$\psi(x, -U_t f) = -\int_0^1 \psi'\big(x, -U_t(\theta f)\big)\frac{\partial}{\partial\theta}U_t(\theta f)(x)\, d\theta. \tag{4.2.8}$$

Notice that

$$\begin{aligned}
&|\psi'(x, -U_t f)| \qquad (4.2.9)\\
&= \Big| -\beta(x) - 2\alpha(x)U_t f(x) + \int_{(0,\infty)} y(1 - e^{yU_t f(x)})\pi(x, dy)\Big|\\
&= \Big| -\beta(x) - 2\alpha(x)\mathbb{N}_x[e^{i\langle W_t, f\rangle} - 1] + \int_{(0,\infty)} y\mathbf{P}_{y\delta_x}[1 - e^{i\langle X_t, f\rangle}]\pi(x, dy)\Big|\\
&\le \|\beta\|_\infty + 2\alpha(x)\mathbb{N}_x[\langle W_t, |f|\rangle] + \int_{(0,\infty)} y\mathbf{P}_{y\delta_x}[2 \wedge \langle X_t, |f|\rangle]\pi(x, dy)
\end{aligned}$$

$$\le \|\beta\|_\infty + 2\|\alpha\|_\infty P_t^\beta|f|(x) + \Big(\sup_{x\in E}\int_{(0,1]} y^2\pi(x,dy)\Big) P_t^\beta|f|(x) + 2\sup_{x\in E}\int_{(1,\infty)} y\pi(x,dy)$$
$$=: C_1 + C_2(P_t^\beta|f|)(x),$$

where C_1, C_2 are constants independent on f, x and t. Now, combining (4.2.8), (4.2.7) and (4.2.9), we get the desired result. □

This lemma also says that if $f \in L^2(\xi)$ then

$$\Pi_x\Big[\int_0^t \psi(\xi_s, -U_{t-s}f)ds\Big] \in \mathbb{C}, \quad x \in E, t \ge 0.$$

is well defined. In fact, using Jensen's inequality and the Markov property, we have

$$\begin{aligned}
&\Pi_x\Big[\int_0^t \big|\psi\big(\xi_s, -U_{t-s}f\big)\big|ds\Big] \qquad (4.2.10)\\
&\le \Pi_x\Big[\int_0^t \big(C_1P_{t-s}^\beta|f|(\xi_s) + C_2P_{t-s}^\beta|f|(\xi_s)^2\big)ds\Big]\\
&\le \int_0^t \big(C_1e^{t\|\beta\|}\Pi_x\big[\Pi_{\xi_s}[|f(\xi_{t-s})|]\big] + C_2e^{2t\|\beta\|}\Pi_x\big[\Pi_{\xi_s}[|f(\xi_{t-s})|]^2\big]\big)\, ds\\
&\le \int_0^t (C_1e^{t\|\beta\|}\Pi_x[|f(\xi_t)|] + C_2e^{2t\|\beta\|}\Pi_x[|f(\xi_t)|^2])\, ds < \infty.
\end{aligned}$$

4.3 Proof of the main result

To prove Proposition 4.1.1, we will need the generalized spine decomposition theorem from [65]. Let $f \in \mathcal{B}_b(E, \mathbb{R}_+)$, $T > 0$ and $x \in E$. Suppose that $\mathbf{P}_{\delta_x}[\langle X_T, f\rangle] = \mathbb{N}_x[\langle W_T, f\rangle] = P_T^\beta f(x) \in (0,\infty)$, then we can define the following probability transforms:

$$d\mathbf{P}_{\delta_x}^{\langle X_T,f\rangle} := \frac{\langle X_T, f\rangle}{P_T^\beta f(x)} d\mathbf{P}_{\delta_x}; \quad d\mathbb{N}_x^{\langle W_T,f\rangle} := \frac{\langle W_T, f\rangle}{P_T^\beta f(x)} d\mathbb{N}_x.$$

Following the definition in Chapter 3, we say that $\{\xi, \mathbf{n}; \mathbf{Q}_x^{(f,T)}\}$ is a spine representation of $\mathbb{N}_x^{\langle W_T,f\rangle}$ if

- the spine process $\{(\xi_t)_{0\le t\le T}; \mathbf{Q}_x^{(f,T)}\}$ is a copy of $\{(\xi_t)_{0\le t\le T}; \Pi_x^{(f,T)}\}$, where

$$d\Pi_x^{(f,T)} := \frac{f(\xi_T)e^{\int_0^T \beta(\xi_s)ds}}{P_T^\beta f(x)} d\Pi_x;$$

- given $\{(\xi_t)_{0\le t\le T}; \mathbf{Q}_x^{(f,T)}\}$, the immigration measure

$$\{\mathbf{n}(\xi, ds, dw); \mathbf{Q}_x^{(f,T)}[\cdot|(\xi_t)_{0\le t\le T}]\}$$

is a Poisson random measure on $[0,T]\times\mathbb{W}$ with intensity

$$\mathbf{m}(\xi,ds,dw):=2\alpha(\xi_s)ds\cdot\mathbb{N}_{\xi_s}(dw)+ds\cdot\int_{y\in(0,\infty)}y\mathbf{P}_{y\delta_{\xi_s}}(X\in dw)\pi(\xi_s,dy);\quad(4.3.1)$$

- $\{(Y_t)_{0\le t\le T};\mathbf{Q}_x^{(f,T)}\}$ is an $\mathcal{M}(E)$-valued process defined by

$$Y_t:=\int_{(0,t]\times\mathbb{W}}w_{t-s}\mathbf{n}(\xi,ds,dw),\quad 0\le t\le T.$$

According to the spine decomposition theorem in [65], we have that

$$\{(X_s)_{s\ge0};\mathbf{P}_{\delta_x}^{\langle X_T,f\rangle}\}\overset{f.d.d.}{=}\{(X_s+W_s)_{s\ge0};\mathbf{P}_{\delta_x}\otimes\mathbb{N}_x^{\langle W_T,f\rangle}\}\qquad(4.3.2)$$

and

$$\{(W_s)_{0\le s\le T};\mathbb{N}_x^{\langle W_T,f\rangle}\}\overset{f.d.d.}{=}\{(Y_s)_{s\ge0};\mathbf{Q}_x^{(f,T)}\}.\qquad(4.3.3)$$

Proof of Proposition 4.1.1. Assume that $f\in\mathcal{B}_b(E)$. Fix $t>0,r\in[0,t),x\in E$ and a strictly positive $g\in\mathcal{B}_b(E)$. Denote by $\{\xi,\mathbf{n};\mathbf{Q}_x^{(g,t)}\}$ the spine representation of $\mathbb{N}_x^{\langle W_t,g\rangle}$. Conditioned on $\{\xi;\mathbf{Q}_x^{(g,t)}\}$, denote by $\mathbf{m}(\xi,ds,dw)$ the conditional intensity of $\mathbf{n}$ in (4.3.1). Denote by $\Pi_{r,x}$ the probability of Hunt process $\{\xi;\Pi\}$ initiated at time r and position x. From Lemma 4.2.1, we have $\mathbf{Q}_x^{(g,t)}$-almost surely

$$\begin{aligned}&\int_{[0,t]\times\mathbb{W}}|e^{i\langle w_{t-s},f\rangle}-1|\mathbf{m}(\xi,ds,dw)\le\int_{[0,t]\times\mathbb{W}}\big(|\langle w_{t-s},f\rangle|\wedge2\big)\mathbf{m}(\xi,ds,dw)\\&\le\int_0^t\Big(2\alpha(\xi_s)\mathbb{N}_{\xi_s}\big(\langle W_{t-s},|f|\rangle\big)+\int_{(0,1]}y\mathbf{P}_{y\delta_{\xi_s}}[\langle X_{t-s},|f|\rangle]\pi(\xi_s,dy)\\&\qquad+2\int_{(1,\infty)}y\pi(\xi_s,dy)\Big)ds\\&\le\int_0^t(P^\beta_{t-s}|f|)(\xi_s)\Big(2\alpha(\xi_s)+\int_{(0,1]}y^2\pi(\xi_s,dy)\Big)ds+2t\sup_{x\in E}\int_{(1,\infty)}y\pi(x,dy)\\&\le\Big(2\|\alpha\|_\infty+\sup_{x\in E}\int_{(0,1]}y^2\pi(x,dy)\Big)te^{t\|\beta\|_\infty}\|f\|_\infty+2t\sup_{x\in E}\int_{(1,\infty)}y\pi(x,dy)<\infty.\end{aligned}$$

Using this, Fubini's theorem, (4.2.4) and (4.2.6) we have $\mathbf{Q}_x^{(g,t)}$-almost surely,

$$\begin{aligned}&\int_{[0,t]\times\mathbb{N}}(e^{i\langle w_{t-s},f\rangle}-1)\mathbf{m}(\xi,ds,dw)\\&=\int_0^t\Big(2\alpha(\xi_s)\mathbb{N}_{\xi_s}(e^{i\langle W_{t-s},f\rangle}-1)+\int_{(0,\infty)}y\mathbf{P}_{y\delta_{\xi_s}}[e^{i\langle X_{t-s},f\rangle}-1]\pi(\xi_s,dy)\Big)ds\\&=\int_0^t\Big(2\alpha(\xi_s)U_{t-s}f(\xi_s)+\int_{(0,\infty)}y(e^{yU_{t-s}f(\xi_s)}-1)\pi(\xi_s,dy)\Big)ds\\&=-\int_0^t\psi_0'\big(\xi_s,-U_{t-s}f\big)ds.\end{aligned}$$

Therefore, according to (4.3.3), Campbell's formula and above, we have that

$$\begin{aligned}\mathbb{N}_x^{\langle W_t,g\rangle}[e^{i\langle W_t,f\rangle}] &= \mathbf{Q}_x^{(g,t)}\Big[\exp\Big\{\int_{[0,t]\times\mathbb{N}}(e^{i\langle w_{t-s},f\rangle}-1)\mathbf{m}(\xi,ds,dw)\Big\}\Big] \\ &= \Pi_x^{(g,t)}[e^{-\int_0^t \psi_0'(\xi_s,-U_{t-s}f)ds}] \\ &= \frac{1}{P_t^\beta g(x)}\Pi_x[g(\xi_t)e^{-\int_0^t \psi'(\xi_s,-U_{t-s}f)ds}].\end{aligned} \tag{4.3.4}$$

Let $\epsilon > 0$. Define $f^+ = (f\vee 0)+\epsilon$ and $f^- = (-f)\vee 0+\epsilon$, then $f^\pm$ are strictly positive and $f = f^+ - f^-$. According to (4.3.2), we have that

$$\frac{\mathbf{P}_{\delta_x}[\langle X_t,f^\pm\rangle e^{i\langle X_t,f\rangle}]}{\mathbf{P}_{\delta_x}[\langle X_t,f^\pm\rangle]} = \mathbf{P}_{\delta_x}[e^{i\langle X_t,f\rangle}]\mathbb{N}_x^{\langle W_t,f^\pm\rangle}[e^{i\langle X_t,f\rangle}].$$

Using (4.3.4) and the above, we have

$$\begin{aligned}\frac{\mathbf{P}_{\delta_x}[\langle X_t,f\rangle e^{i\langle X_t,f\rangle}]}{\mathbf{P}_{\delta_x}[e^{i\langle X_t,f\rangle}]} &= \mathbf{P}_{\delta_x}[\langle X_t,f^+\rangle]\mathbb{N}_x^{\langle W_t,f^+\rangle}[e^{i\langle X_t,f\rangle}] - \mathbf{P}_{\delta_x}[\langle X_t,f^-\rangle]\mathbb{N}_x^{\langle W_t,f^-\rangle}[e^{i\langle X_t,f\rangle}] \\ &= \Pi_x[f(\xi_t)e^{-\int_0^t \psi'(\xi_s,-U_{t-s}f)ds}].\end{aligned}$$

Therefore, we have

$$\frac{\partial}{\partial\theta}U_t(\theta f)(x) = \frac{\mathbf{P}_{\delta_x}[i\langle X_t,f\rangle e^{i\langle X_t,f\rangle}]}{\mathbf{P}_{\delta_x}[e^{i\langle X_t,f\rangle}]} = \Pi_x[if(\xi_t)e^{-\int_0^t \psi'(\xi_s,-U_{t-s}(\theta f))ds}].$$

Since $\{(\xi_{r+t})_{t\geq 0};\Pi_{r,x}\} \overset{d}{=} \{(\xi_t)_{t\geq 0};\Pi_x\}$, we have

$$\begin{aligned}&\frac{\partial}{\partial\theta}U_{t-r}(\theta f)(x) = \Pi_x[if(\xi_{t-r})e^{-\int_0^{t-r} \psi'(\xi_s,-U_{t-r-s}(\theta f))ds}] \\ &= \Pi_{r,x}[if(\xi_t)e^{-\int_0^{t-r} \psi'(\xi_{r+s},-U_{t-r-s}(\theta f))ds}] = \Pi_{r,x}[if(\xi_t)e^{-\int_r^{t} \psi'(\xi_s,-U_{t-s}(\theta f))ds}].\end{aligned}$$

From (4.2.9), we know that for each $\theta\in\mathbb{R}$, $(t,x)\mapsto|\psi'(x,-U_tf(x))|$ is locally bounded (i.e. bounded on $[0,T]\times E$ for each $T\geq 0$). Therefore, we can apply Lemma 4.2.4 and get that

$$\frac{\partial}{\partial\theta}U_{t-r}(\theta f)(x) + \Pi_{r,x}\Big[\int_r^t \psi'\big(\xi_s,-U_{t-s}(\theta f)\big)\frac{\partial}{\partial\theta}U_{t-s}(\theta f)(\xi_s)\,ds\Big] = \Pi_{r,x}[if(\xi_t)]$$

and

$$\begin{aligned}&\frac{\partial}{\partial\theta}U_{t-r}(\theta f)(x) + \Pi_{r,x}\Big[\int_r^t e^{\int_r^s \beta(\xi_u)du}\psi_0'\big(\xi_s,-U_{t-s}(\theta f)\big)\frac{\partial}{\partial\theta}U_{t-s}(\theta f)(\xi_s)\,ds\Big] \\ &= \Pi_{r,x}[ie^{\int_r^t \beta(\xi_s)ds}f(\xi_t)].\end{aligned}$$

Integrating the two displays above with respect to θ on [0,1], using (4.2.8), (4.2.9), (4.2.7) and

Fubini's theorem, we get

$$U_{t-r}f(x) - \Pi_{r,x}\Big[\int_r^t \psi\big(\xi_s, -U_{t-s}f\big)\, ds\Big] = i\theta\Pi_{r,x}[f(\xi_t)]$$

and

$$U_{t-r}f(x) - \Pi_{r,x}\Big[\int_r^t e^{\int_r^s \beta(\xi_u)du}\psi_0\big(\xi_s, -U_{t-s}f\big)\, ds\Big] = i\Pi_{r,x}[e^{\int_r^t \beta(\xi_u)du}f(\xi_t)].$$

Taking $r = 0$, we get that (4.1.1) and (4.1.2) are true if $f \in \mathcal{B}_b(E)$.

The rest of the proof is to evaluate (4.1.1) and (4.1.2) for all $f \in L_2(\xi)$. We only do this for (4.1.1) since the argument for (4.1.2) is similar. Let $n \in \mathbb{N}$. Writing $f_n := (f^+ \wedge n) - (f^- \wedge n)$, then $f_n \xrightarrow[n\to\infty]{} f$ pointwise. From what we have proved, we have

$$U_t f_n(x) - \Pi_x\Big[\int_0^t \psi\big(\xi_s, -U_{t-s}f_n\big)\, ds\Big] = i\Pi_x[f_n(\xi_t)]. \tag{4.3.5}$$

Notice the following statements are true.

- It is clear that $\Pi_x[f_n(\xi_t)] \xrightarrow[n\to\infty]{} \Pi_x[f(\xi_t)]$.
- $U_t f_n(x) \xrightarrow[n\to\infty]{} U_t f(x)$ due to (4.2.4), the dominated convergence theorem and the fact that

$$|e^{iW_t(f_n)} - 1| \le \langle W_t, |f|\rangle; \quad \mathbb{N}_x[\langle W_t, |f|\rangle] = (P_t^\beta|f|)(x) < \infty.$$

- $\Pi_x[\int_0^t \psi(\xi_s, -U_{t-s}f_n)ds] \xrightarrow[n\to\infty]{} \Pi_x[\int_0^t \psi(\xi_s, -U_{t-s}f)ds]$ due to the dominated convergence theorem, (4.2.10) and the fact (see (4.2.5)) that

$$\big|\psi(\xi_s, -U_{t-s}f_n)\big| \le C_1P_{t-s}^\beta|f|(\xi_s) + C_2P_{t-s}^\beta|f|(\xi_s)^2.$$

Using the above arguments, letting $n \to \infty$ in (4.3.5), we get the desired result. □

Chapter 5 Spine decomposition of critical superprocesses: Slack type result

5.1 Introduction

5.1.1 Background

As mentioned in Chapter 1, it is well known that for a critical Galton-Watson process $\{(Z_n)_{n\geq 0}; P\}$, we have

$$nP(Z_n > 0) \xrightarrow[n\to\infty]{} \frac{2}{\sigma^2} \tag{5.1.1}$$

and

$$\left\{\frac{Z_n}{n}; P(\cdot|Z_n > 0)\right\} \xrightarrow[n\to\infty]{\text{law}} \frac{\sigma^2}{2}\mathbf{e}, \tag{5.1.2}$$

where σ^2 is the variance of the offspring distribution and $\mathbf{e}$ is an exponential random variable with mean 1. The result (5.1.1) is due to Kolmogorov [48], and the result (5.1.2) is due to Yaglom [81]. For further references to these results, see [38, 46]. Since then, lots of analogous results have been obtained for more general critical branching processes with finite 2nd moment, see [4, 6, 5, 44] for example.

Notice that (5.1.1) and (5.1.2) are still valid when $\sigma^2 = \infty$, see [46] for example. In this case, the limits in (5.1.1) and (5.1.2) are degenerate, and thus more appropriate scalings are needed. Research in this direction was first conducted by Zolotarev [82] in a simplified continuous time set-up, which is then extended by Slack [75] to discrete time critical Galton-Watson processes allowing infinite variance. The main result of [75] can be stated as follows. Consider a critical Galton-Watson process $\{(Z_n)_{n\geq 0}; P\}$ with infinite variance. Assume that the generating function $f(s)$ of the offspring distribution is of the form

$$f(s) = s + (1-s)^{1+\alpha} l(1-s), \quad s \geq 0, \tag{5.1.3}$$

where $\alpha \in (0, 1]$ and l is a function slowly varying at 0. Then

$$P(Z_n > 0) = n^{-1/\alpha} L(n), \tag{5.1.4}$$

where L is a function slowly varying at ∞, and

$$\left\{P(Z_n > 0)Z_n; P(\cdot|Z_n > 0)\right\} \xrightarrow[n\to\infty]{\text{law}} \mathbf{z}^{(\alpha)}, \tag{5.1.5}$$

where $\mathbf{z}^{(\alpha)}$ is a positive random variable with Laplace transform

$$E[e^{-u\mathbf{z}^{(\alpha)}}] = 1 - (1 + u^{-\alpha})^{-1/\alpha}, \quad u \geq 0. \tag{5.1.6}$$

In [76], Slack also considered the converse of this problem: In order for $\left\{P(Z_n > 0)Z_n; P(\cdot|Z_n > 0)\right\}$ to have a non-degenerate weak limit, the generating function of the offspring distribution must be of the form of (5.1.3) for some $0 < \alpha \leq 1$. For shorter and more unified approaches to these results, we refer our readers to [11, 61].

Goldstein and Hoppe [35] considered the asymptotic behavior of multitype critical Galton-Watson processes without the 2nd moment condition. Their main result can be stated as follows. Let $\mathbf{Z}_n = (Z_n^{(1)}, \dots, Z_n^{(d)})$ be a critical, d-type, nonsingular and positively regular Galton-Watson process. Denote by $\mathbf{F}(\mathbf{s}) = (\mathbf{F}_1(\mathbf{s}), \dots, \mathbf{F}_d(\mathbf{s}))$ the generating function of the offspring distribution, and by $\mathbf{F}^{(n)}(\mathbf{s})$, $n > 1$, its nth iterates. Let M be the mean matrix of $\mathbf{Z}$. Let $\mathbf{v}$ and $\mathbf{u}$ be the left and right eigenvectors of M, respectively, corresponding to the maximal eigenvalue 1, and normalized so that $\mathbf{v} \cdot \mathbf{u} = 1$ and $\mathbf{1} \cdot \mathbf{u} = 1$, with $\mathbf{1}$ being the vector $(1, \dots, 1)$. Suppose that

$$\mathbf{v}G(\mathbf{1} - x\mathbf{u})\mathbf{u} = x^{\alpha} l(x), \quad x > 0, \tag{5.1.7}$$

where $0 < \alpha \leq 1$; l is slowly varying at 0; and the matrix $G(\mathbf{s})$ is defined by

$$\mathbf{1} - \mathbf{F}(\mathbf{s}) = (M - G(\mathbf{s}))(\mathbf{1} - \mathbf{s}), \quad \mathbf{s} \in \mathbb{R}_+^d.$$

Let $a_n := \mathbf{v} \cdot (\mathbf{1} - \mathbf{F}^{(n)}(\mathbf{0}))$, with $\mathbf{0} \in \mathbb{R}_+^d$ being the vector $(0, \dots, 0)$. It was shown in [35] that, for each $\mathbf{i} \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$,

$$nl(a_n)\,\mathrm{P}(\mathbf{Z}_n \neq \mathbf{0}|Z_0 = \mathbf{i})^{\alpha} \xrightarrow[n\to\infty]{} \frac{(\mathbf{i} \cdot \mathbf{u})^{\alpha}}{\alpha}, \tag{5.1.8}$$

and for each $\mathbf{j} \in \mathbb{N}_0^d$,

$$\{a_n \mathbf{Z}_n \cdot \mathbf{j}; P(\cdot|\mathbf{Z}_n \neq \mathbf{0}, \mathbf{Z}_0 = \mathbf{i})\} \xrightarrow[n\to\infty]{\text{law}} (\mathbf{v} \cdot \mathbf{j})\mathbf{z}^{(\alpha)}, \tag{5.1.9}$$

where $\mathbf{z}^{(\alpha)}$ is a random variable with Laplace transform given by (5.1.6). For the converse of this problem, Vatutin [78] showed that in order for the left side of (5.1.9) to have a non-degenerate weak limit, one must have (5.1.7) for some $0 < \alpha \leq 1$. Vatutin [78] also considered analogous results for the continuous time multitype critical Galton-Watson processes.

Asmussen and Hering [4, Sections 6.3 and 6.4] discussed similar questions for critical branching Markov processes (Y_t) in a general space E under some ergodicity condition (the so-called condition (M), see [4, p. 156]) on the mean semigroup of (Y_t). When the second moment is infinite, under a condition parallel to (5.1.7) (the so-called condition (S) [4, p. 207]), results parallel to (5.1.8) and (5.1.9) were proved in [4, Theorem 6.4.2] for critical branching Markov processes.

In this chapter, we are interested in a class of measure-valued branching Markov process known as (ξ, ψ)-superprocesses: ξ, the spatial motion of the superprocess, is a Hunt process on a locally compact separable metric space E; ψ, the branching mechanism of the superprocess, is a function on $E \times [0, \infty)$ of the form

$$\psi(x, z) := -\beta(x)z + \alpha(x)z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(x, dy), \quad x \in E, z \geq 0, \quad (5.1.10)$$

where $\beta \in \mathscr{B}_b(E)$, $\alpha \in \mathscr{B}_b^+(E)$ and $\pi(x, dy)$ is a kernel from E to $(0, \infty)$ such that $\sup_{x \in E} \int_{(0,\infty)} (y \wedge y^2)\pi(x, dy) < \infty$. For the precise definition and properties of superprocesses, see [56].

Results parallel to (5.1.1) and (5.1.2) have been obtained for some critical superprocesses by Evans and Perkins [31] and Ren, Song and Zhang [68]. Evans and Perkins [31] considered critical superprocesses with branching mechanism of the form $(x, z) \mapsto z^2$ and with the spatial motion satisfying some ergodicity conditions. Ren, Song and Zhang [68] extended the results of [31] to a class of critical superprocesses with general branching mechanism and general spatial motions. The main results of [68] are as follows. Let $\{(X_t)_{t\geq 0}; \mathbf{P}_\mu\}$ be a critical superprocess starting from a finite measure μ on E. Suppose the spatial motion ξ is intrinsically ultracontractive with respect to some reference measure m, and the branching mechanism ψ satisfies the following second moment condition

$$\sup_{x \in E} \int_{(0,\infty)} y^2 \pi(x, dy) < \infty. \quad (5.1.11)$$

For any finite measure μ on E and any measurable function f on E, we use $\langle \mu, f \rangle$ to denote the integral of f with respect to μ. Put $\|\mu\| = \langle \mu, 1 \rangle$. Under some other mild assumptions, it was proved in [68] that

$$t\mathbf{P}_\mu(\|X_t\| \neq 0) \xrightarrow[t\to\infty]{} c^{-1}\langle \mu, \phi \rangle, \quad (5.1.12)$$

and for a large class of testing functions f on E,

$$\{t^{-1}X_t(f); \mathbf{P}_\mu(\cdot | \|X_t\| \neq 0)\} \xrightarrow[t\to\infty]{\text{law}} c\langle \phi^*, f \rangle_m \mathbf{e}. \quad (5.1.13)$$

Here, the constant $c > 0$ is independent of the choice of μ and f; $\langle \cdot, \cdot \rangle_m$ denotes the inner product in $L^2(E,m)$; $\mathbf{e}$ is an exponential random variable with mean 1; and ϕ (respectively, ϕ^*) is the principal eigenfunction of (respectively, the dual of) the generator of the mean semigroup of X. In [65], we provided an alternative probabilistic approach to (5.1.12) and (5.1.13).

It is natural to ask whether results parallel to (5.1.4) and (5.1.5) are still valid for some critical superprocesses without the second moment condition (5.1.11). A simpler version of this question has already been answered in the context of continuous-state branching processes (CSBPs) which can be viewed as superprocesses without spatial movements. Kyprianou and Pardo [50] considered CSBPs $\{(Y_t)_{t\geq 0}; P\}$ with stable branching mechanism $\psi(z) = cz^\gamma$, where $c > 0$ and $\gamma \in (1,2]$. He showed that for all $x \geq 0$, with $c_t := (c(\gamma-1)t)^{1/(\gamma-1)}$,

$$\{c_t^{-1}Y_t; P(\cdot|Y_t > 0, Y_0 = x)\} \xrightarrow[t\to\infty]{\text{law}} \mathbf{z}^{(\gamma-1)}, \tag{5.1.14}$$

where $\mathbf{z}^{(\gamma-1)}$ is a random variable with Laplace transform given by (5.1.6) (with $\alpha = \gamma - 1$). Recently, Ren, Yang and Zhao [70] studied CSBPs $\{(Y_t)_{t\geq 0}; P\}$ with branching mechanism

$$\psi(z) = cz^\gamma l(z), \quad z \geq 0, \tag{5.1.15}$$

where $c > 0$, $\gamma \in (1,2]$ and l is a function slowly varying at 0. It was proved in [70] that for all $x \geq 0$, with $\lambda_t := P_1(Y_t > 0)$,

$$\{\lambda_t Y_t; P(\cdot|Y_t > 0, Y_0 = x)\} \xrightarrow[t\to\infty]{\text{law}} \mathbf{z}^{(\gamma-1)}. \tag{5.1.16}$$

Later, Iyer, Leger and Pego [43] considered the converse problem: Suppose $\{(Y_t)_{t\geq 0}; P\}$ is a CSBP with critical branching mechanism ψ satisfying Grey's condition. In order for the left side of (5.1.16) to have a non-trivial weak limit for some positive constants $(\lambda_t)_{t\geq 0}$, one must have (5.1.15) for some $1 < \gamma \leq 2$.

In this chapter, we will establish a result parallel to (5.1.14) for some critical (ξ,ψ)-superprocess $\{X; \mathbf{P}\}$ with spatially dependent stable branching mechanism. In particular, we assume that the spatial motion ξ is intrinsically ultracontractive with respect to some reference measure m, and the branching mechanism takes the form

$$\psi(x,z) = -\beta(x)z + \kappa(x)z^{\gamma(x)}, \quad x \in E, z \geq 0,$$

where $\beta \in \mathscr{B}_b(E)$, $\gamma \in \mathscr{B}_b^+(E)$, $\kappa \in \mathscr{B}_b^+(E)$ with $1 < \gamma(\cdot) < 2$, $\gamma_0 := \operatorname{ess\,inf}_{m(dx)} \gamma(x) > 1$ and $\operatorname{ess\,inf}_{m(dx)} \kappa(x) > 0$. Let μ be an arbitrary finite initial measure on E. We will show that $\mathbf{P}_\mu(\|X_t\| \neq 0)$ converges to 0 as $t \to \infty$ and is regularly varying at infinity with index $\frac{1}{\gamma_0 - 1}$.

Furthermore, if $m(x : \gamma(x) = \gamma_0) > 0$, we will show that

$$\lim_{t\to\infty} \eta_t^{-1} \mathbf{P}_\mu(\|X_t\| \neq 0) = \mu(\phi),$$

and for a large class of non-negative testing functions f,

$$\{\eta_t X_t(f); \mathbf{P}_\mu(\cdot|\|X_t\| \neq 0)\} \xrightarrow[t\to\infty]{\text{law}} \langle f, \phi^* \rangle_m \mathbf{z}^{(\gamma_0-1)}, \tag{5.1.17}$$

where $\eta_t := \big(C_X(\gamma_0 - 1)t\big)^{-\frac{1}{\gamma_0-1}}$, $C_X := \langle \mathbf{1}_{\gamma(\cdot)=\gamma_0} \kappa \cdot \phi^{\gamma_0}, \phi^* \rangle_m$ and $\mathbf{z}^{(\gamma_0-1)}$ is a random variable with Laplace transform given by (5.1.6) (with $\alpha = \gamma_0 - 1$). Precise statements of the assumptions and the results are presented in the next subsection. It is interesting to mention here that, even though the stable index $\gamma(x)$ is spatially dependent, the limiting behavior of the critical superprocess $\{X; \mathbf{P}\}$ depends primarily on the lowest index γ_0.

5.1.2 Model and results

For any measurable space $(E, \mathscr{E})$, we denote by $\mathscr{E}$ the collection of all real-valued measurable functions on E. Define $\mathscr{E}_b := \{f \in \mathscr{E} : \sup_{x\in E} |f(x)| < \infty\}$, $\mathscr{E}^+ := \{f \in \mathscr{E} : \forall x \in E,\ f(x) \geq 0\}$ and $\mathscr{E}^{++} := \{f \in \mathscr{E} : \forall x \in E,\ f(x) > 0\}$. Define $\mathscr{E}_b^+ := \mathscr{E}_b \cap \mathscr{E}^+$ and $\mathscr{E}_b^{++} := \mathscr{E}_b \cap \mathscr{E}^{++}$. Denote by $\mathcal{M}_E$ the collection of all measures on $(E, \mathscr{E})$. Denote by $\mathcal{M}_E^\sigma$ the collection of all σ-finite measures on $(E, \mathscr{E})$. For simplicity, we write $\mu(f)$ and sometimes $\langle \mu, f \rangle$ for the integration of a function f with respect to a measure μ. We also write $\langle f, g \rangle_m$ for $\int_E fgdm$ to emphasize that it is the inner product in the Hilbert space $L^2(E, m)$. For any $f \in \mathscr{E}^+$, define $\mathcal{M}_E^f := \{\mu \in \mathcal{M}_E : \mu(f) < \infty\}$. In particular, $\mathcal{M}_E^1$ is the collection of all finite measures on E. If E is a topological space, we denote by $\mathscr{B}(E)$ the collection of all Borel subsets of E.

We now give the definition of a (ξ, ψ)-superprocess: Let E be a locally compact separable metric space, the spatial motion $\xi = \{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in E}\}$ be an E-valued Hunt process with its lifetime denoted by ζ, and the branching mechanism ψ be a function on $E \times [0, \infty)$ given by (5.1.10). We say an $\mathcal{M}_E^1$-valued Hunt process $X = \{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_E^1}\}$ is a (ξ, ψ)-*superprocess* if for each $t \geq 0, \mu \in \mathcal{M}_E^1$ and $f \in \mathscr{B}_b^+(E)$, we have

$$\mathbf{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)},$$

where the function $(t, x) \mapsto V_t f(x)$ on $[0, \infty) \times E$ is the unique locally bounded positive solution to the equation

$$V_t f(x) + \Pi_x\Big[\int_0^{t\wedge\zeta} \psi(\xi_s, V_{t-s} f) ds\Big] = \Pi_x[f(\xi_t)\mathbf{1}_{t<\zeta}], \quad t \geq 0, x \in E. \tag{5.1.18}$$

(In this chapter, for any real-valued function F on $E\times[0,\infty)$ and real-valued function f on E, we write $F(x,f) := F(x,f(x))$ for simplicity.)

Define the *Feynman-Kac semigroup*

$$P_t^\beta f(x) := \Pi_x\left[e^{\int_0^t \beta(\xi_r)dr} f(\xi_t)\mathbf{1}_{t<\zeta}\right], \quad t\geq 0, x\in E, f\in \mathscr{B}_b(E).$$

(Notice that if $\beta\equiv 0$, then $P_t := P_t^0$ is the *transition semigroup* of the process ξ.) It is known, see [56, Proposition 2.27] for example, (P_t^β) is *the mean semigroup* of the superprocess $\{X;\mathbf{P}\}$, in the sense that

$$\mathbf{P}_\mu[X_t(f)] = \mu(P_t^\beta f), \quad \mu\in\mathcal{M}_E^1, t\geq 0, f\in\mathscr{B}_b(E).$$

The mean semigroup plays a central role in the study of the asymptotic behavior of superprocesses. As discussed in [31], in order to have a result like (5.1.13) or (5.1.17), we have to establish the asymptotic behavior of the mean semigroup first. This can be done under the following assumptions on the spatial motion ξ:

Assumption 5.1. There exist an $m\in\mathcal{M}_E^\sigma$ with full support on the state space E and a family of strictly positive, bounded continuous functions $\{p_t(\cdot,\cdot): t>0\}$ on $E\times E$ such that

$$\Pi_x[f(\xi_t)\mathbf{1}_{t<\zeta}] = \int_E p_t(x,y)f(y)m(dy), \quad t>0, x\in E, f\in\mathscr{B}_b(E);$$

$$\int_E p_t(y,x)m(dy)\leq 1, \quad t>0, x\in E;$$

$$\int_E\int_E p_t(x,y)^2 m(dx)m(dy)<\infty, \quad t>0;$$

and the functions $x\mapsto\int_E p_t(x,y)^2m(dy)$ and $x\mapsto \int_E p_t(y,x)^2 m(dy)$ are both continuous.

Under Assumption 5.1, it is proved in [68, 67] that there exists a function $p_t^\beta(x,y)$ on $(0,\infty)\times E\times E$ which is continuous in (x,y) for each $t>0$ such that

$$e^{-\|\beta\|_\infty t}p_t(x,y)\leq p_t^\beta(x,y)\leq e^{\|\beta\|_\infty t}p_t(x,y), \quad t>0, x,y\in E,$$

and that for any $t>0, x\in E$ and $f\in\mathscr{B}_b(E)$,

$$P_t^\beta f(x) = \int_E p_t^\beta(x,y)f(y)m(dy).$$

$(p_t^\beta)_{t\geq 0}$ is called the *density* of the semigroup $(P_t^\beta)_{t\geq 0}$. Define the dual semigroup $(P_t^{\beta*})_{t\geq 0}$ by

$$P_0^{\beta*} = I; \quad P_t^{\beta*}f(x) := \int_E p_t^\beta(y,x)f(y)m(dy), \quad t>0, x\in E, f\in\mathscr{B}_b(E).$$

It is proved in [68, 67] that $(P_t^\beta)_{t\geq 0}$ and $(P_t^{\beta*})_{t\geq 0}$ are both strongly continuous semigroups of compact operators in $L^2(E,m)$. Let L and L^* be the generators of the semigroups $(P_t^\beta)_{t\geq 0}$ and $(P_t^{\beta*})_{t\geq 0}$, respectively. Denote by $\sigma(L)$ and $\sigma(L^*)$ the spectra of L and L^*, respectively. According to [73, Theorem V.6.6], $\lambda := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(L^*))$ is a common eigenvalue of multiplicity 1 for both L and L^*. Using the argument in [68], the eigenfunctions ϕ of L and ϕ^* of L^* associated with the eigenvalue λ can be chosen to be strictly positive and continuous everywhere on E. We further normalize ϕ and ϕ^* by $\langle \phi, \phi \rangle_m = \langle \phi, \phi^* \rangle_m = 1$ so that they are unique. Moreover, for each $t \geq 0$ and $x \in E$, we have $P_t^\beta \phi^*(x) = e^{\lambda t}\phi(x)$ and $P_t^{\beta*}\phi(x) = e^{\lambda t}\phi^*(x)$. We refer to ϕ (resp. ϕ^*) and λ the *principal eigenfunction* and the *principal eigenvalue* of L (resp. L^*).

Now, from

$$\mathbf{P}_\mu[X_t(\phi)] = e^{\lambda t}\mu(\phi), \quad t \geq 0,$$

we see that, if $\lambda > 0$, the mean of $X_t(\phi)$ will increase exponentially; if $\lambda < 0$, the mean of $X_t(\phi)$ will decrease exponentially; and if $\lambda = 0$, the mean of $X_t(\phi)$ will be a constant. Therefore, we say X is *supercritical, critical or subcritical*, according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Since we are only interested in the critical case, we assume the following:

Assumption 5.2. The superprocess X is critical, i.e., $\lambda = 0$.

Let φ (resp. φ^*) be the principal eigenfunction of (resp. the dual of) the transition semigroup (P_t) of the spatial process ξ. Our second assumption on the spatial process ξ is the following:

Assumption 5.3. φ is bounded, and $(P_t)_{t\geq 0}$ is *intrinsically ultracontractive*, that is, for each $t > 0$, there is a constant $c_t > 0$ such that for each $x, y \in E$, $p_t(x,y) \leq c_t\varphi(x)\varphi^*(y)$.

Under Assumption 5.3, it is proved in [68, 67] that the principal eigenfunction ϕ of the Feynman-Kac semigroup $(P_t^\beta)_{t\geq 0}$ is also bounded. Moreover, $(P_t^\beta)_{t\geq 0}$ is also *intrinsically ultracontractive*, in the sense that for each $t > 0$, there is a constant $c_t > 0$ such that for each $x, y \in E$, $p_t^\beta(x,y) \leq c_t\phi(x)\phi^*(y)$. In fact, it is proved in [47] that for each $t > 0$, $(p_t^\beta(x,y))_{x,y\in E}$ is comparable to $(\phi(x)\phi^*(y))_{x,y\in E}$ in the sense that there is a constant $c_t > 1$ such that

$$c_t^{-1} \leq \frac{p_t^\beta(x,y)}{\phi(x)\phi^*(y)} \leq c_t, \quad x, y \in E. \tag{5.1.19}$$

It is also shown in [47] that there are constants $c_0, c_1 > 0$ such that

$$\sup_{x,y\in E}\left|\frac{p_t^\beta(x,y)}{\phi(x)\phi^*(y)} - 1\right| \leq c_0 e^{-c_1 t}, \quad t > 1. \tag{5.1.20}$$

We refer our readers to [68] for a list of examples of processes satisfying Assumption 5.1 and 5.3.

Our assumption on the branching mechanism is the following:

Assumption 5.4. The branching mechanism ψ is of the form:

$$\begin{aligned}\psi(x,z) &= -\beta(x)z + \kappa(x)\int_0^\infty (e^{-zy} - 1 + zy)\frac{dy}{\Gamma(-\gamma(x))y^{1+\gamma(x)}}\\ &= -\beta(x)z + \kappa(x)z^{\gamma(x)}, \quad x \in E, z \geq 0,\end{aligned}$$

where $\beta \in \mathscr{B}_b(E), \gamma \in \mathscr{B}_b^+(E)$, $\kappa \in \mathscr{B}_b^{++}(E)$ with $1 < \gamma(\cdot) < 2$, $\gamma_0 := \operatorname{ess\,inf}_{m(dx)} \gamma(x) > 1$ and $\kappa_0 := \operatorname{ess\,inf}_{m(dx)} \kappa(x) > 0$.

Here we use the definition of the Gamma function on the negative half line:

$$\Gamma(x) := \int_0^\infty t^{x-1}\Big(e^{-t} - \sum_{k=0}^{n-1}\frac{(-t)^k}{k!}\Big)dt, \quad -n < x < -n+1, n \in \mathbb{N}. \tag{5.1.21}$$

We now present the main results of this chapter:

Theorem 5.1.1. *Suppose that* $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu \in \mathcal{M}_E^1}\}$ *is a* (ξ,ψ)*-superprocess satisfying Assumptions 5.1–5.4. Then,*

(1) $\{X;\mathbf{P}\}$ *is non-persistent, that is, for each* $t > 0$ *and* $x \in E$, $\mathbf{P}_{\delta_x}(\|X_t\| = 0) > 0$*;*

(2) for each $\mu \in \mathcal{M}_E^1$, $\mathbf{P}_\mu(\|X_t\| \neq 0)$ *converges to* 0 *as* $t \to \infty$ *and is regularly varying at infinity with index* $-(\gamma_0 - 1)^{-1}$*. Furthermore, if* $m(x : \gamma(x) = \gamma_0) > 0$*, then*

$$\lim_{t\to\infty} \eta_t^{-1}\mathbf{P}_\mu(\|X_t\| \neq 0) = \mu(\phi);$$

(3) suppose $m(x : \gamma(x) = \gamma_0) > 0$*. Let* $f \in \mathscr{B}^+(E)$ *be such that* $\langle f, \phi^*\rangle_m > 0$ *and* $\|\phi^{-1}f\|_\infty < \infty$*. Then for each* $\mu \in \mathcal{M}_E^1$,

$$\{\eta_t X_t(f); \mathbf{P}_\mu(\cdot | \|X_t\| \neq 0)\} \xrightarrow[t\to\infty]{\text{law}} \langle f, \phi^*\rangle_m \mathbf{z}^{(\gamma_0 - 1)}.$$

Here, $\eta_t := \big(C_X(\gamma_0 - 1)t\big)^{-\frac{1}{\gamma_0 - 1}}$, $C_X := \langle \mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\cdot\phi^{\gamma_0}, \phi^*\rangle_m$ *and* $\mathbf{z}^{(\gamma_0-1)}$ *is a random variable with Laplace transform given by* (5.1.6) *(with* $\alpha = \gamma_0 - 1$*).*

5.1.3 Methods and overview

To establish Theorem 5.1.1(2) and Theorem 5.1.1(3), we use a spine decomposition theorem for X. Roughly speaking, the spine is the trajectory of an immortal moving particle and the spine decomposition theorem says that, after a size-biased transform, the transformed superprocess can be decomposed in law as the sum of a copy of the original superprocess and an immigration process along this spine, see [25, 28, 57]. The family of functions used for the size-biased transform is $(e^{-\lambda t}X_t(\phi))_{t\geq 0}$, which is a martingale. Therefore, this size-biased transform can be viewed as a martingale change of measure. Under Assumptions 5.1 and 5.3, the spine process $\{\xi; \Pi^{(\phi)}\}$ is an ergodic process. We take advantage of this ergodicity to study the asymptotic behavior of the superprocess.

Similar idea has already been used by Powell [62] to establish results parallel to (5.1.12) and (5.1.13) for a class of critical branching diffusion processes. Let $\{(Y_t)_{t\geq 0}; P\}$ be a branching diffusion process, in a bounded domain, with finite second moment. As have been discussed in [62], a direct study of the partial differential equation satisfied by the survival probability $(t,x) \mapsto P_{\delta_x}(\|Y_t\| \neq 0)$ is tricky. Instead, by using a spine decomposition approach, Powell [62] showed that the survival probability decays like $a(t)\phi(x)$, where $\phi(x)$ is the principal eigenfunction of the mean semigroup of (Y_t) and $a(t)$ is a function capturing the uniform speed. Then, the problem is reduced to the study of a single ordinary differential equation satisfied by $a(t)$. Later, inspired by [62], we gave in [65] a similar proof of (5.1.12) for a class of general critical superprocesses with finite second moment. In this chapter, we will generalize these arguments to a class of general critical superprocesses without finite second moment and establish Theorem 5.1.1(2). For the conditional weak convergence result, i.e., Theorem 5.1.1(3), we use a fact that the Laplace transform given in (5.1.6) can be characterized by a non-linear delay equation (see Lemma 5.3.5). Using the spine method, we show that the Laplace transform of the one-dimensional distributions of the superprocess, after a proper rescaling, can be characterized by a similar equation (see (5.3.23)). Then, the desired convergence of the distributions can be established by a comparison between the equations. Again, the ergodicity of the spine process plays a central role in the comparison.

A similar idea of establishing weak convergence through a comparison of the equations satisfied by the distributions has already been used by us in [63, 65]. We characterized the exponential distribution using its double size-biased transform; and to help us make the comparison, we investigated the double size-biased transform of the corresponding processes. However, the double-size-biased transform of a random variable requires its second moment being finite. Since we do not assume the second moment condition in this chapter, we can not use the method of double size-biased transform.

In [62] (for critical branching diffusions in a bounded domain with finite variance) and in [65, 68] (for general critical superprocesses with finite variance), the conditional weak convergence was proved in two steps. First, a convergence result was established for ϕ, the principal eigenfunction of the mean semigroup of the corresponding process, and then the second moment condition was used to extend the result to more general testing functions. However, in the present case, since we are not assuming the second moment condition, this type of argument does not work. Instead, we use a generalized spine decomposition theorem, which is developed in [65], to establish Theorem 5.1.1(3) for a large class of general testing functions in one stroke.

The rest of this chapter is organized as follows: In Subsections 5.2.1, 5.2.2 and 5.2.3, we give some preliminary results about the asymptotic equivalence, regularly varying functions and superprocesses, respectively. In Subsection 5.2.4, we present the generalized spine decomposition theorem. In Subsection 5.2.5, we discuss the ergodicity of the spine process. In Subsections 5.3.1 and 5.3.2 we give the poofs of Theorem 5.1.1(1) and 5.1.1(2), respectively. In Subsection 5.3.3, we give the equation that characterize the one-dimensional distributions. In Subsection 5.3.4, we give the equation that characterize the distribution with Laplace transform (5.1.6). Finally, in Subsection 5.3.5, we make comparison of these two equations and give the proof of Theorem 5.1.1(3).

5.2 Preliminaries

5.2.1 Asymptotic equivalence

In this subsection, we give a lemma on asymptotic equivalence. Let $t_0 \in [-\infty, \infty]$. For any $f_0, f_1 \in \mathscr{B}^{++}(\mathbb{R})$, we say f_0 and f_1 are *asymptotically equivalent at* t_0, if $\left|\frac{f_0(t)}{f_1(t)} - 1\right| \xrightarrow[t\to t_0]{} 0$; and in this case, we write $f_0(t) \underset{t\to t_0}{\sim} f_1(t)$. Let E be a measurable space. For any $g_0, g_1 \in \mathscr{B}^{++}(\mathbb{R}\times E)$, we say g_0 and g_1 are *uniformly asymptotically equivalent at* t_0, if $\sup_{x\in E}\left|\frac{g_0(t,x)}{g_1(t,x)} - 1\right| \xrightarrow[t\to t_0]{} 0$; and in this case, we write $g_0(t,x) \overset{x\in E}{\underset{t\to t_0}{\sim}} g_1(t,x)$.

Lemma 5.2.1. *Suppose that* $f_0, f_1 \in \mathscr{B}_b^{++}(\mathbb{R}\times E)$ *and* $f_0(t,x) \overset{x\in E}{\underset{t\to t_0}{\sim}} f_1(t,x)$. *If* $m \in \mathcal{M}_E^1$, *then*

$$\int_E f_0(t,x)m(dx) \underset{t\to t_0}{\sim} \int_E f_1(t,x)m(dx).$$

Proof. Since

$$\begin{aligned}\left|\frac{\int_E f_0(t,x)m(dx)}{\int_E f_1(t,x)m(dx)} - 1\right| &= \left|\int_E \frac{f_0(t,x)}{f_1(t,x)}\frac{f_1(t,x)m(dx)}{\int_E f_1(t,y)m(dy)} - 1\right| \\ &\le \int_E \left|\frac{f_0(t,x)}{f_1(t,x)} - 1\right|\frac{f_1(t,x)m(dx)}{\int_E f_1(t,y)m(dy)} \le \sup_{x\in E}\left|\frac{f_0(t,x)}{f_1(t,x)} - 1\right| \xrightarrow[t\to t_0]{} 0,\end{aligned}$$

the assertion is valid. □

5.2.2 Regular variation

In this subsection, we give some preliminary results on regular variation. We refer the reader to [10] for more results on regular variation. For $f \in \mathscr{B}^{++}((0,\infty))$, we say f is regularly varying at ∞ (resp. at 0) with index $\gamma \in (-\infty, \infty)$ if for any $\lambda \in (0,\infty)$,

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\gamma \quad \Big(\text{resp. } \lim_{t\to 0} \frac{f(\lambda t)}{f(t)} = \lambda^\gamma\Big).$$

In this case we write $f \in \mathcal{R}_\gamma^\infty$ (resp. $f \in \mathcal{R}_\gamma^0$). Further, if $\gamma = 0$, then we say f is slowly varying. According to [10, Theorem 1.3.1], if L is a function slowly varying at ∞, then it can be written in the form

$$L(t) = c(t) \exp\Big\{ \int_{t_0}^t \epsilon(u) \frac{du}{u} \Big\}, \quad t \geq t_0,$$

for some $t_0 > 0$, where $(c(t))_{t\geq t_0}$ and $(\epsilon(t))_{t\geq t_0}$ are measurable functions with $c(t) \xrightarrow[t\to\infty]{} c \in (0,\infty)$ and $\epsilon(t) \xrightarrow[t\to\infty]{} 0$. In particular, we know that, there is $t_0 > 0$ large enough such that L is locally bounded on $[t_0, \infty)$.

Lemma 5.2.2 ([10, Propositions 1.5.8 and 1.5.10])**.** *Suppose that $L \in \mathcal{R}_0^\infty$.*

- *Let $t_0 \in (0,\infty)$ be large enough so that L is locally bounded on $[t_0,\infty)$. If $\alpha > 0$, then*

$$\int_{t_0}^t L(u) du^\alpha \underset{t\to\infty}{\sim} t^\alpha L(t).$$

- *If $\alpha < 0$ then $\int_t^\infty L(u)du^\alpha < \infty$ for t large enough, and*

$$-\int_t^\infty L(u) du^\alpha \underset{t\to\infty}{\sim} t^\alpha L(t).$$

Corollary 5.2.3. *Suppose that $l \in \mathcal{R}_0^0$.*

- *Let $s_0 \in (0,\infty)$ be small enough so that l is locally bounded on $(0, s_0]$. If $\alpha < 0$, then*

$$-\int_s^{s_0} l(u) du^\alpha \underset{s\to 0}{\sim} s^\alpha l(s).$$

- *If $\alpha > 0$, then $\int_0^s l(u)du^\alpha < \infty$ for s small enough, and*

$$\int_0^s l(u) du^\alpha \underset{s\to 0}{\sim} s^\alpha l(s).$$

Proof. Since $l \in \mathcal{R}_0^0$, we know that, if one defines $L(t) := l(t^{-1})$ for each $t \in (0,\infty)$, then $L \in \mathcal{R}_0^\infty$. Therefore, there exists $t_0 \in (0,\infty)$ such that L is locally bounded on $[t_0,\infty)$. Taking $s_0 := t_0^{-1}$, we then immediately get that l is locally bounded on $(0, s_0]$. If $\alpha < 0$, then according to Lemma 5.2.2, we have

$$\int_{t_0}^{t} L(u)du^{-\alpha} \underset{t\to\infty}{\sim} t^{-\alpha}L(t).$$

Replacing t with s^{-1}, we have

$$-\int_{s}^{s_0} l(u)du^{\alpha} = \int_{s_0^{-1}}^{s^{-1}} L(u)du^{-\alpha} \underset{s\to 0}{\sim} (s^{-1})^{-\alpha}L(s^{-1}) = s^{\alpha}l(s),$$

as desired. The second assertion can be proved similarly. □

Lemma 5.2.4 ([10, Theorem 1.5.12]). *If $f \in \mathcal{R}_{\alpha}^{\infty}$ with $\alpha > 0$, there exists $g \in \mathcal{R}_{1/\alpha}^{\infty}$ with*

$$g(f(t)) \underset{t\to\infty}{\sim} f(g(t)) \underset{t\to\infty}{\sim} t.$$

Here g is determined uniquely up to asymptotic equivalence as $t \to \infty$.

Corollary 5.2.5. *If $f \in \mathcal{R}_{\alpha}^{0}$ with $\alpha < 0$, there exists $g \in \mathcal{R}_{1/\alpha}^{\infty}$ with*

$$g(f(t)) \underset{t\to 0}{\sim} t; \quad f(g(t)) \underset{t\to\infty}{\sim} t. \tag{5.2.1}$$

Here g is determined uniquely up to asymptotic equivalence as $t \to \infty$.

Proof. Since $f \in \mathcal{R}_{\alpha}^{0}$, we know that $\tilde{f} \in \mathcal{R}_{-\alpha}^{\infty}$ with $\tilde{f}(t) := f(t^{-1})$. Noticing that $-\alpha > 0$, according to Lemma 5.2.4, there exists $h \in \mathcal{R}_{-1/\alpha}^{\infty}$ such that

$$h(\tilde{f}(t)) \underset{t\to\infty}{\sim} t; \quad \tilde{f}(h(t)) \underset{t\to\infty}{\sim} t. \tag{5.2.2}$$

Denoting by $g := h^{-1} \in \mathcal{R}_{1/\alpha}^{\infty}$, the above translates to (5.2.1).

Now, suppose that there is another $g_0 \in \mathcal{R}_{1/\alpha}^{\infty}$ satisfies (5.2.1) with g replaced by g_0. Denoting by $h_0 := g_0^{-1}$, we can verify that (5.2.2) is valid with h replaced by h_0. According to Lemma 5.2.4, h and h_0 are asymptotic equivalent at ∞. Hence, so are g and g_0. □

Lemma 5.2.6. *Let E be a measurable space with a non-degenerate measure $m \in \mathcal{M}_E^1$. Let $\gamma \in \mathscr{B}_b(E)$ with*

$$\gamma_0 := \operatorname*{ess\,inf}_{m(dx)} \gamma(x) := \sup\{r : m(x : \gamma(x) < r) = 0\}.$$

Then $\left(\int_E t^{\gamma(x)}m(dx)\right)_{t\in(0,\infty)} \in \mathcal{R}_{\gamma_0}^{0}$. Further, if $m\{x : \gamma(x) = \gamma_0\} > 0$, then

$$\int_E t^{\gamma(x)}m(dx) \underset{t\to 0}{\sim} m\{x : \gamma(x) = \gamma_0\}t^{\gamma_0}.$$

Proof. If $\lambda \in (0, 1]$, then we have

$$\frac{\int_E \lambda^{\gamma(x)}t^{\gamma(x)}m(dx)}{\int_E t^{\gamma(x)}m(dx)} \leq \frac{\int_E \lambda^{\gamma_0}t^{\gamma(x)}m(dx)}{\int_E t^{\gamma(x)}m(dx)} = \lambda^{\gamma_0}, \quad t \in (0,\infty).$$

This implies that

$$\limsup_{(0,\infty)\ni t\to 0} \frac{\int_E \lambda^{\gamma(x)} t^{\gamma(x)} m(dx)}{\int_E t^{\gamma(x)} m(dx)} \le \lambda^{\gamma_0}.$$

Also, for any $\epsilon \in (0,\infty)$, we have

$$\begin{aligned}
\frac{\int_E \lambda^{\gamma(x)} t^{\gamma(x)} m(dx)}{\int_E t^{\gamma(x)} m(dx)} &\ge \frac{\int_{\gamma(x)\le\gamma_0+\epsilon} \lambda^{\gamma(x)} t^{\gamma(x)} m(dx)}{\int_E t^{\gamma(x)} m(dx)} \\
&\ge \lambda^{\gamma_0+\epsilon} \frac{\int_{\gamma(x)\le\gamma_0+\epsilon} t^{\gamma(x)} m(dx)}{\int_{\gamma(x)\le\gamma_0+\epsilon} t^{\gamma(x)} m(dx) + \int_{\gamma(x)>\gamma_0+\epsilon} t^{\gamma(x)} m(dx)} \\
&= \lambda^{\gamma_0+\epsilon} \frac{1}{1 + \frac{\int_{\gamma(x)>\gamma_0+\epsilon} t^{\gamma(x)-(\gamma_0+\epsilon)} m(dx)}{\int_{\gamma(x)\le\gamma_0+\epsilon} t^{\gamma(x)-(\gamma_0+\epsilon)} m(dx)}}, \quad t \in (0,\infty), \\
&\xrightarrow[(0,\infty)\ni t\to 0]{} \lambda^{\gamma_0+\epsilon},
\end{aligned}$$

where the last convergence is due to the monotone convergence theorem. Therefore

$$\liminf_{(0,\infty)\ni t\to 0} \frac{\int_E \lambda^{\gamma(x)} t^{\gamma(x)} m(dx)}{\int_E t^{\gamma(x)} m(dx)} \ge \lambda^{\gamma_0}.$$

Summarizing the above, we get

$$\lim_{(0,\infty)\ni t\to 0} \frac{\int_E \lambda^{\gamma(x)} t^{\gamma(x)} m(dx)}{\int_E t^{\gamma(x)} m(dx)} = \lambda^{\gamma_0}, \quad \lambda \in (0,1].$$

If $\lambda \in (1,\infty)$, taking $f(x,t) := t^{\gamma(x)}$, from what we have proved, we also have that

$$\lim_{(0,\infty)\ni t\to 0} \frac{\int_E f(x,\lambda t) m(dx)}{\int_E f(x,t) m(dx)} = \lim_{(0,\infty)\ni t\to 0} \frac{\int_E f(x,t) m(dx)}{\int_E f(x,\lambda^{-1} t) m(dx)} = \left((\lambda^{-1})^{\gamma_0}\right)^{-1} = \lambda^{\gamma_0}.$$

This proved the first part of the lemma. If further we have $m(x : \gamma(x) = \gamma_0) > 0$, then by the monotone convergence theorem it is easy to see that

$$\frac{\int_E t^{\gamma(x)} m(dx)}{t^{\gamma_0}} \xrightarrow[(0,\infty)\ni t\to 0]{} m(x : \gamma(x) = \gamma_0) \in (0,\infty).$$

□

5.2.3 Superprocesses

In this subsection, we recall some known results on the (ξ,ψ)-superprocess $\{X; \mathbf{P}\}$. It is known, see [56, Theorem 2.23] for example, that (5.1.18) can be written as

$$V_t f(x) + \int_0^t P^\beta_{t-r} \psi_0(x, V_r f) dr = P^\beta_t f(x), \quad f \in \mathscr{B}^+_b(E), t \ge 0, x \in E, \tag{5.2.3}$$

where

$$\psi_0(x,z) := \alpha(x)z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(x,dy), \quad x \in E, z \geq 0.$$

Suppose that Assumptions 5.1–5.2 hold. Integrating both sides of (5.2.3) with respect to $\phi^* dm$, we get that

$$\langle V_t f, \phi^* \rangle_m + \int_s^t \langle \psi_0(\cdot, V_r f), \phi^* \rangle_m dr = \langle V_s f, \phi^* \rangle_m, \quad t \geq s \geq 0, f \in \mathscr{B}_b^+(E). \tag{5.2.4}$$

Let $\mathbb{W}$ be the collection of all $\mathcal{M}_E^1$-valued càdlàg paths on $[0,\infty)$. We refer to $\mathbb{W}$ as the *canonical space of* $(X_t)_{t\geq 0}$. In fact, (X_t) can be viewed as a $\mathbb{W}$-valued random variable. We denote the *coordinate process of* $\mathbb{W}$ by $(W_t)_{t\geq 0}$.

We say that $(X_t)_{t\geq 0}$ is *non-persistent* if $\mathbf{P}_{\delta_x}(\|X_t\| = 0) > 0$ for all $x \in E$ and $t > 0$. Suppose that $(X_t)_{t\geq 0}$ is non-persistent, then according to [56, Section 8.4], there is a family of measures $(\mathbb{N}_x)_{x\in E}$ on $\mathbb{W}$ such that

- $\mathbb{N}_x(\forall t \geq 0, \|W_t\| = 0) = 0$;
- $\mathbb{N}_x(\|W_0\| \neq 0) = 0$;
- for any $\mu \in \mathcal{M}_E^1$, if $\mathcal{N}$ is a Poisson random measure defined on some probability space with intensity $\mathbb{N}_\mu(\cdot) := \int_E \mathbb{N}_x(\cdot)\mu(dx)$, then the superprocess $\{X; \mathbf{P}_\mu\}$ can be realized by $\widetilde{X}_0 := \mu$ and $\widetilde{X}_t(\cdot) := \mathcal{N}[W_t(\cdot)]$ for each $t > 0$.

We refer to $(\mathbb{N}_x)_{x\in E}$ as the *Kuznetsov measures* of X. For the existence and further properties of such measures, we refer our readers to [56].

From Campbell's formula, see the proof of [49, Theorem 2.7] for example, we have

$$-\log \mathbf{P}_\mu[e^{-X_t(f)}] = \mathbb{N}_\mu[1 - e^{-W_t(f)}], \quad \mu \in \mathcal{M}_E^1, t > 0, f \in \mathscr{B}_b^+(E). \tag{5.2.5}$$

For each $x \in E$ and $t \geq 0$, taking $\mu = \delta_x$ and $f = \lambda \mathbf{1}_E$ with $\lambda > 0$ in the above equation, and letting $\lambda \to \infty$, we get

$$v_t(x) := \lim_{\lambda\to\infty} V_t(\lambda \mathbf{1}_E)(x) = -\log \mathbf{P}_{\delta_x}(\|X_t\| = 0) = \mathbb{N}_x(\|W_t\| \neq 0). \tag{5.2.6}$$

For each $\mu \in \mathcal{M}_E^1$ and $t > 0$, by (5.2.5), (5.2.6) and the monotone convergence theorem, we have

$$\begin{aligned} \mathbb{N}_\mu(\|W_t\| \neq 0) &= -\log \mathbf{P}_\mu(\|X_t\| = 0) = \lim_{\lambda\to\infty}(-\log \mathbf{P}_\mu[e^{-\lambda X_t(\mathbf{1}_E)}]) \\ &= \lim_{\lambda\to\infty} \langle \mu, V_t(\lambda \mathbf{1}_E) \rangle = \mu(v_t). \end{aligned} \tag{5.2.7}$$

It is know that for any $f \in \mathscr{B}_b^+(E)$,

$$\mathbb{N}_\mu[W_t(f)] = \mathbf{P}_\mu[X_t(f)] = \mu(P_t^\beta f), \quad t \geq 0, \tag{5.2.8}$$

see [65, Lemma 3.3] for example.

5.2.4 Spine decompositions

Let $(\Omega, \mathscr{F})$ be a measurable space with a σ-finite measure μ. For any $F \in \mathscr{F}$, we say *μ can be size-biased by F* if $\mu(F < 0) = 0$ and $\mu(F) \in (0, \infty)$. In this case, we define the *F-transform of μ* as the probability μ^F on $(\Omega, \mathscr{F})$ such that

$$d\mu^F = \frac{F}{\mu(F)} d\mu.$$

Let $\{X; \mathbf{P}\}$ be a non-persistent superprocess. Let $\mu \in \mathcal{M}_E^1$ and $T > 0$. Suppose that $g \in \mathscr{B}^+(E)$ satisfies that $\mu(P_T^\beta g) \in (0, \infty)$. Then, according to (5.2.8), $\mathbf{P}_\mu$ (resp. $\mathbb{N}_\mu$) can be size-biased by $X_T(g)$ (resp. $W_T(g)$). Denote by $\mathbf{P}_\mu^{X_T(g)}$ (resp. $\mathbb{N}_\mu^{W_T(g)}$) the $X_T(g)$-transform of $\mathbf{P}_\mu$ (resp. the $W_T(g)$-transform of $\mathbb{N}_\mu$). The spine decomposition theorem characterizes the law of $\{(X_t)_{t \geq 0}; \mathbf{P}_\mu^{X_T(g)}\}$ in two steps. The first step of the theorem says that $\{(X_t)_{t \geq 0}; \mathbf{P}_\mu^{X_T(g)}\}$ can be decomposed in law as the sum of two independent measure-valued processes:

Theorem 5.2.7 (Size-biased decomposition, [65])**.**

$$\{(X_t)_{t \geq 0}; \mathbf{P}_\mu^{X_T(g)}\} \overset{f.d.d.}{=} \{(X_t + W_t)_{t \geq 0}; \mathbf{P}_\mu \otimes \mathbb{N}_\mu^{W_T(g)}\}.$$

The second step of the spine decomposition theorem says that $\{(W_t)_{0 \leq t \leq T}; \mathbb{N}_\mu^{W_T(g)}\}$ has a spine representation: We say $\{(\xi_t)_{0 \leq t \leq T}, \mathbf{n}_T, (Y_t)_{0 \leq t \leq T}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is a *spine representation of* $\mathbb{N}_\mu^{W_T(g)}$ if,

- the *spine process* $\{(\xi_t)_{0 \leq t \leq T}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is a copy of $\{(\xi_t)_{0 \leq t \leq T}; \Pi_\mu^{(g,T)}\}$, where $\Pi_\mu^{(g,T)}$ is the $g(\xi_T) \exp\{\int_0^T \beta(\xi_s) ds\}$-transform of the measure $\Pi_\mu(\cdot) := \int_E \mu(dx) \Pi_x(\cdot)$;
- given $\{(\xi_t)_{0 \leq t \leq T}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$, the *immigration measure* $\{\mathbf{n}_T; \dot{\mathbf{P}}_\mu^{(g,T)}[\cdot | (\xi_t)_{0 \leq t \leq T}]\}$ is a Poisson random measure on $[0, T] \times \mathbb{W}$ with intensity

$$\mathbf{m}_T^\xi(ds, dw) := 2\alpha(\xi_s) ds \cdot \mathbb{N}_{\xi_s}(dw) + ds \cdot \int_{(0,\infty)} y \mathbf{P}_{y\delta_{\xi_s}}(X \in dw) \pi(\xi_s, dy);$$

- $\{(Y_t)_{0 \leq t \leq T}; \dot{\mathbf{P}}_\mu^{(g,T)}\}$ is an $\mathcal{M}_E^1$-valued process defined by

$$Y_t := \int_{(0,t] \times \mathbb{W}} w_{t-s} \mathbf{n}_T(ds, dw), \quad 0 \leq t \leq T.$$

Theorem 5.2.8 (Spine representation, [65]). *Let* $\{(Y_t)_{0\le t\le T};\dot{\mathbf{P}}_\mu^{(g,T)}\}$ *be the spine representation of* $\mathbb{N}_\mu^{W_T(g)}$ *defined above. Then we have*

$$\{(Y_t)_{0\le t\le T};\dot{\mathbf{P}}_\mu^{(g,T)}\}\overset{f.d.d.}{=}\{(W_t)_{0\le t\le T};\mathbb{N}_\mu^{W_T(g)}\}.$$

Notice that $\mathbf{P}_\mu^{X_T(g)}(X_0=\mu)=1$. Also notice that $\mathbb{N}_\mu$ is not a probability measure, but after the transform, $\mathbb{N}_\mu^{W_T(g)}$ is a probability measure. Since $\mathbb{N}_\mu(\|W_0\|\neq 0)=0$, we have $\mathbb{N}_\mu^{W_T(g)}(\|W_0\|=0)=1$. Similarly, Π_μ is not typically a probability measure, but after the transform, $\Pi_\mu^{(T,g)}$ is a probability measure. We note that

$$\begin{aligned}\Pi_\mu^{(T,g)}[f(\xi_0)]&=\frac{1}{\mu(P_T^\beta g)}\Pi_\mu\Big[g(\xi_T)\exp\Big\{\int_0^T\beta(\xi_s)ds\Big\}f(\xi_0)\Big]\\&=\frac{1}{\mu(P_T^\beta g)}\int_E(P_T^\beta g)(x)\cdot f(x)\mu(dx),\end{aligned}$$

which says that

$$\Pi_\mu^{(T,g)}(\xi_0\in dx)=\frac{1}{\mu(P_T^\beta g)}(P_T^\beta g)(x)\mu(dx),\quad x\in E. \tag{5.2.9}$$

Now, suppose that $\{\xi;\Pi\}$ satisfies Assumption 5.1. Recall that ϕ is the principal eigenfunction of the mean semigroup of X. The classical spine decomposition theorem, see [25], [28] and [57] for example, considered the case when $g=\phi$ only. In this case, the family of probabilities $(\Pi_\mu^{(\phi,T)})_{T\ge 0}$ is consistent in the sense of Kolmogorov's extension theorem, that is, the process $\{(\xi_t)_{0\le t\le T};\Pi_\mu^{(\phi,T)}\}$ can be realized as the restriction of some process, say $\{(\xi_t)_{t\ge 0};\Pi_\mu^{(\phi)}\}$, on the finite time interval $[0,T]$. In fact, one can also check that this consistency property is satisfied by $(\mathbf{P}_\mu^{X_T(\phi)})_{T\ge0}$, $(\mathbb{N}_\mu^{W_T(\phi)})_{T\ge0}$ and $(\dot{\mathbf{P}}_\mu^{(\phi,T)})_{T\ge0}$. Therefore, the actual statement of the classical spine decomposition theorem is different from merely replacing g with ϕ in Theorem 5.2.7 and 5.2.8: There is no need to restrict the corresponding processes on the finite time interval $[0,T]$. Because of its theoretical importance, we state the classical spine decomposition theorem explicitly here:

Corollary 5.2.9. *For each* $\mu\in\mathcal{M}_E^\phi\cap\mathcal{M}_E^1$, *we have*

$$\{(X_t)_{t\ge0};\mathbf{P}_\mu^{(\phi)}\}\overset{f.d.d.}{=}\{(X_t+W_t)_{t\ge0};\mathbf{P}_\mu\otimes\mathbb{N}_\mu^{(\phi)}\}.$$

Here, the probability $\mathbf{P}_\mu^{(\phi)}$ *is Doob's h-transform of* $\mathbf{P}_\mu$ *whose restriction on the natural filtration* $(\mathscr{F}_t^X)$ *of the process* $(X_t)_{t\ge0}$ *is*

$$d(\mathbf{P}_\mu^{(\phi)}|_{\mathscr{F}_t^X})=\frac{X_t(\phi)}{\mu(\phi)}d(\mathbf{P}_\mu|_{\mathscr{F}_t^X}),\quad t\ge 0;$$

and $\mathbb{N}_\mu^{(\phi)}$ *is a probability measure on* $\mathbb{W}$ *whose restriction on the natural filtration* $(\mathscr{F}_t^W)$ *of the* *process* $(W_t)_{t\geq 0}$ *is*

$$d(\mathbb{N}^{(\phi)}_\mu|_{\mathscr{F}^W_t}) = \frac{W_t(\phi)}{\mu(\phi)} d(\mathbb{N}_\mu|_{\mathscr{F}^W_t}), \quad t \geq 0.$$

Let $\mu \in \mathcal{M}^{(\phi)}_\mu$, we say $\{(\xi_t)_{t\geq 0}, \mathbf{n}, (Y_t)_{t\geq 0}; \dot{\mathbf{P}}^{(\phi)}_\mu\}$ is a *spine representation of* $\mathbb{N}^{(\phi)}_\mu$ if:

- the *spine process* $\{(\xi_t)_{t\geq 0}; \dot{\mathbf{P}}^{(\phi)}_\mu\}$ is a copy of $\{(\xi_t)_{t\geq 0}; \Pi^{(\phi)}_\mu\}$ where the probability $\Pi^{(\phi)}_\mu$ is Doob's h-transform of Π_μ whose restriction on the natural filtration $(\mathscr{F}^\xi_t)$ of the process $(\xi_t)_{t\geq 0}$ is

$$d(\Pi^{(\phi)}_\mu|_{\mathscr{F}^\xi_t}) = \frac{\phi(\xi_t) e^{\int_0^t \beta(\xi_s) ds}}{\mu(\phi)} d(\Pi_\mu|_{\mathscr{F}^\xi_t}), \quad t \geq 0;$$

- conditioned on $\{(\xi_t)_{t\geq 0}; \dot{\mathbf{P}}^{(\phi)}_\mu\}$, *the immigration measure* $\{\mathbf{n}; \dot{\mathbf{P}}^{(\phi)}_\mu[\cdot|(\xi_t)_{t\geq 0}]\}$ is a Poisson random measure on $[0,\infty) \times \mathbb{W}$ with intensity

$$\mathbf{m}^\xi(ds, dw) := 2\alpha(\xi_s) ds \cdot \mathbb{N}_{\xi_s}(dw) + ds \cdot \int_{(0,\infty)} y \mathbf{P}_{y\delta_{\xi_s}}(X \in dw) \pi(\xi_s, dy);$$

- $\{(Y_t)_{t\geq 0}; \dot{\mathbf{P}}^{(\phi)}_\mu\}$ is an $\mathcal{M}^1_E$-valued process defined by

$$Y_t := \int_{(0,t]\times\mathbb{W}} w_{t-s} \mathbf{n}(ds, dw), \quad t \geq 0.$$

Corollary 5.2.10. *Let* $\{(Y_t)_{t\geq 0}; \dot{\mathbf{P}}^{(\phi)}_\mu\}$ *be the spine representation of* $\mathbb{N}^{(\phi)}_\mu$ *defined above. Then we have*

$$\{(Y_t)_{t\geq 0}; \dot{\mathbf{P}}^{(\phi)}_\mu\} \overset{f.d.d.}{=} \{(W_t)_{t\geq 0}; \mathbb{N}^{(\phi)}_\mu\}.$$

For the sake of generality, the spine decomposition theorems above are all stated with respect to a general initial configuration μ. If $\mu = \delta_x$ for some $x \in E$, then by (5.2.9), we have $\Pi^{(T,g)}_{\delta_x}(\xi_0 = x) = 1$, so sometimes we write $\Pi^{(T,g)}_x$ for $\Pi^{(T,g)}_{\delta_x}$. Similarly, we write $\Pi^{(\phi)}_x$ for $\Pi^{(\phi)}_{\delta_x}$.

5.2.5 Ergodicity of the spine process

In this subsection, we discuss the ergodicity of the spine process $\{(\xi_t)_{t\geq 0}; (\Pi^{(\phi)}_x)_{x\in E}\}$ under Assumptions 5.1–5.3. According to [47], $\{\xi; \Pi^{(\phi)}_x\}$ is a time homogeneous Hunt process and its transition density with respect to the measure m is

$$q_t(x,y) := \frac{\phi(y)}{\phi(x)} p^\beta_t(x,y), \quad x, y \in E, t > 0.$$

Let $c_0 > 0$ and $c_1 > 0$ be the constants in (5.1.20), then we have

$$\sup_{x\in E} \left| \frac{q_t(x,y)}{\phi(y)\phi^*(y)} - 1 \right| \leq c_0 e^{-c_1 t}, \quad t > 1. \tag{5.2.10}$$

This implies that the process $\{\xi; \Pi_x^{(\phi)}\}$ is ergodic. One can easily get from (5.2.10) that $(\phi\phi^*)(x)m(dx)$ is the unique invariant probability measure of $\{\xi; \Pi_x^{(\phi)}\}$. The following two lemmas are also simple consequences of (5.2.10). They will be needed in the proof of Theorem 5.1.1(3).

Lemma 5.2.11 ([65, Lemma 5.6]). *If F is a bounded Borel function on $E \times [0,1] \times [0,\infty)$ such that $F(y,u) := \lim_{t\to\infty} F(y,u,t)$ exists for each $y \in E$ and $u \in [0,1]$, then*

$$\int_0^1 F(\xi_{(1-u)t}, u, t)du \xrightarrow[t\to\infty]{L^2(\Pi_x^{(\phi)})} \int_0^1 \langle F(\cdot,u), \phi\phi^*\rangle_m du, \quad x \in E.$$

Lemma 5.2.12. *Let F be a non-negative bounded Borel function on $E \times [0,1] \times [0,\infty)$. Define $F(y,u) := \limsup_{t\to\infty} F(y,u,t)$ for each $y \in E$ and $u \in [0,1]$. Then, for each $x \in E$ and $p \geq 1$,*

$$\limsup_{t\to\infty} \left\| \int_0^1 F(\xi_{(1-u)t}, u, t)du \right\|_{\Pi_x^{(\phi)}; L^p} \leq \int_0^1 \langle F(\cdot,u), \phi\phi^*\rangle_m du, \quad x \in E.$$

Proof. For each $(y,u,t) \in E \times [0,1] \times [0,\infty)$, define $\bar{F}(y,u,t) := \sup_{s:s\geq t} F(y,u,s)$. Then $\bar{F}$ is a bounded Borel function on $E \times [0,1] \times [0,\infty)$ such that

$$F(x,u) = \lim_{t\to\infty} \bar{F}(x,u,t), \quad x \in E, u \in [0,1].$$

From Lemma 5.2.11, we know that

$$\int_0^1 \bar{F}(\xi_{(1-u)t}, u, t)du \xrightarrow[t\to\infty]{L^2(\Pi_x^{(\phi)})} \int_0^1 \langle F(\cdot,u), \phi\phi^*\rangle_m du, \quad x \in E,$$

which implies convergence in probability. The bounded convergence theorem then gives that, for each $p \geq 1$,

$$\int_0^1 \bar{F}(\xi_{(1-u)t}, u, t)du \xrightarrow[t\to\infty]{L^p(\Pi_x^{(\phi)})} \int_0^1 \langle F(\cdot,u), \phi\phi^*\rangle_m du, \quad x \in E.$$

Finally, noting that $0 \leq F \leq \bar{F}$, we get

$$\begin{aligned}\limsup_{t\to\infty} \left\| \int_0^1 F(\xi_{(1-u)t}, u, t)du \right\|_{\Pi_x^{(\phi)}; L^p} &\leq \limsup_{t\to\infty} \left\| \int_0^1 \bar{F}(\xi_{(1-u)t}, u, t)du \right\|_{\Pi_x^{(\phi)}; L^p} \\ &= \int_0^1 \langle F(\cdot,u), \phi\phi^*\rangle_m du, \quad x \in E\end{aligned}$$

□

5.3 Proofs

5.3.1 Proof of Theorem 5.1.1(1)

Let $\{X;\mathbf{P}\}$ be a (ξ,ψ)-superprocess satisfying Assumptions 5.1–5.4. In this subsection, we will prove the following result stronger than non-persistency:

Proposition 5.3.1. *For each $t > 0$, $\inf_{x\in E} \mathbf{P}_{\delta_x}(\|X_t\| = 0) > 0$.*

Proof. Recall that $\kappa_0 = \operatorname{ess\,inf}_{m(dx)} \kappa(x)$ and $\gamma_0 = \operatorname{ess\,inf}_{m(dx)} \gamma(x)$. For each $x \in E$, let $\tilde{\kappa}(x) := \kappa(x)\mathbf{1}_{\kappa(x)\geq\kappa_0} + \kappa_0\mathbf{1}_{\kappa(x)<\kappa_0}$ and $\tilde{\gamma}(x) := \gamma(x)\mathbf{1}_{\gamma(x)\geq\gamma_0} + \gamma_0\mathbf{1}_{\gamma(x)<\gamma_0}$. Then, we know that $m(\tilde{\kappa} \neq \kappa) = 0$ and $m(\tilde{\gamma} \neq \gamma) = 0$. Define $\widetilde{\psi}(x,z) := -\beta(x)z + \tilde{\kappa}(x)z^{\tilde{\gamma}(x)}$ for each $x \in E$ and $z \geq 0$, then for each $z \geq 0$, $\widetilde{\psi}(\cdot,z) = \psi(\cdot,z)$, m-almost everywhere.

If we replace ψ with $\tilde{\psi}$ in (5.1.18), the solution $V_t f(x)$ of equation (5.1.18) is also the solution of

$$V_t f(x) + \Pi_x\Big[\int_0^{t\wedge\zeta} \widetilde{\psi}(\xi_s, V_{t-s}f)ds\Big] = \Pi_x\big[f(\xi_t)\mathbf{1}_{t<\zeta}\big].$$

So, we can consider $\{X;\mathbf{P}\}$ as a superprocess with branching mechanism $\tilde{\psi}$. Define

$$\widehat{\psi}(z) := -(\|\beta\|_\infty + \kappa_0)z + \kappa_0 z^{\gamma_0}, \quad z \geq 0.$$

Using the fact that $\gamma_0 > 1$ and $\kappa_0 > 0$, it is easy to verify that

$$\inf_{x\in E} \widetilde{\psi}(x,z) \geq \hat{\psi}(z), \quad z \geq 0; \quad \int_1^\infty \frac{1}{\widehat{\psi}(z)}dz < \infty; \quad \hat{\psi}(+\infty) = +\infty.$$

Therefore $\widetilde{\psi}$ satisfies the condition of [68, Lemma 2.3]. As a consequence, we have the desired result. □

5.3.2 Proof of Theorem 5.1.1(2)

Let $\{X;\mathbf{P}\}$ be a (ξ,ψ)-superprocess satisfying Assumptions 5.1–5.4. From Proposition 5.3.1, we know that our superprocess $\{X;\mathbf{P}\}$ is non-persistent, that is,

$$\mathbf{P}_{\delta_x}(\|X_t\| = 0) > 0, \quad t > 0, x \in E.$$

Notice that $\mathbf{P}_{\delta_x}[X_t(\phi)] = \phi(x) > 0$, so we have

$$\mathbf{P}_{\delta_x}(\|X_t\| = 0) < 1, \quad t > 0, x \in E.$$

From these and (5.2.6), we have that $v_t \in \mathscr{B}_b^{++}(E)$ for each $t > 0$. According to (5.2.6) and (5.2.3), by monotonicity, we see that $(v_t)_{t>0}$ satisfies the equation

$$v_{s+t}(x) + \int_0^t P_{t-r}^{\beta}\psi_0(x, v_{s+r})dr = P_t^{\beta} v_s(x) \in [0,\infty), \quad s > 0, t \geq 0, x \in E.$$

Notice that, under Assumption 5.1, according to (5.1.19), $d\nu := \phi^* dm$ defines a finite measure on E. Therefore, $\langle v_t, \phi^* \rangle_m < \infty$ for each $t > 0$.

According to (5.2.4), (5.2.6) and the monotone convergence theorem, $(v_t)_{t>0}$ also satisfies the equation

$$\langle v_t, \phi^* \rangle_m + \int_s^t \langle \psi_0(\cdot, v_t), \phi^* \rangle_m dr = \langle v_s, \phi^* \rangle_m \in [0, \infty), \quad s, t > 0. \tag{5.3.1}$$

One of the consequences of this equation is that, see [65, Lemma 5.1] for example,

$$\|\phi^{-1} v_t\|_\infty \xrightarrow[t\to\infty]{} 0. \tag{5.3.2}$$

Therefore, to prove Theorem 5.1.1(2), we only need to consider the speed of this convergence. This is answered in two steps. The first step says that $(\phi^{-1} v_t)(x)$ will converge to 0 in the same speed as $\langle v_t, \phi^* \rangle_m$, uniformly in $x \in E$:

Proposition 5.3.2. $(\phi^{-1} v_t)(x) \overset{x\in E}{\underset{t\to\infty}{\sim}} \langle v_t, \phi^* \rangle_m$.

The second step characterizes this speed:

Proposition 5.3.3. *$(\langle v_t, \phi^* \rangle_m)_{t>0}$ is regularly varying at ∞ with index $-\frac{1}{\gamma_0 - 1}$. Furthermore, if $m(x : \gamma(x) = \gamma_0) > 0$, then*

$$\langle v_t, \phi^* \rangle_m \underset{t\to\infty}{\sim} \big(C_X(\gamma_0 - 1)t\big)^{-\frac{1}{\gamma_0 - 1}},$$

where $C_X := \langle \mathbf{1}_{\gamma=\gamma_0} \kappa \phi^{\gamma_0}, \phi^* \rangle_m$.

Proof of Proposition 5.3.2. We use an argument similar to that used in [65] for critical superprocesses with finite 2nd moment. For each $\mu \in \mathcal{M}_E^{\phi}$, denote by $\{(Y_t), (\xi_t), \mathbf{n}; \dot{\mathbf{P}}_\mu^{(\phi)}\}$ the spine representation of $\mathbb{N}_\mu^{(\phi)}$. According to (5.2.7), (5.2.8) and Theorem 5.2.8, we have that for each $t > 0$,

$$\langle \mu, \phi \rangle \dot{\mathbf{P}}_\mu^{(\phi)}[Y_t(\phi)^{-1}] = \mathbb{N}_\mu[W_t(\phi)] \mathbb{N}_\mu^{W_t(\phi)}[W_t(\phi)^{-1}] = \mathbb{N}_\mu(W_t(\phi) > 0) = \mu(v_t). \tag{5.3.3}$$

Taking $\mu = \delta_x$ in (5.3.3), we get $(\phi^{-1} v_t)(x) = \dot{\mathbf{P}}_{\delta_x}^{(\phi)}[Y_t(\phi)^{-1}]$. Taking $\mu = \nu$ in (5.3.3), we get $\langle v_t, \phi^* \rangle_m = \dot{\mathbf{P}}_\nu^{(\phi)}[Y_t(\phi)^{-1}]$. Therefore, to complete the proof, we only need to show that

$$\dot{\mathbf{P}}_{\delta_x}^{(\phi)}[Y_t(\phi)^{-1}] \overset{x\in E}{\underset{t\to\infty}{\sim}} \dot{\mathbf{P}}_\nu^{(\phi)}[Y_t(\phi)^{-1}].$$

For any $t > 0$ and any $G \in \mathscr{B}((0,t])$, define

$$Y_t^G := \int_{G\times\mathbb{W}} w_{t-s}\mathbf{n}(ds, dw).$$

Then for any $0 < t_0 < t$, we can decompose Y_t into

$$Y_t = Y_t^{(0,t_0]} + Y_t^{(t_0,t]}.$$

Using this decomposition, for each $0 < t_0 < t < \infty$ and $x \in E$, we have

$$\dot{\mathbf{P}}^{(\phi)}_{\delta_x}[Y_t(\phi)^{-1}] = \dot{\mathbf{P}}^{(\phi)}_{\nu}[Y_t^{(t_0,t]}(\phi)^{-1}] + \epsilon_x^1(t_0,t) + \epsilon_x^2(t_0,t), \tag{5.3.4}$$

where

$$\begin{aligned}\epsilon_x^1(t_0,t) &:= \dot{\mathbf{P}}^{(\phi)}_{\delta_x}[Y_t^{(t_0,t]}(\phi)^{-1}] - \dot{\mathbf{P}}^{(\phi)}_{\nu}[Y_t^{(t_0,t]}(\phi)^{-1}];\\ \epsilon_x^2(t_0,t) &:= \dot{\mathbf{P}}^{(\phi)}_{\delta_x}[Y_t(\phi)^{-1} - Y_t^{(t_0,t]}(\phi)^{-1}].\end{aligned}$$

By the construction and the Markov property of $\{Y, \xi; \dot{\mathbf{P}}^{\phi}\}$, we have that

$$\dot{\mathbf{P}}^{(\phi)}[Y_t^{(t_0,t]}(\phi)^{-1}|\mathscr{F}^{\xi}_{t_0}] = \dot{\mathbf{P}}^{(\phi)}_{\delta_{\xi_{t_0}}}[Y_{t-t_0}(\phi)^{-1}] = (\phi^{-1}v_{t-t_0})(\xi_{t_0});$$

$$\dot{\mathbf{P}}^{(\phi)}_{\nu}[Y_t^{(t_0,t]}(\phi)^{-1}] = \Pi^{(\phi)}_{\nu}[(\phi^{-1}v_{t-t_0})(\xi_{t_0})] = \langle v_{t-t_0}, \phi^*\rangle_m; \tag{5.3.5}$$

$$\dot{\mathbf{P}}^{(\phi)}_{\delta_x}[Y_t^{(t_0,t]}(\phi)^{-1}] = \Pi^{(\phi)}_{x}[(\phi^{-1}v_{t-t_0})(\xi_{t_0})] = \int_E q_{t_0}(x,y)(\phi^{-1}v_{t-t_0})(y)m(dy). \tag{5.3.6}$$

Let $c_0, c_1 > 0$ be the constants in (5.1.20). We claim that

$$|\epsilon_x^1(t_0,t)| \le c_0 e^{-c_1 t_0}\langle v_{t-t_0}, \phi^*\rangle_m, \quad t_0 > 1. \tag{5.3.7}$$

In fact,

$$\begin{aligned}|\epsilon_x^1(t_0,t)| &= \left|\dot{\mathbf{P}}^{(\phi)}_{\delta_x}[Y_t^{(t_0,t]}(\phi)^{-1}] - \dot{\mathbf{P}}^{(\phi)}_{\nu}[Y_t^{(t_0,t]}(\phi)^{-1}]\right|\\ &= \left|\int_E q_{t_0}(x,y)(\phi^{-1}v_{t-t_0})(y)m(dy) - \langle v_{t-t_0}, \phi^*\rangle_m\right|\\ &\le \int_{y\in E}\left|q_{t_0}(x,y) - (\phi\phi^*)(y)\right|(\phi^{-1}v_{t-t_0})(y)m(dy)\\ &\le c_0 e^{-c_1 t_0}\langle v_{t-t_0}, \phi^*\rangle_m.\end{aligned}$$

We now claim that, if $t_0 > 1$ and $t - t_0$ is large enough, then

$$|\epsilon_x^2(t_0,t)| \le t_0\|\kappa\gamma\phi^{\gamma-1}\|_\infty \cdot \|\phi^{-1}v_{t-t_0}\|_\infty^{\gamma_0-1}(1 + c_0 e^{-c_1 t_0})\langle v_{t-t_0}, \phi^*\rangle_m. \tag{5.3.8}$$

In fact, using the Markov property of the spine process and the property of Poisson random measures, we have

$$\begin{aligned}|\epsilon_x^2(t_0,t)| &= \left|\dot{\mathbf{P}}_{\delta_x}^{(\phi)}[Y_t(\phi)^{-1} - Y_t^{(t_0,t]}(\phi)^{-1}]\right| \\ &= \dot{\mathbf{P}}_{\delta_x}^{(\phi)}[Y_t^{(0,t_0]}(\phi)\cdot Y_t(\phi)^{-1}\cdot Y_t^{(t_0,t]}(\phi)^{-1}] \\ &\le \dot{\mathbf{P}}_{\delta_x}^{(\phi)}[\mathbf{1}_{Y_t^{(0,t_0]}(\phi)\neq 0}\cdot Y_t^{(t_0,t]}(\phi)^{-1}] \\ &= \dot{\mathbf{P}}_{\delta_x}^{(\phi)}\left[\dot{\mathbf{P}}_{\delta_x}^{(\phi)}[\mathbf{1}_{Y_t^{(0,t_0]}(\phi)\neq 0}|\mathscr{F}_{t_0}^{\xi}]\cdot \dot{\mathbf{P}}_{\delta_x}^{(\phi)}[Y_t^{(t_0,t]}(\phi)^{-1}|\mathscr{F}_{t_0}^{\xi}]\right].\end{aligned} \tag{5.3.9}$$

On one hand, according to (5.2.10) and (5.3.6), we know that

$$\dot{\mathbf{P}}_{\delta_x}^{(\phi)}[Y_t^{(t_0,t]}(\phi)^{-1}] \le (1+c_0e^{-c_1t_0})\langle v_{t-t_0},\phi^*\rangle_m. \tag{5.3.10}$$

On the other hand, since $\phi^{-1}v_s$ converges to 0 uniformly when $s\to\infty$, we can choose $s_0>0$ such that for any $s\ge s_0$, we have $\|\phi^{-1}v_s\|_\infty\le 1$. Then, if $t-s>t-t_0\ge s_0$, using the fact that v_t is non-increasing in t, we get

$$\kappa(x)\gamma(x)v_{t-s}(x)^{\gamma(x)-1} \le \|\kappa\gamma\phi^{\gamma-1}\|_\infty\cdot\|\phi^{-1}v_{t-s}\|_\infty^{\gamma_0-1} \le \|\kappa\gamma\phi^{\gamma-1}\|_\infty\cdot\|\phi^{-1}v_{t-t_0}\|_\infty^{\gamma_0-1}.$$

Therefore, using Campbell's formula, (5.1.21) and the fact that $e^{-x}\ge 1-x$, we have, for $t-t_0\ge s_0$,

$$\begin{aligned}&\dot{\mathbf{P}}_{\delta_x}^{(\phi)}[\mathbf{1}_{\|Y_t^{(0,t_0]}\|\neq 0}|\mathscr{F}_{t_0}^{\xi}] \le -\log\left(1-\dot{\mathbf{P}}_{\delta_x}^{(\phi)}[\mathbf{1}_{\|Y_t^{(0,t_0]}\|\neq 0}|\mathscr{F}_{t_0}^{\xi}]\right) \\ &= -\log\lim_{\lambda\to\infty}\dot{\mathbf{P}}_{\delta_x}^{(\phi)}[e^{-\lambda Y_t^{(0,t_0]}(\mathbf{1}_E)}|\mathscr{F}_{t_0}^{\xi}] \\ &= -\log\lim_{\lambda\to\infty}\exp\left\{-\int_{[0,t]\times\mathbb{W}}\left(1-\exp\{-\mathbf{1}_{s\le t_0}w_{t-s}(\lambda\mathbf{1}_E)\}\right)\mathbf{m}^{\xi}(ds,dw)\right\} \\ &= \int_{[0,t]\times\mathbb{W}}\mathbf{1}_{s\le t_0}\mathbf{1}_{\|w_{t-s}\|\neq 0}\mathbf{m}^{\xi}(ds,dw) = \int_0^{t_0}ds\int_{(0,\infty)}y\mathbf{P}_{y\delta_{\xi_s}}[\mathbf{1}_{\|X_{t-s}\|\neq 0}]\pi(\xi_s,dy) \\ &= \int_0^{t_0}ds\int_{(0,\infty)}y(1-e^{-yv_{t-s}(\xi_s)})\frac{\kappa(\xi_s)dy}{\Gamma(-\gamma(\xi_s))y^{1+\gamma(x)}} = \int_0^{t_0}\left(\kappa\gamma v_{t-s}^{\gamma-1}\right)(\xi_s)ds \\ &\le t_0\|\kappa\gamma\phi^{\gamma-1}\|_\infty\cdot\|\phi^{-1}v_{t-t_0}\|_\infty^{\gamma_0-1}.\end{aligned}$$

Combining this with (5.3.9) and (5.3.10), we get (5.3.8).

Now, for $0<t_0<t<\infty$ and $x\in E$, if $t_0>1$ and $t-t_0$ is large enough, according to (5.3.4), (5.3.5), (5.3.6), (5.3.7) and (5.3.10), we have

$$\begin{aligned}\left|\frac{(\phi^{-1}v_t)(x)}{\langle v_{t-t_0},\phi^*\rangle_m}-1\right| &\le \frac{|\epsilon_x^1(t_0,t)|}{\langle v_{t-t_0},\phi^*\rangle_m}+\frac{|\epsilon_x^2(t_0,t)|}{\langle v_{t-t_0},\phi^*\rangle_m} \\ &\le c_0e^{-c_1t_0}+t_0\|\kappa(x)\gamma(x)\phi(x)^{\gamma(x)-1}\|_\infty\cdot\|\phi^{-1}v_{t-t_0}\|_\infty^{\gamma_0-1}(1+c_0e^{-c_1t_0}).\end{aligned} \tag{5.3.11}$$

According to (5.3.2), there exists a map $t \mapsto t_0(t)$ such that,

$$t_0(t) \xrightarrow[t\to\infty]{} \infty; \quad t_0(t)\|\phi^{-1}v_{t-t_0(t)}\|_\infty^{\gamma_0-1} \xrightarrow[t\to\infty]{} 0.$$

Plugging this choice of $t_0(t)$ back into (5.3.11), we have that

$$\sup_{x\in E}\left|\frac{(\phi^{-1}v_t)(x)}{\langle v_{t-t_0(t)},\phi^*\rangle_m} - 1\right| \xrightarrow[t\to\infty]{} 0. \tag{5.3.12}$$

Notice that

$$\begin{aligned}\left|\frac{\langle v_t,\phi^*\rangle_m}{\langle v_{t-t_0(t)},\phi^*\rangle_m} - 1\right| &\le \int \left|\frac{(\phi^{-1}v_t)(x)}{\langle v_{t-t_0(t)},\phi^*\rangle} - 1\right|\phi\phi^*(x)m(dx) \\ &\le \sup_{x\in E}\left|\frac{(\phi^{-1}v_t)(x)}{\langle v_{t-t_0(t)},\phi^*\rangle_m} - 1\right| \xrightarrow[t\to\infty]{} 0.\end{aligned} \tag{5.3.13}$$

Now, by (5.3.12), (5.3.13) and the property of uniform convergence, we get

$$\sup_{x\in E}\left|\frac{(\phi^{-1}v_t)(x)}{\langle v_t,\phi^*\rangle_m} - 1\right| \xrightarrow[t\to\infty]{} 0,$$

as desired. □

Proof of Proposition 5.3.3. From (5.3.1) we know that $\langle v_t,\phi^*\rangle_m$ is continuous and strictly decreasing in $t\in(0,\infty)$. Since the superprocess $(X_t)_{t\ge0}$ is right continuous in the weak topology with the null measure as an absorbing state, we have that, for each $\mu\in\mathcal{M}_E^1$, $\mathbf{P}_\mu(\|X_t\|\neq 0) \xrightarrow[t\to0]{} 1$. Taking $\mu=\nu$, according to (5.2.7), we have that $\langle v_t,\phi^*\rangle_m \xrightarrow[t\to0]{} +\infty$. On the other hand, according to (5.3.2), we have $\langle v_t,\phi^*\rangle_m \xrightarrow[t\to\infty]{} 0$. Therefore, the map $t\mapsto\langle v_t,\phi^*\rangle$ has an inverse on $(0,\infty)$ which is denoted by

$$R:(0,\infty)\to(0,\infty).$$

Now, if we denote by

$$\epsilon_t(x) := \frac{v_t(x)}{\langle v_t,\phi^*\rangle\phi(x)} - 1, \quad t>0, x\in E.$$

Then, we have

$$v_t(x) = \big(1+\epsilon_{R(\langle v_t,\phi^*\rangle)}(x)\big)\langle v_t,\phi^*\rangle\phi(x), \quad t>0, x\in E. \tag{5.3.14}$$

Further, by Proposition 5.3.2 and the fact that $R(u) \xrightarrow[u\to0]{} \infty$, we have

$$\sup_{x\in E}|\epsilon_{R(u)}(x)| \xrightarrow[u\to0]{} 0. \tag{5.3.15}$$

Now, by (5.3.1), we have

$$\frac{d\langle v_r, \phi^*\rangle_m}{dr} = -\langle \psi_0(\cdot, v_r), \phi^*\rangle_m > 0 \quad a.e..$$

Therefore,

$$\begin{aligned} s - t &= \int_t^s dr = \int_s^t \langle \psi_0(\cdot, v_r), \phi^*\rangle_m^{-1} d\langle v_r, \phi^*\rangle_m \\ &\overset{\text{by (5.3.14)}}{=} \int_s^t \big\langle \psi_0\big(\cdot, (1 + \epsilon_{R(\langle v_r, \phi^*\rangle_m)})\langle v_r, \phi^*\rangle \phi\big), \phi^*\big\rangle_m^{-1} d\langle v_r, \phi^*\rangle_m \\ &= \int_{\langle v_s, \phi^*\rangle}^{\langle v_t, \phi^*\rangle} \big\langle \psi_0\big(\cdot, (1 + \epsilon_{R(u)}) u\phi\big), \phi^*\big\rangle_m^{-1} du. \end{aligned}$$

Letting $t \to 0$, we get

$$s = \int_{\langle v_s, \phi^*\rangle}^{\infty} \big\langle \psi_0\big(\cdot, (1 + \epsilon_{R(u)}) u\phi\big), \phi^*\big\rangle_m^{-1} du, \quad s \in (0, \infty).$$

Since R is the inverse of $t \mapsto \langle v_t, \phi^*\rangle$, the above implies that

$$R(r) = \int_r^{\infty} \big\langle \psi_0\big(\cdot, (1 + \epsilon_{R(u)}) u\phi\big), \phi^*\big\rangle_m^{-1} du, \quad r \in (0, \infty). \tag{5.3.16}$$

We now check the regularly varying property of $R(r)$ at $r = 0$. This can be done by considering the regularly varying property of $u \to \big\langle \psi_0\big(\cdot, (1 + \epsilon_{R(u)}) u\phi\big), \phi^*\big\rangle_m$ at 0. According to (5.3.15), $1 + \epsilon_{R(u)}(x) \underset{u\to 0}{\overset{x\in E}{\sim}} 1$. Since $\gamma(\cdot)$ is bounded, we have $\big(1 + \epsilon_{R(u)}(x)\big)^{\gamma(x)} \underset{u\to 0}{\overset{x\in E}{\sim}} 1$. Therefore, from Lemma 5.2.1, we have that

$$\begin{aligned} &\big\langle \psi_0\big(\cdot, (1 + \epsilon_{R(u)}) u\phi\big), \phi^*\big\rangle_m \\ &= \big\langle \kappa(x)\big(1 + \epsilon_{R(u)}(x)\big)^{\gamma(x)} u^{\gamma(x)} \phi(x)^{\gamma(x)}, \phi^*(x)\big\rangle_{m(dx)} \\ &\underset{u\to 0}{\sim} \langle u^{\gamma(x)}, \kappa(x)\phi(x)^{\gamma(x)} \phi^*(x)\rangle_{m(dx)}. \end{aligned} \tag{5.3.17}$$

According to Lemma 5.2.6, and using the fact that $\kappa(x)\phi(x)^{\gamma(x)}$ is bounded and the measure $\phi^* dm$ is finite, we have that $\langle \psi_0\big(\cdot, (1 + \epsilon_{R(u)}) u\phi\big), \phi^*\rangle_m$ is regularly varying at $u = 0$ with index γ_0. Noticing that $-(\gamma_0 - 1) < 0$, according to Corollary 5.2.3 and (5.3.16), R is regularly varying at 0 with index $-(\gamma_0 - 1)$. Therefore, from $R(\langle v_s, \phi^*\rangle_m) = s$ and Corollary 5.2.5, we have that $(\langle v_s, \phi^*\rangle_m)_{s\in(0,\infty)}$ is regularly varying at ∞ with index $-(\gamma_0 - 1)^{-1}$.

Further, if $m\{x : \gamma(x) = \gamma_0\} > 0$, then according to Lemma 5.2.6 and (5.3.17), we know that

$$\begin{aligned} \big\langle \psi_0\big(\cdot, (1 + \epsilon_{R(u)}) u\phi\big), \phi^*\big\rangle_m &\underset{u\to 0}{\sim} \langle u^{\gamma(x)}, \kappa(x)\phi(x)^{\gamma(x)} \phi^*(x)\rangle_{m(dx)} \\ &\underset{u\to 0}{\sim} \langle \mathbf{1}_{\gamma(x)=\gamma_0}, \kappa(x)\phi(x)^{\gamma_0} \phi^*(x)\rangle_{m(dx)} u^{\gamma_0} =: C_X u^{\gamma_0}. \end{aligned}$$

Therefore, we have $\big\langle \psi_0\big(\cdot,(1+\epsilon_{R(u)})u\phi\big),\phi^*\big\rangle_m^{-1} = u^{-\gamma_0}l(u)$, where $l(u)$ converges to the constant C_X^{-1} when $u\to 0$. Now according to Corollary 5.2.3 and (5.3.16) we have that

$$\begin{aligned} R(r) &= \int_r^\infty \big\langle \psi_0\big(\cdot,(1+\epsilon_{R(u)})u\phi\big),\phi^*\big\rangle_m^{-1} du = \int_r^\infty u^{-\gamma_0}l(u)du \\ &= -\frac{1}{\gamma_0-1}\int_r^\infty l(u)du^{-(\gamma_0-1)} \\ &\underset{r\to 0}{\sim} C_X^{-1}(\gamma_0-1)^{-1}r^{-(\gamma_0-1)}. \end{aligned}$$

Now since $r\mapsto \langle v_r,\phi^*\rangle_m$ is the inverse of $r\mapsto R(r)$, from [10, Proposition 1.5.15.] and the above, we have

$$\langle v_r,\phi^*\rangle_m \underset{r\to\infty}{\sim} \big(C_X(\gamma_0-1)r\big)^{-\frac{1}{\gamma_0-1}}.$$

□

Proof of Theorem 5.1.1(2). According to (5.2.7) and (5.3.2),

$$-\log \mathbf{P}_\mu(\|X_t\|=0) = \mu(v_t) \le \mu(\phi)\|\phi^{-1}v_t\|_\infty \xrightarrow[t\to\infty]{} 0.$$

Therefore, $\mathbf{P}_\mu(\|X_t\|\neq 0)\xrightarrow[t\to\infty]{} 0$.

Noticing that $x \underset{x\to 0}{\sim} -\log(1-x)$, according to (5.2.7), Lemma 5.2.1 and Proposition 5.3.2, we have

$$\mathbf{P}_\mu(\|X_t\|\neq 0) \underset{t\to\infty}{\sim} -\log \mathbf{P}_\mu(\|X_t\|=0) = \mu(\phi\phi^{-1}v_t) \underset{t\to\infty}{\sim} \mu(\phi)\langle v_t,\phi^*\rangle_m.$$

Therefore, according to Proposition 5.3.3, we get the desired result. □

5.3.3 Characterization of the one dimensional distribution

Let $\{(X_t)_{t\ge 0};\mathbf{P}\}$ be a (ξ,ψ)-superprocess satisfying Assumptions 5.1–5.4. Suppose $m(x:\gamma(x)=\gamma_0)>0$. Recall that we want to find a proper normalization $(\eta_t)_{t\ge 0}$ such that $\big\{(\eta_t X_t(f))_{t\ge 0};\mathbf{P}_\mu(\cdot|\|X_t\|\neq 0)\big\}$ converges weakly to a non-degenerate distribution for a large class of functions f and initial configurations μ. Our guess of (η_t) is

$$\eta_t := (C_X(\gamma_0-1)t)^{-\frac{1}{\gamma_0-1}},\quad t\ge 0, \tag{5.3.18}$$

because in this case

$$\mathbf{P}_{\delta_x}[\eta_t X_t(f)|\|X_t\|\neq 0] = \frac{\mathbf{P}_{\delta_x}[\eta_t X_t(f)\mathbf{1}_{\|X_t\|\neq 0}]}{\mathbf{P}_{\delta_x}(\|X_t\|\neq 0)} = \frac{\eta_t}{\mathbf{P}_{\delta_x}(\|X_t\|\neq 0)}P_t^\beta f(x) \underset{t\to\infty}{\sim} \langle f,\phi^*\rangle_m.$$

Here we have used Theorem 5.1.1(2) and the fact that (see (5.1.20))

$$P_t^\beta f(x) = \int_E p_t^\beta(x,y)f(y)dy \xrightarrow[t\to\infty]{} \phi(x)\langle f,\phi^*\rangle_m.$$

From the point of view of Laplace transforms, the desired result that, for any $f \in \mathscr{B}_b^+(E)$ and $\mu \in \mathcal{M}_E^1$, $\{(\eta_t X_t(f))_{t\geq 0}; \mathbf{P}_\mu(\cdot|\|X_t\| \neq 0)\}$ converge weakly to some probability distribution F_f is equivalent to the following convergence:

$$\mathbf{P}_\mu[1 - e^{-\theta\eta_t X_t(f)}|\|X_t\| \neq 0] = \frac{1 - \exp\{-\mu(V_t(\theta\eta_t f))\}}{\mathbf{P}_\mu(\|X_t\| \neq 0)} \xrightarrow[t\to\infty]{} \int_{[0,\infty)} (1 - e^{-\theta u})F_f(du).$$

According to Theorem 5.1.1(2) and $1 - e^{-x} \underset{x\to 0}{\sim} x$, this is equivalent to

$$\frac{\mu(V_t(\theta\eta_t f))}{\eta_t} \xrightarrow[t\to\infty]{} \mu(\phi)\int_{[0,\infty)} (1 - e^{-\theta u})F_f(du). \tag{5.3.19}$$

Therefore, to establish the weak convergence of $\{(\eta_t X_t(f))_{t\geq 0}; \mathbf{P}_\mu(\cdot|\|X_t\| \neq 0)\}$, one only needs to verify (5.3.19).

In order to investigate the convergence of $\mu(V_t(\theta\eta_t f))/\eta_t$, we need to investigate the properties of $\theta \to V_t(\theta f)$. (Note that (5.2.3) only gives the the dynamics of $t \to V_t(\theta f)$.) This is done in the following proposition:

Proposition 5.3.4. *For any $f \in \mathscr{B}_b^+(E), \theta \geq 0, x \in E$ and $T > 0$, we have*

$$V_T(\theta f)(x) = \phi(x)\int_0^\theta \Pi_x^{(\phi)}\Big[\frac{f(\xi_T)}{\phi(\xi_T)}\exp\Big\{-\int_0^T \big(\kappa\gamma V_{T-s}(rf)^{\gamma-1}\big)(\xi_s)ds\Big\}\Big]dr. \tag{5.3.20}$$

Proof. It follows from Theorem 5.2.7 and 5.2.8 that

$$\frac{\mathbf{P}_{\delta_x}[X_T(f)e^{-\theta X_T(f)}]}{\mathbf{P}_{\delta_x}[X_T(f)]} = \mathbf{P}_{\delta_x}^{X_T(f)}[e^{-\theta X_T(f)}] = \mathbf{P}_{\delta_x}[e^{-\theta X_T(f)}]\dot{\mathbf{P}}_x^{(T,f)}[e^{-\theta Y_T(f)}],$$

where $\{(\xi)_{0\leq t\leq T}, \mathbf{n}_T, (Y)_{0\leq t\leq T}; \dot{\mathbf{P}}_x^{(f,T)}\}$ is a spine representation of $\mathbb{N}_x^{W_T(f)}$ with $\mathbf{m}_T^\xi$ being the intensity of the immigration measure $\mathbf{n}_T$ conditioned on $\{(\xi)_{0\leq t\leq T}; \dot{\mathbf{P}}_x^{(f,T)}\}$. From this, we have

$$\frac{\partial}{\partial\theta}(-\log\mathbf{P}_{\delta_x}[e^{-\theta X_T(f)}]) = \frac{\mathbf{P}_{\delta_x}[X_T(f)e^{-\theta X_T(f)}]}{\mathbf{P}_{\delta_x}[e^{-\theta X_T(f)}]} = P_T^\beta f(x)\dot{\mathbf{P}}_x^{(T,f)}[e^{-\theta Y_T(f)}]. \tag{5.3.21}$$

On the other hand, if we write $F(s,w) := \mathbf{1}_{s\leq T} w_{T-s}(f)$, then by Assumption 5.4, Campbell's formula and (5.1.21), we have

$$\begin{aligned} -\log\dot{\mathbf{P}}_x^{(T,f)}[e^{-\theta\mathbf{n}_T(F)}|\mathbf{m}_T^\xi] &= \mathbf{m}_T^\xi(1 - e^{-\theta F}) \qquad (5.3.22)\\ &= \int_0^T ds\int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}[1 - e^{-\theta X_{T-s}(f)}]\pi(\xi_s,y)\\ &= \int_0^T ds\cdot\kappa(\xi_s)\int_{(0,\infty)} (1 - e^{-yV_{T-s}(\theta f)(\xi_s)})\frac{dy}{\Gamma(-\gamma(\xi_s))y^{\gamma(\xi_s)}} \end{aligned}$$

$$= \int_0^T \left(\kappa\gamma V_{T-s}(\theta f)^{\gamma-1}\right)(\xi_s)ds.$$

Note that, since $\mathbf{n}_T(F) = Y_T(f)$, we can derive from (5.3.21) and (5.3.22) that

$$\begin{aligned} V_T(\theta f)(x) &= -\log \mathbf{P}_{\delta_x}[e^{-\theta X_T(f)}] = \int_0^\theta P_T^\beta f(x)\dot{\mathbf{P}}_x^{(T,f)}[e^{-rY_T(f)}]dr \\ &= P_T^\beta f(x) \int_0^\theta \Pi_x^{(T,f)}\Big[\exp\Big\{-\int_0^T \left(\kappa\gamma V_{T-s}(rf)^{\gamma-1}\right)(\xi_s)\, ds\Big\}\Big]dr \\ &= \phi(x)\int_0^\theta \Pi_x^{(\phi)}\Big[\frac{f(\xi_T)}{\phi(\xi_T)}\exp\Big\{-\int_0^T \left(\kappa\gamma V_{T-s}(rf)^{\gamma-1}\right)(\xi_s)ds\Big\}\Big]dr, \end{aligned}$$

as required. □

Replacing θ with $\theta\eta_T$ in (5.3.20), we have

$$\begin{aligned} &\frac{V_T(\theta\eta_T f)(x)}{\eta_T} \qquad (5.3.23) \\ &= \phi(x)\frac{1}{\eta_T}\int_0^{\theta\eta_T} \Pi_x^{(\phi)}\Big[\frac{f(\xi_T)}{\phi(\xi_T)}\exp\Big\{-\int_0^T \left(\kappa\gamma V_{T-s}(rf)^{\gamma-1}\right)(\xi_s)ds\Big\}\Big]dr \\ &= \phi(x)\int_0^{\theta} \Pi_x^{(\phi)}\Big[\frac{f(\xi_T)}{\phi(\xi_T)}\exp\Big\{-\int_0^T \left(\kappa\gamma V_{T-s}(r\eta_T f)^{\gamma-1}\right)(\xi_s)ds\Big\}\Big]dr \\ &= \phi(x)\int_0^{\theta} \Pi_x^{(\phi)}\Big[\frac{f(\xi_T)}{\phi(\xi_T)}\exp\Big\{-T\int_0^1 \left(\kappa\gamma V_{uT}(r\eta_T f)^{\gamma-1}\right)(\xi_{(1-u)T})du\Big\}\Big]dr. \end{aligned}$$

5.3.4 Distribution with Laplace transform (5.1.6)

The distribution with Laplace transform (5.1.6) can be characterized by the following result.

Lemma 5.3.5. *The non-linear delay equation*

$$G(\theta) = \int_0^\theta \exp\Big\{-\frac{\gamma_0}{\gamma_0-1}\int_0^1 G(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u}\Big\}dr, \quad \theta \ge 0, \qquad (5.3.24)$$

has a unique solution:

$$G(\theta) = \Big(\frac{1}{1+\theta^{-(\gamma_0-1)}}\Big)^{\frac{1}{\gamma_0-1}}, \quad \theta \ge 0. \qquad (5.3.25)$$

We first introduce some notation: If f is a measurable function which is L^p integrable on the measure space $(S, \mathscr{S}, \mu)$ with $p > 0$, then we write

$$\|f\|_{\mu;p} := \Big(\int_S |f|^p d\mu\Big)^{\frac{1}{p}}.$$

Notice that, when $p \ge 1$, $\|f\|_{\mu;p}$ is simply the L^p norm of f with respect to the measure μ. In order to prove the above lemma, we will need the following:

Lemma 5.3.6. *Suppose that F is a non-negative function on $[0,\infty)$ satisfying the property that there exists a constant $C > 0$ such that $F(\theta) \le C\theta$ for all $\theta \ge 0$ and*

$$F(\theta) \le C \int_0^\theta \|F(ru^{\frac{1}{\gamma_0-1}})\|_{\mathbf{1}_{0<u<1}\frac{du}{u};\gamma_0-1} dr, \quad \theta \ge 0.$$

Then $F \equiv 0$.

Proof. We prove this lemma by contradiction. Assume that

$$\rho := \sup\{x : F(\theta) = 0, \theta \in [0,x)\} < \infty. \tag{5.3.26}$$

Write $F_\alpha(\theta) := F(\alpha+\theta)$ for each $\alpha, \theta \ge 0$. We first claim that

$$F_\alpha(\theta) \le C(\rho C + 1)\theta, \quad \theta \le \frac{1}{C}, \alpha \le \rho.$$

In fact, if $\theta \le \frac{1}{C}$ and $\alpha \le \rho$, then

$$\begin{aligned} F_\alpha(\theta) &\le C \int_\alpha^{\alpha+\theta} \|F(ru^{\frac{1}{\gamma_0-1}})\|_{\mathbf{1}_{0<u<1}\frac{du}{u};(\gamma_0-1)} dr \le C \int_\alpha^{\alpha+\theta} \|Cru^{\frac{1}{\gamma_0-1}}\|_{\mathbf{1}_{0<u<1}\frac{du}{u};\gamma_0-1} dr \\ &\le C^2(\alpha+\theta)\theta \|u^{\frac{1}{\gamma_0-1}}\|_{\mathbf{1}_{0<u<1}\frac{du}{u};\gamma_0-1} \le C(\rho C + 1)\theta. \end{aligned}$$

We then claim that, if

$$F_\alpha(\theta) \le C^k(\rho C + 1)\theta^k, \quad \theta \le \frac{1}{C}, \alpha \le \rho, \tag{5.3.27}$$

for some $k \in \mathbb{N}$, then

$$F_\alpha(\theta) \le C^{k+1}(\rho C + 1)\theta^{k+1}, \quad \theta \le \frac{1}{C}, \alpha \le \rho.$$

In fact, if (5.3.27) is true, then for each $\theta \le \frac{1}{C}$ and $\alpha \le \rho$,

$$\begin{aligned} F_\alpha(\theta) &\le C \int_\alpha^{\alpha+\theta} \|F(ru^{\frac{1}{\gamma_0-1}})\|_{\mathbf{1}_{0<u<1}\frac{du}{u};\gamma_0-1} dr \\ &= C \int_0^\theta \left\|F\left((\alpha+r)u^{\frac{1}{\gamma_0-1}}\right)\right\|_{\mathbf{1}_{0<u<1}\frac{du}{u};\gamma_0-1} dr \\ &= C \int_0^\theta \|F_{\alpha u^{1/(\gamma_0-1)}}(ru^{\frac{1}{\gamma_0-1}})\|_{\mathbf{1}_{0<u<1}\frac{du}{u};\gamma_0-1} dr \\ &\le C \int_0^\theta \|C^k(\rho C+1)r^k u^{\frac{k}{\gamma_0-1}}\|_{\mathbf{1}_{0<u<1}\frac{du}{u};\gamma_0-1} dr \\ &\le C^{k+1}(\rho C+1)\theta^{k+1}\|u^{\frac{k}{\gamma_0-1}}\|_{\mathbf{1}_{0<u<1}\frac{du}{u};\gamma_0-1} \\ &\le C^{k+1}(\rho C+1)\theta^{k+1}. \end{aligned}$$

Therefore, by induction, we have

$$F_\alpha(\theta) \le C^k(\rho C+1)\theta^k, \quad \theta \le \frac{1}{C}, \alpha \le \rho, k \in \mathbb{N}.$$

As a consequence, we must have $F(\theta) = 0$ if $\theta < \rho + \frac{1}{C}$. This, however, contradicts (5.3.26). □

Proof of Lemma 5.3.5. We first verify that (5.3.25) is a solution of (5.3.24). In fact, if $G(\theta) = (\frac{1}{1+\theta^{-(\gamma_0-1)}})^{\frac{1}{\gamma_0-1}}$, then

$$\begin{aligned}
&\int_0^\theta \exp\Big\{-\frac{\gamma_0}{\gamma_0-1}\int_0^1 G(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u}\Big\}dr \\
&= \int_0^\theta \exp\Big\{-\frac{\gamma_0}{\gamma_0-1}\int_0^1 \frac{du}{u+r^{-(\gamma_0-1)}}\Big\}dr = \int_0^\theta \exp\Big\{-\frac{\gamma_0}{\gamma_0-1}\log\frac{1+r^{-(\gamma_0-1)}}{r^{-(\gamma_0-1)}}\Big\}dr \\
&= \int_0^\theta (\frac{1+r^{-(\gamma_0-1)}}{r^{-(\gamma_0-1)}})^{-\frac{\gamma_0}{\gamma_0-1}}dr = \int_0^\theta \big(1+r^{-(\gamma_0-1)}\big)^{-\frac{\gamma_0}{\gamma_0-1}} r^{-\gamma_0}dr = G(\theta).
\end{aligned}$$

The last equality is due to $G(0) = 0$ and

$$\begin{aligned}
\frac{d}{d\theta}G(\theta) &= -\frac{1}{\gamma_0-1}\big(1+\theta^{-(\gamma_0-1)}\big)^{-\frac{1}{\gamma_0-1}-1}\frac{d}{d\theta}\theta^{-(\gamma_0-1)} \\
&= \big(1+\theta^{-(\gamma_0-1)}\big)^{-\frac{\gamma_0}{\gamma_0-1}}\theta^{-\gamma_0}.
\end{aligned}$$

Now assume that G_0 is another solution to the equation (5.3.24), we then only have to show that $G_0 = G$. This can be done by showing that $F(\theta) = 0$ where

$$F(\theta) := |G(\theta)^{\gamma_0-1} - G_0(\theta)^{\gamma_0-1}|^{\frac{1}{\gamma_0-1}}, \quad \theta \ge 0.$$

We claim that the non-negative function F satisfies the following inequality:

$$F(\theta) \le C_0 \int_0^\theta \|F(ru^{\frac{1}{\gamma_0-1}})\|_{\mathbf{1}_{0<u<1}\frac{du}{u};\gamma_0-1}dr, \quad \theta \ge 0, \tag{5.3.28}$$

for some constant $C_0 > 0$. In fact, by the L^p Minkowski inequality with $p = \frac{1}{\gamma_0-1} > 1$, we have

$$\begin{aligned}
&|G(\theta)^{\gamma_0-1} - G_0(\theta)^{\gamma_0-1}| \\
&= \Big|\|e^{-\gamma_0\int_0^1 G(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u}}\|_{\mathbf{1}_{0<r<\theta}dr;\frac{1}{\gamma_0-1}} - \|e^{-\gamma_0\int_0^1 G_0(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u}}\|_{\mathbf{1}_{0<r<\theta}dr;\frac{1}{\gamma_0-1}}\Big| \\
&\le \|e^{-\gamma_0\int_0^1 G(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u}} - e^{-\gamma_0\int_0^1 G_0(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u}}\|_{\mathbf{1}_{0<r<\theta}dr;\frac{1}{\gamma_0-1}} \\
&\le \Big\|\gamma_0\int_0^1 G(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u} - \gamma_0\int_0^1 G_0(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u}\Big\|_{\mathbf{1}_{0<r<\theta}dr;\frac{1}{\gamma_0-1}} \\
&\le \gamma_0\bigg(\int_0^\theta \Big(\int_0^1 |G(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1} - G_0(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}|\frac{du}{u}\Big)^{\frac{1}{\gamma_0-1}}dr\bigg)^{\gamma_0-1}.
\end{aligned}$$

In other words, there is a constant $C_0 := \gamma_0^{\frac{1}{\gamma_0-1}} > 0$ such that (5.3.28) is true. On the other hand, according to (5.3.24), we have that $G(\theta) \leq \theta$ and $G_0(\theta) \leq \theta$. Therefore, we also have that there is a constant $C_1 > 0$ such that $F(\theta) \leq C_1\theta$. Therefore, according to Lemma 5.3.6 and (5.3.28), we have $F \equiv 0$ as desired. □

5.3.5 Proof of Theorem 5.1.1(3)

Consider the (ξ,ψ)-superprocess $\{X;\mathbf{P}\}$ which satisfies Assumptions 5.1–5.4. Suppose that $m(x : \gamma(x) = \gamma_0) > 0$. Let $f \in \mathscr{B}^+(E)$ be such that $\langle f, \phi^*\rangle_m > 0$ and $c_f := \|\phi^{-1} f\|_\infty < \infty$.

Without loss of generality, we assume that $\langle f, \phi^*\rangle_m = 1$. We claim that, in order to prove Theorem 5.1.1(3), we only need to show that

$$g(t,\theta,x) := \frac{V_t(\theta\eta_t f)(x)}{\eta_t\phi(x)} \xrightarrow[t\to\infty]{} G(\theta) := \Big(\frac{1}{1+\theta^{-(\gamma_0-1)}}\Big)^{\frac{1}{\gamma_0-1}}, \quad x \in E, \theta \geq 0. \tag{5.3.29}$$

In fact, by (5.3.23), we have $\|V_t(\theta\eta_t f)/\eta_t\|_\infty \leq \theta\|\phi\|_\infty\|\phi^{-1}f\|_\infty$. Therefore, if (5.3.29) is true, then by the bounded convergence theorem, for each $\mu \in \mathcal{M}_E^1$,

$$\frac{\mu\big(V_t(\theta\eta_t f)\big)}{\eta_t} \xrightarrow[t\to\infty]{} \mu(\phi)G(\theta),$$

which, by the discussion in Subsection 5.3.3, is equivalent to Theorem 5.1.1(3).

From Lemma 5.3.5, we have that G satisfies

$$G(\theta) = \int_0^\theta e^{-\frac{1}{\gamma_0-1}J_G(r)}dr, \quad \theta \geq 0, \tag{5.3.30}$$

where

$$J_G(r) := \gamma_0\int_0^1 G(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1}\frac{du}{u}, \quad r \geq 0. \tag{5.3.31}$$

According to (5.3.23), we know that g satisfies

$$g(t,\theta,x) = \int_0^\theta \Pi_x^{(\phi)}[(\phi^{-1}f)(\xi_t)e^{-\frac{1}{\gamma_0-1}J_g(t,r,\xi)}]dr, \quad t \geq 0, \theta \geq 0, x \in E, \tag{5.3.32}$$

where, for each $t \geq 0$ and $r \geq 0$,

$$J_g(t,r,\xi) := (\gamma_0-1)t\int_0^1 \big(\kappa\gamma\cdot(\phi\eta_{ut})^{\gamma-1}g(ut, ru^{\frac{1}{\gamma_0-1}},\cdot)^{\gamma-1}\big)(\xi_{(1-u)t})du. \tag{5.3.33}$$

For each $t \geq 0$ and $r \geq 0$, define

$$J'_G(t,r,\xi) := \gamma_0(\gamma_0-1)t\int_0^1 \big(\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\cdot(\phi\eta_{ut})^{\gamma_0-1}G\big(ru^{\frac{1}{\gamma_0-1}}\big)^{\gamma_0-1}\big)(\xi_{(1-u)t})du \tag{5.3.34}$$

and

$$J'_g(t,r,\xi) := \gamma_0(\gamma_0-1)t\int_0^1 \big(\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\cdot(\phi\eta_{ut})^{\gamma_0-1}g\big(ut, ru^{\frac{1}{\gamma_0-1}},\cdot\big)^{\gamma_0-1}\big)(\xi_{(1-u)t})du. \tag{5.3.35}$$

The underlying idea of the proof is to show that J_G, J'_G, J_g and J'_g are approximately equal in some sense when $t \to \infty$.

Step 1: We will give upper bounds for G, g, J_G, J'_G, J_g and J'_g respectively. From (5.3.30) we have

$$G(r) \leq r, \quad r \geq 0. \tag{5.3.36}$$

From (5.3.31) and (5.3.36), we have

$$J_G(r) \leq \gamma_0 r^{\gamma_0 - 1}, \quad r \geq 0. \tag{5.3.37}$$

From (5.3.32), we have

$$g(t, r, x) \leq c_f r, \quad t \geq 0, r \geq 0, x \in E. \tag{5.3.38}$$

From (5.3.18), (5.3.33), (5.3.38) and the fact that $\gamma(\cdot) - 1 < 1$, we have

$$\begin{aligned}
J_g(t, r, \xi) &\leq \|\kappa \cdot (c_f \phi)^{\gamma - 1}\|_\infty \int_0^1 \big(t\eta_{ut}^{\gamma - 1}(ru^{\frac{1}{\gamma_0 - 1}})^{\gamma - 1}\big)\big(\xi_{(1-u)t}\big)du \\
&= \|\kappa \cdot (c_f \phi)^{\gamma - 1}\|_\infty \int_0^1 \big(r^{\gamma - 1}t^{1 - \frac{\gamma - 1}{\gamma_0 - 1}}\big(C_X(\gamma_0 - 1)\big)^{-\frac{\gamma - 1}{\gamma_0 - 1}}\big)\big(\xi_{(1-u)t}\big)du \\
&\leq \max\{1, r\} \cdot \|\kappa \cdot (c_f \phi)^{\gamma - 1}\|_\infty \Big\|\big(C_X(\gamma_0 - 1)\big)^{-\frac{\gamma - 1}{\gamma_0 - 1}}\Big\|_\infty \\
&:= c_2 \cdot \max\{1, r\}, \quad t \geq 1, r \geq 0.
\end{aligned}$$

From (5.3.18), (5.3.35) and (5.3.38), we have

$$\begin{aligned}
J'_g(t, r, \xi) &\leq \gamma_0(\gamma_0 - 1)t \int_0^1 \big(\mathbf{1}_{\gamma(\cdot) = \gamma_0}\kappa \cdot (\phi\eta_{ut})^{\gamma_0 - 1}(c_f r u^{\frac{1}{\gamma_0 - 1}})^{\gamma_0 - 1}\big)(\xi_{(1-u)t})du \\
&\leq \gamma_0(\gamma_0 - 1)c_f^{\gamma_0 - 1} r^{\gamma_0 - 1}\|\mathbf{1}_{\gamma(\cdot) = \gamma_0}\kappa\phi^{\gamma_0 - 1}\|_\infty \int_0^1 t\big(C_X(\gamma_0 - 1)ut\big)^{-1}udu \\
&=: c_3 \cdot r^{\gamma_0 - 1}, \quad t \geq 0, r \geq 0.
\end{aligned}$$

From (5.3.18), (5.3.34) and (5.3.36), we have

$$\begin{aligned}
J'_G(t, r, \xi) &\leq \gamma_0(\gamma_0 - 1)t \int_0^1 \big(\mathbf{1}_{\gamma(\cdot) = \gamma_0}\kappa \cdot (\phi\eta_{ut})^{\gamma_0 - 1}(r u^{\frac{1}{\gamma_0 - 1}})^{\gamma_0 - 1}\big)(\xi_{(1-u)t})du \qquad (5.3.39) \\
&\leq \gamma_0(\gamma_0 - 1) r^{\gamma_0 - 1}\big\|\mathbf{1}_{\gamma(\cdot) = \gamma_0}\kappa\phi^{\gamma_0 - 1}\big\|_\infty \int_0^1 t\big(C_X(\gamma_0 - 1)ut\big)^{-1}udu \\
&=: c_4 \cdot r^{\gamma_0 - 1}, \quad t \geq 0, r \geq 0.
\end{aligned}$$

Step 2: We will show that, for each $t \geq 0, \theta \geq 0$, and $x \in E$

$$|G(\theta)^{\gamma_0 - 1} - g(t, \theta, x)^{\gamma_0 - 1}|$$

$$\le I_1(t,\theta,x) + c_f^{\gamma_0-1} I_2(t,\theta,x) + c_f^{\gamma_0-1} I_3(t,\theta,x) + c_f^{\gamma_0-1} I_4(t,\theta,x),$$

where

$$I_1(t,\theta,x) := \Big\| e^{-J_G(r)} - \|(\phi^{-1}f)(\xi_t)^{\gamma_0-1} e^{-J_G(r)}\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}},$$

$$I_2(t,\theta,x) := \Big\| \|J_G(r) - J'_G(t,r,\xi)\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}},$$

$$I_3(t,\theta,x) := \Big\| \|J'_G(t,r,\xi) - J'_g(t,r,\xi)\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}},$$

and

$$I_4(t,\theta,x) := \Big\| \|J'_g(t,r,\xi) - J_g(t,r,\xi)\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}}.$$

In fact, we can rewrite (5.3.30) and (5.3.32) as:

$$G(\theta)^{\gamma_0-1} = \|e^{-J_G(r)}\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}}, \quad \theta \ge 0,$$

and

$$g(t,\theta,x)^{\gamma_0-1} = \Big\| \|(\phi^{-1}f)(\xi_t)^{\gamma_0-1} e^{-J_g(t,r,\xi)}\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}}, \quad t\ge 0, \theta\ge 0, x\in E.$$

Therefore, by Minkowski's inequality we have that, for each $t \ge 0, \theta \ge 0$ and $x \in E$,

$$\begin{aligned}
&|G(\theta)^{\gamma_0-1} - g(t,\theta,x)^{\gamma_0-1}| \\
&\le \Big\| e^{-J_G(r)} - \|(\phi^{-1}f)(\xi_t)^{\gamma_0-1} e^{-J_g(t,r,\xi)}\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}} \\
&\le I_1(t,\theta,x) + \Big\| \|(\phi^{-1}f)(\xi_t)^{\gamma_0-1} e^{-J_G(r)}\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} - \\
&\qquad\qquad \|(\phi^{-1}f)(\xi_t)^{\gamma_0-1} e^{-J_g(t,r,\xi)}\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}} \\
&\le I_1(t,\theta,x) + \Big\| \|(\phi^{-1}f)(\xi_t)^{\gamma_0-1} (e^{-J_G(r)} - e^{-J_g(t,r,\xi)})\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}} \\
&\le I_1(t,\theta,x) + c_f^{\gamma_0-1} \Big\| \|J_G(r) - J_g(t,r,\xi)\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}} \\
&\le I_1(t,\theta,x) + c_f^{\gamma_0-1} I_2(t,\theta,x) + c_f^{\gamma_0-1} I_3(t,\theta,x) + c_f^{\gamma_0-1} I_4(t,\theta,x).
\end{aligned}$$

Step 3: We will show that, for each $\theta \ge 0$ and $x \in E$, $I_1(t,\theta,x) \xrightarrow[t\to\infty]{} 0$. Notice that, by (5.1.20),

$$\Pi_x^{(\phi)}[(\phi^{-1}f)(\xi_t)] = \phi(x)^{-1}\Pi_x[f(\xi_t)e^{-\int_0^t \beta(\xi_s)ds}] = \phi(x)^{-1}P_t^\beta f(x) \xrightarrow[t\to\infty]{} 1, \quad x \in E.$$

Therefore,

$$\begin{aligned}
&e^{-J_G(r)} - \|(\phi^{-1} f)(\xi_t)^{\gamma_0-1} e^{-J_G(r)}\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \\
&\quad = e^{-J_G(r)}\Big(1 - \Pi_x^{(\phi)}[(\phi^{-1} f)(\xi_t)]^{\gamma_0-1}\Big) \xrightarrow[t\to\infty]{} 0, \quad x \in E, r \geq 0.
\end{aligned}$$

We also have the following bound:

$$\Big|e^{-J_G(r)} - \|(\phi^{-1} f)(\xi_t)^{\gamma_0-1} e^{-J_G(r)}\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}}\Big| \leq 1 + c_f^{\gamma_0-1}.$$

Therefore, by the bounded convergence theorem, we have that, for each $\theta \geq 0$ and $x \in E$, $I_1(t,\theta,x) \xrightarrow[t\to\infty]{} 0$.

Step 4: We will show that, for each $\theta \geq 0$ and $x \in E$, $I_2(t,\theta,x) \xrightarrow[t\to\infty]{} 0$. Notice that, according to (5.3.31) and (5.3.34), for each $t \geq 0$ and $r \geq 0$,

$$\begin{aligned}
&J_G(r) - J'_G(t,r,\xi) \\
&\quad = \int_0^1 \gamma_0 G\big(ru^{\frac{1}{\gamma_0-1}}\big)^{\gamma_0-1}\big(1 - (\gamma_0 - 1)\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\phi^{\gamma_0-1} tu\eta_{ut}^{\gamma_0-1}\big)(\xi_{(1-u)t})\frac{du}{u} \\
&\quad = \int_0^1 \gamma_0 G\big(ru^{\frac{1}{\gamma_0-1}}\big)^{\gamma_0-1}\big(1 - C_X^{-1}\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\phi^{\gamma_0-1}\big)(\xi_{(1-u)t})\frac{du}{u}.
\end{aligned}$$

Also notice that, according to (5.3.36), for each $r \geq 0$, $u \in [0,1]$ and $x \in E$,

$$\begin{aligned}
&\big|\gamma_0 G\big(ru^{\frac{1}{\gamma_0-1}}\big)^{\gamma_0-1}\big(1 - C_X^{-1}\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\phi^{\gamma_0-1}\big)(x)\frac{1}{u}\big| \\
&\quad \leq \frac{\gamma_0}{u} G\big(ru^{\frac{1}{\gamma_0-1}}\big)^{\gamma_0-1}\big|\big(1 - C_X^{-1}\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\phi^{\gamma_0-1}\big)(x)\big| \\
&\quad \leq \gamma_0 r^{\gamma_0-1}\big(1 + \big\|C_X^{-1}\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\phi^{\gamma_0-1}\big\|_\infty\big).
\end{aligned}$$

Therefore, according to Lemma 5.2.11 and the definition of C_X, we have that, for each $r \geq 0$ and $x \in E$,

$$J_G(r) - J'_G(t,r,\xi) \xrightarrow[t\to\infty]{L^2(\Pi_x^{(\phi)})} \int_0^1 \frac{\gamma_0}{u} G\big(ru^{\frac{1}{\gamma_0-1}}\big)^{\gamma_0-1}\big\langle 1 - C_X^{-1}\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\phi^{\gamma_0-1}, \phi\phi^*\big\rangle_m du = 0.$$

According to (5.3.37) and (5.3.39), we have that, for each $r \geq 0$ and $t \geq 0$,

$$\big|J_G(r) - J'_G(t,r,\xi)\big| \leq (\gamma_0 + c_4) r^{\gamma_0-1}. \tag{5.3.40}$$

Therefore, according to the bounded convergence theorem, we have that, for each $r \geq 0$ and $x \in E$,

$$\big\|J_G(r) - J'_G(t,r,\xi)\big\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \xrightarrow[t\to\infty]{} 0.$$

According to (5.3.40), we have that, for each $\theta \geq 0$, $r \in [0,\theta]$ and $x \in E$,

$$\left\| J_G(r) - J'_G(t,r,\xi) \right\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \leq (\gamma_0 + c_4)\theta^{\gamma_0-1}.$$

Finally, according to the bounded convergence theorem, we have that, for each $\theta \geq 0$ and $x \in E$, $I_2(t,\theta,x) \xrightarrow[t\to\infty]{} 0$.

Step 5: We will show that, for each $\theta \geq 0$ and $x \in E$, $I_4(t,\theta,x) \xrightarrow[t\to\infty]{} 0$. We first note that, for each $t \geq 0$ and $r \geq 0$, we have

$$J_g(t,r,\xi) - J'_g(t,r,\xi) = (\gamma_0 - 1)t \int_0^1 \big(\mathbf{1}_{\gamma(\cdot)>\gamma_0}\kappa\gamma \cdot (\phi\eta_{ut})^{\gamma-1} g(ut, ru^{\frac{1}{\gamma_0-1}}, \cdot)^{\gamma-1}\big)\big(\xi_{(1-u)t}\big)du. \tag{5.3.41}$$

We then note that, according (5.3.38) and the definition of η_t, for each $r \geq 0$, $u \in (0,1)$ and $x \in E$, we have

$$\begin{aligned}
&(\gamma_0 - 1)t\mathbf{1}_{\gamma(x)>\gamma_0}\kappa(x)\gamma(x)\big(\phi(x)\eta_{ut}\big)^{\gamma(x)-1} g\big(ut, ru^{\frac{1}{\gamma_0-1}}, x\big)^{\gamma(x)-1} \\
&\quad\leq (\gamma_0 - 1)\big\|\kappa\gamma \cdot (c_f r\phi)^{\gamma-1}\big\|_\infty \mathbf{1}_{\gamma(x)>\gamma_0} t\eta_{ut}^{\gamma(x)-1} u^{\frac{\gamma(x)-1}{\gamma_0-1}} \\
&\quad= (\gamma_0 - 1)\big\|\kappa\gamma \cdot (c_f r\phi)^{\gamma-1}\big\|_\infty \mathbf{1}_{\gamma(x)>\gamma_0} t\big(C_X(\gamma_0 - 1)ut\big)^{-\frac{\gamma(x)-1}{\gamma_0-1}} u^{\frac{\gamma(x)-1}{\gamma_0-1}} \\
&\quad\leq (\gamma_0 - 1)\mathbf{1}_{\gamma(x)>\gamma_0} t^{1-\frac{\gamma(x)-1}{\gamma_0-1}} \big\|\kappa\gamma \cdot (c_f r\phi)^{\gamma-1}\big\|_\infty \sup_{x\in E}\big(C_X(\gamma_0 - 1)\big)^{-\frac{\gamma(x)-1}{\gamma_0-1}} \\
&\quad\xrightarrow[t\to\infty]{} 0.
\end{aligned} \tag{5.3.42}$$

This also gives an upper bound: For each $r \geq 0$, $u \in (0,1)$, $x \in E$ and $t \geq 1$, we have

$$\begin{aligned}
&(\gamma_0 - 1)t\mathbf{1}_{\gamma(x)>\gamma_0}\kappa(x)\gamma(x)\big(\phi(x)\eta_{ut}\big)^{\gamma(x)-1} g\big(ut, ru^{\frac{1}{\gamma_0-1}}, x\big)^{\gamma(x)-1} \\
&\quad\leq (\gamma_0 - 1)\big\|\kappa\gamma \cdot (c_f r\phi)^{\gamma-1}\big\|_\infty \sup_{x\in E}\big(C_X(\gamma_0 - 1)\big)^{-\frac{\gamma(x)-1}{\gamma_0-1}}.
\end{aligned} \tag{5.3.43}$$

Now, with (5.3.41), (5.3.42) and (5.3.44), we can apply Lemma 5.2.11 to the function

$$(y,u,t) \mapsto (\gamma_0 - 1)t\mathbf{1}_{\gamma(y)>\gamma_0}\kappa(y)\gamma(y)\big(\phi(y)\eta_{ut}\big)^{\gamma(y)-1} g\big(ut, ru^{\frac{1}{\gamma_0-1}}, y\big)^{\gamma(y)-1},$$

which says that, for each $r \geq 0$,

$$J_g(t,r,\xi) - J'_g(t,r,\xi) \xrightarrow[t\to\infty]{L^2(\Pi_x^{(\phi)})} 0.$$

According to (5.3.41) and (5.3.43), for each $r \geq 0$ and $t \geq 1$, we have that

$$\big|J_g(t,r,\xi) - J'_g(t,r,\xi)\big| \leq (\gamma_0 - 1)\big\|\kappa\gamma \cdot (c_f r\phi)^{\gamma-1}\big\|_\infty \sup_{x\in E}\big(C_X(\gamma_0 - 1)\big)^{-\frac{\gamma(x)-1}{\gamma_0-1}}. \tag{5.3.44}$$

Therefore, according to the bounded convergence theorem, for each $r \geq 0$ and $x \in E$, we have that

$$\left\|J_g'(t,r,\xi) - J_g(t,r,\xi)\right\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \xrightarrow[t\to\infty]{} 0.$$

According to (5.3.44), for each $\theta \geq 0$, $r \in [0,\theta]$, $t \geq 1$ and $x \in E$, we have that

$$\left\|J_g'(t,r,\xi) - J_g(t,r,\xi)\right\|_{\Pi_x^{(\phi)};\frac{1}{\gamma_0-1}} \leq (\gamma_0 - 1)\left\|\kappa\gamma \cdot (c_f\theta\phi)^{\gamma-1}\right\|_\infty \sup_{x\in E}\big(C_X(\gamma_0 - 1)\big)^{-\frac{\gamma(x)-1}{\gamma_0-1}}.$$

Therefore, according to the bounded convergence theorem, for each $\theta \geq 0$ and $x \in E$, we have that $I_4(t,\theta,x) \xrightarrow[t\to\infty]{} 0$.

Step 6: We will show that

$$\limsup_{t\to\infty} I_3(t,\theta,x) \leq \gamma_0\Big(\int_0^\theta \|M(ru^{\frac{1}{\gamma_0-1}})\|_{\mathbf{1}_{0\leq u\leq 1}\frac{du}{u};\gamma_0-1}dr\Big)^{\gamma_0-1}, \quad \theta \geq 0, x \in E,$$

where

$$M(t,r,x) := |G(r)^{\gamma_0-1} - g(t,r,x)^{\gamma_0-1}|^{\frac{1}{\gamma_0-1}}, \quad t \geq 0, r \geq 0, x \in E,$$

and

$$M(r,x) := \limsup_{t\to\infty} M(t,r,x); \quad M(r) := \sup_{x\in E} M(r,x), \quad r \geq 0, x \in E.$$

Notice that, according to (5.3.36) and (5.3.38), we have the following bound:

$$M(t,r,x) \leq |r^{\gamma_0-1} + c_f^{\gamma_0-1}r^{\gamma_0-1}|^{\frac{1}{\gamma_0-1}} =: c_6r, \tag{5.3.45}$$

where the constant c_6 is independent of t and x. Therefore, we have

$$M(r,x) \leq M(r) \leq c_6r, \quad r \geq 0, x \in E.$$

From the definition of J_G', J_g' and η_t, we have for each $t \geq 0$ and $r \geq 0$,

$$\begin{aligned} &|J_G'(t,r,\xi) - J_g'(t,r,\xi)| \\ &\leq \gamma_0(\gamma_0 - 1)t\int_0^1 \big(\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa \cdot (\phi\eta_{ut})^{\gamma_0-1}M(ut, ru^{\frac{1}{\gamma_0-1}}, \cdot)^{\gamma_0-1}\big)(\xi_{(1-u)t})du \\ &= \gamma_0C_X^{-1}\int_0^1 \big(\mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\phi^{\gamma_0-1}u^{-1}M(ut, ru^{\frac{1}{\gamma_0-1}}, \cdot)^{\gamma_0-1}\big)(\xi_{(1-u)t})du. \end{aligned} \tag{5.3.46}$$

According to (5.3.45), we have the following upper bound:

$$u^{-1}M(ut, ru^{\frac{1}{\gamma_0-1}}, x) \leq c_6ru^{\frac{2-\gamma_0}{\gamma_0-1}} \leq c_6r, \quad u \in (0,1), r \geq 0, t \geq 0, x \in E.$$

Therefore, fixing an $r \geq 0$, we can apply Lemma 5.2.12 to the function

$$(y,u,t) \mapsto \gamma_0C_X^{-1}\mathbf{1}_{\gamma(y)=\gamma_0}\kappa(y)\phi(y)^{\gamma_0-1}u^{-1}M(ut, ru^{\frac{1}{\gamma_0-1}}, y)^{\gamma_0-1}$$

since it is a bounded Borel function on $E \times (0,1) \times [0,\infty)$. Now, according to Lemma 5.2.12, (5.3.46) and the definitions of $M(r,x), M(r)$ and C_X, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \|J'_G(t,r,\xi) - J'_g(t,r,\xi)\|_{\Pi^\phi_x;\frac{1}{\gamma_0-1}} \qquad (5.3.47)\\
&\le \gamma_0 C_X^{-1} \int_0^1 \left\langle \mathbf{1}_{\gamma(\cdot)=\gamma_0} \kappa \phi^{\gamma_0-1} M(ru^{\frac{1}{\gamma_0-1}},\cdot)^{\gamma_0-1}, \phi\phi^* \right\rangle_m \frac{du}{u}\\
&\le \gamma_0 \int_0^1 M(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1} \frac{du}{u}.
\end{aligned}$$

We now recall the reverse Fatou's lemma in L^p with $p \ge 1$: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions defined on a measure space S with σ-finite measure μ. If there exists a non-negative $L^p(\mu)$-integrable function g on S such that $f_n \le g$ for all n, then according to the classical reverse Fatou's lemma, we have

$$\limsup_{n\to\infty} \|f_n\|_{\mu;p} = \Big(\limsup_{n\to\infty} \int f_n^p d\mu\Big)^{\frac{1}{p}} \le \Big(\int \limsup_{n\to\infty} f_n^p d\mu\Big)^{\frac{1}{p}} = \big\|\limsup_{n\to\infty} f_n\big\|_{\mu;p}.$$

Now, use this version of the revers Fatou's lemma and (5.3.47), we have that

$$\begin{aligned}
\limsup_{t\to\infty} I_3(t,\theta,x) &\le \Big\| \limsup_{t\to\infty} \|J'_G(t,r,\xi) - J'_g(t,r,\xi)\|_{\Pi^{(\phi)}_x;\frac{1}{\gamma_0-1}} \Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}}\\
&\le \Big\|\gamma_0 \int_0^1 M(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1} \frac{du}{u}\Big\|_{\mathbf{1}_{0\le r\le\theta}dr;\frac{1}{\gamma_0-1}}\\
&= \gamma_0\Big(\int_0^\theta \Big(\int_0^1 M(ru^{\frac{1}{\gamma_0-1}})^{\gamma_0-1} \frac{du}{u}\Big)^{\frac{1}{\gamma_0-1}} dr\Big)^{\gamma_0-1}\\
&= \gamma_0\Big(\int_0^\theta \|M(ru^{\frac{1}{\gamma_0-1}})\|_{\mathbf{1}_{0\le u\le 1}\frac{du}{u};\gamma_0-1} dr\Big)^{\gamma_0-1}, \quad \theta \ge 0, x \in E.
\end{aligned}$$

Step 7. We will show that $M(\theta) = 0$ for each $\theta \ge 0$. We first claim that

$$M(\theta) \le c_M \int_0^\theta \big\|M(ru^{\frac{1}{\gamma_0-1}})\big\|_{\mathbf{1}_{0\le u\le 1}\frac{du}{u};\gamma_0-1} dr, \quad \theta \ge 0,$$

for some constant $c_M > 0$. In fact, a direct application of Steps 2-6 gives that, for each $t \ge 0$ and $x \in E$:

$$\begin{aligned}
M(r,x)^{\gamma_0-1} &= \limsup_{t\to\infty} M(t,r,x)^{\gamma_0-1} = \limsup_{t\to\infty} |G(r)^{\gamma_0-1} - g(t,r,x)^{\gamma_0-1}|\\
&\le \limsup_{t\to\infty} \big(I_1(t,\theta,x) + c_f^{\gamma_0-1} I_2(t,\theta,x) + c_f^{\gamma_0-1} I_3(t,\theta,x) + c_f^{\gamma_0-1} I_4(t,\theta,x)\big)\\
&= c_f^{\gamma_0-1} \limsup_{t\to\infty} I_3(t,\theta,x) \le c_f^{\gamma_0-1}\gamma_0\Big(\int_0^\theta \big\|M(ru^{\frac{1}{\gamma_0-1}})\big\|_{\mathbf{1}_{0\le u\le 1}\frac{du}{u};\gamma_0-1} dr\Big)^{\gamma_0-1}.
\end{aligned}$$

Therefore, for each $\theta \geq 0$,

$$M(\theta) = \sup_{x \in E} M(r,x) \leq c_f \gamma_0^{\frac{1}{\gamma_0 - 1}} \int_0^\theta \left\| M(ru^{\frac{1}{\gamma_0-1}}) \right\|_{\mathbf{1}_{0 \leq u \leq 1} \frac{du}{u}; \gamma_0 - 1} dr.$$

According to that $M(\theta) \leq c_6 \theta$ for each θ, we can apply Lemma 5.3.6 to the above inequality to get the desired result. Finally, by the definition of M, $M \equiv 0$ implies the desired assertion (5.3.29).

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